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Chance-Constrained Optimization

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I. Introduction

General Uncertainty Problem (UP)

$$\min c(x;\xi) \text{ subject to } g(x;\xi) \leq 0,\ x \in X$$

- $\xi \in \mathbb{R}^S$ ... data element of the problem
- $x \in X \subset \mathbb{R}^n$ ... decision vector
- $c : \mathbb{R}^n \times \mathbb{R}^S \to \mathbb{R}$ ... objective function
- $g : \mathbb{R}^n \times \mathbb{R}^S \to \mathbb{R}^K$ ... constraint function

Characterization of the problem:

- the knowledge of the data is insufficient (uncertain): we only know that $\xi \in \Xi \subset \mathbb{R}^S$ (Ξ ... uncertainty set)
- the value of the objective is the best possible, given an instance (realization) of ξ
- the constraints are to be satisfied as much as possible, given the instance of ξ

WLOG: $c(x;\xi) := c^T x$

- $g(x;\xi) \leq 0$ to be satisfied for all instances $\xi \in \Xi$[1]:

$$\min c^T x \text{ subject to } g(x;\xi) \leq 0, \ x \in X \quad \forall \xi \in \Xi$$

- no other info on ξ needed/used

... worst-case approach

- issues:
  - numerical tractability
  - conservativeness
- some methods developed if a stochastic information is given

... randomized approach

[1] This Ξ can differ from the uncertainty set defined beforehand but the distinction is not important for our purposes.

Chance Constrained Optimization (CCO) Approach

to Solve General Uncertainty Problem

- $g(x;\xi) \leq 0$ to be satisfied with a prescribed, sufficiently high probability:

$$\min c^T x \text{ subject to } \mathbb{P}\{\xi \in \Xi \mid g(x;\xi) \leq 0\} \geq 1 - \varepsilon, \; x \in X$$

- formal assumptions:
  - ξ is a random vector of a know distribution $\mathbb{P}$ with the support Ξ
  - $\varepsilon \in [0;1]$ is the prescribed probability of violating the uncertain constraints
- issues:
  - convexity
  - numerical tractability

- Formalization of the CCO problem

$$H(x) := \{\xi \in \Xi \mid g(x;\xi) \le 0\}$$
$$G(x) := \mathbb{P}\{H(x)\} = \mathbb{P}\{\xi \in \Xi \mid g(x;\xi) \le 0\},$$
$$X(\varepsilon) := \big\{x \in X \mid \mathbb{P}\{\xi \in \Xi \mid g(x,\xi) \le 0\} \ge 1-\varepsilon\big\} = \{x \in X \mid G(x) \ge 1-\varepsilon$$

- The problem can be rewritten also as

$$\min c^T x \text{ subject to } \mathbb{P}\{H(x)\} \ge 1-\varepsilon,\ x \in X$$
$$\min c^T x \text{ subject to } G(x) \ge 1-\varepsilon,\ x \in X,$$
$$\min c^T x \text{ subject to } x \in X(\varepsilon).$$

- Assume X closed convex set and denote

$$\varphi(\varepsilon) \ldots \text{ optimal objective value of CCO}$$
$$X^*(\varepsilon) \ldots \text{ optimal solution set of CCO}$$

- Sometimes $p = 1-\varepsilon$ used instead.

- $g(x;\xi) := \xi - g(x)$ where $g : \mathbb{R}^n \to \mathbb{R}^K$

$$\min c^T x \text{ subject to } \mathbb{P}\{\xi \in \Xi \mid g(x) \geq \xi\} \geq 1 - \varepsilon, \ \ x \in X$$

- In this case

$$G(x) = F(g(x))$$

where F is K-dimensional cdf (with marginals F_k and corresponding densities f_k).

- $g(x;\xi) := h - Tx$ where $T \in \mathbb{R}^{K\times n}, h \in \mathbb{R}^K$

$$\min c^T x \text{ subject to } \mathbb{P}\{\xi \in \Xi \mid Tx \geq h\} \geq 1 - \varepsilon,\ x \in X$$

- WLOG: h can be deterministic
- if T is deterministic ($\xi = h$ only), the problem is a special case of the RHS problem with the linear $g(x) = Tx$

II. Convexity (theory)

Definition 1

A function $f\colon C \to \mathbb{R}$ is said to be *r-concave* for some $r \in \overline{\mathbb{R}}$ if

1. C is a convex set
2. for each $x, y \in C$ and each $\lambda \in [0;1]$

$$f\big(\lambda x + (1-\lambda)y\big) \geq \big[\lambda f^r(x) + (1-\lambda) f^r(y)\big]^{1/r}$$

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Cases $r = -\infty, 0, +\infty$ treated by continuity.

Proposition 2

1. *If f is r-concave (for some $r \in \overline{\mathbb{R}}$) then it is r'-concave for each $r' \leq r$.*
2. *If f is r-convex (for some $r \in \overline{\mathbb{R}}$) then it is r'-convex for each $r' \geq r$.*

Prominent One-Dimensional Examples

function	r-concave for $r \in$	r-convex for $r \in$	dom f	Note
$\sqrt[3]{x}$	$[-\infty; 3]$	$[3; +\infty]$	$\mathbb{R}$	
$\sqrt{x}$	$[-\infty; 2]$	$[2; +\infty]$	$\mathbb{R}_+$	
x	$[-\infty; 1]$	$[1; +\infty]$	$\mathbb{R}$	ordinary concave/convex
x^2	$[-\infty; \frac{1}{2}]$	$[\frac{1}{2}; +\infty]$	$\mathbb{R}$	
x^3	$[-\infty; \frac{1}{3}]$	$[\frac{1}{3}; +\infty]$	$\mathbb{R}$	
e^x	$[-\infty; 0]$	$[0; +\infty]$	$\mathbb{R}$	log-concave/convex
x^{-3}	$[-\infty; -\frac{1}{3}]$	$[-\frac{1}{3}; +\infty]$	$\mathbb{R}_{++}$	
x^{-2}	$[-\infty; -\frac{1}{2}]$	$[-\frac{1}{2}; +\infty]$	$\mathbb{R}_{++}$	
x^{-1}	$[-\infty; -1]$	$[-1; +\infty]$	$\mathbb{R}_{++}$	
$x^{-1/2}$	$[-\infty; -2]$	$[-2; +\infty]$	$\mathbb{R}_{++}$	

- 0-**concave** function f is also characterized by the inequality

$$f\big(\lambda x + (1-\lambda)y\big) \geq f^{\lambda}(x) \cdot f^{(1-\lambda)}(y).$$

  It is called log-concave as $\ln f$ is a concave function.
- 0-**convex** (log-convex) functions are treated similarly.

Special Cases II
Quasi-Concave and Quasi-Convex Functions

- $-\infty$-**concave** function f is also characterized by the inequality

  $$f\big(\lambda x + (1-\lambda)y\big) \geq \min f(x), f(y).$$

  It is called quasi-concave function. Equivalently,

  $$\mathrm{lev}_{\geq\alpha} := x \mid f(x) \geq \alpha$$

  are convex.

- $+\infty$-**convex** function f is also characterized by the inequality

  $$f\big(\lambda x + (1-\lambda)y\big) \leq \max f(x), f(y).$$

  It is called quasi-convex function. Equivalently,

  $$\mathrm{lev}_{\leq\alpha} := x \mid f(x) \leq \alpha$$

  are convex.

Proposition 3 (Boyd, Vandenberghe (2004))

A continuous function $f : \mathbb{R} \to \mathbb{R}$ is ***quasi-concave*** *iif at least one of the following assertions holds*

1. *f is nondecreasing*
2. *f is nonincreasing*
3. $\exists c \in \operatorname{dom} f$ *such that*
   - *$f(t)$ is nondecreasing if $t \leq c$*
   - *$f(t)$ is nonincreasing if $t \geq c$*

   (i. e., c can by any of the global maximizers).

Proposition 4 (Boyd, Vandenberghe (2004))

A continuous function $f\colon \mathbb{R} \to \mathbb{R}$ is ***quasi-convex*** *iif at least one of the following assertions holds*

1. *f is nondecreasing*
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3. *$\exists c \in \operatorname{dom} f$ such that*
   - *$f(t)$ is nonincreasing if $t \leq c$*
   - *$f(t)$ is nondecreasing if $t \geq c$*

   (i. e., c can by any of the global minimizers).

Quasi-concave and/or quasi-convex functions are sometimes called unimodal.

Definition 5

$\mathbb{P}$ is *r-concave* if for any Borel convex sets A, B with $\mathbb{P}(A), \mathbb{P}(B) > 0$ and every $\lambda \in [0; 1]$ one has

$$\mathbb{P}\big(\lambda A + (1-\lambda)B\big) \geq [\lambda \mathbb{P}^r(A) + (1-\lambda)\mathbb{P}^r(B)]^{1/r}. \qquad (1)$$

cases $r = -\infty, 0, +\infty$ treated by continuity.

Proposition 6 (Borell (1975))

1. *An r-concave probability measure induces an r-concave distribution function.*
2. *If $\mathbb{P}$ is a quasi-concave measure on $\mathbb{R}^S$ and $\dim \operatorname{supp} \mathbb{P} = S$ then $\mathbb{P}$ has a density (with respect to the Lebesgue measure).*

Proposition 7 (Brascamp, Lieb (1976))

Let Ξ be convex such that $\dim \operatorname{aff} \Xi = S'$. Then $\mathbb{P}$ is r'-concave with $r' \in [-\infty; \frac{1}{S'}] \Leftrightarrow$ its probability density (wrt. Lebesgue measure on $\operatorname{aff} \Xi$) is r-concave where

$$r := \begin{cases} -\frac{1}{S'} & \text{if } r' = -\infty, \\ \frac{r'}{1-S'r'} & \text{if } r' \in (-\infty; \frac{1}{S'}), \\ +\infty & \text{if } r' = \frac{1}{S'}. \end{cases}$$

- In particular:
  - if a density is (at least) $(-\frac{1}{S'})$-concave then the corresponding probability measure is (at least) quasi-concave;
  - log-concave density induces log-concave distribution and vice-versa

III. Convexity and CCO

Convexity of the General CCO Problem

Proposition 8

$X(\varepsilon)$ is convex $\Leftrightarrow$ $G(x)$ is quasi-concave on X.

(still assuming X convex). Recall

$$G(x) := \mathbb{P}\{\xi \in \Xi \mid g(x;\xi) \leq 0\},$$
$$X(\varepsilon) := \{x \in X \mid G(x) \geq 1 - \varepsilon\}$$

Theorem 9 (Prékopa (1995))

If

1. $g_k(x;\xi)$ ... *quasi-concave functions of* x *and* ξ *(components of* g*);*
2. ξ ... *r.v. with r-concave density;*
3. $r \geq -\frac{1}{S}$ *(S is the dimension of* ξ*);*

Then $G(x)$ *is* $\gamma = \frac{r}{1+rS}$*-concave function on the set*

$$D := \{x \mid \exists z \in \mathbb{R}^s : g(x;z) \geq 0\}$$

- (2) implies ξ have γ-concave probability
- $r = 0$ ... $G(x)$ is log-concave ($\Rightarrow X(\varepsilon)$ is convex)
- $r = -\frac{1}{s}$ ... $G(x)$ is quasi-concave ($\Rightarrow X(\varepsilon)$ is convex)

- $g(x;\xi) := \xi - g(x)$

$$\min c^T x \text{ subject to } \mathbb{P}\{\xi \in \Xi \mid g(x) \geq \xi\} \geq 1 - \varepsilon, \ x \in X$$

- needed: $g(\cdot;\cdot)$ quasi-concave
- problem: quasi-concavity not preserved under addition (only for $r \geq 1$)
- classical sufficient assumption (Prékopa (1971)): g is concave

Henrion, Strugarek (2008), Henrion, Strugarek (2011), Cheng, Houda, Lisser (2014), van Ackooij (2015)

- general idea of the results: show that marginal constraints $F_k \circ g_k$ are concave, then take convenient copula (independent / log-exp-concave / Archimedean / δ-γ-concave) to obtain a concave $G(x)$

Definition 10

For some $r \in \mathbb{R}$, a function $f : \mathbb{R} \to \mathbb{R}$ is called r-decreasing with the threshold $t^*(r) > 0$ if the function $t^r f(t)$ is strictly decreasing $\forall t > t^*(r)$.

Lemma 11

If a density f is $(r+1)$-decreasing (with a threshold $t^*(r)$) for $r > 0$ then $F \circ [\cdot]^{-1/r}$ is concave (on $(0, t^*(r)^{-r})$).

- The concavity of $F_k \circ g_k$ follows by the trick

$$F_k \circ g_k = \left(F_k \circ [\cdot]^{-1/r_k}\right) \circ \left([\cdot]^{-r_k} \circ g_k\right)$$

Theorem 12

If

1. g_k *are* $(-r_k)$*-concave function for some* $r_k > 0$*;*
2. ξ *has independent components with* $(r_k + 1)$*-decreasing densities with the thresholds* $t_k^*(r_k + 1) > 0$
3. $\varepsilon < \varepsilon^* := 1 - \max F_k(t_k^*(r_k + 1))$

then $X(\varepsilon)$ *is convex.*

- (2) can be replaced by a tighter condition: the reversed hazard rate functions $\frac{f_k}{F_k}$ are $(r_k + 1)$-decreasing (with some thresholds $t_k^*(r_k + 1) > 0$).

Theorem 13

If

1. g_k *are* $(-r_k)$*-concave function for some* $r_k > 0$*;*
2. ξ_k *have* $(r_k + 1)$*-decreasing densities with thresholds* $t_k^*(r_k + 1) > 0$
3. *the joint distribution of* ξ *is driven by a copula* C *which*
   - (a) *is either Archimedean,*
   - (b) *or for which* $\ln \circ C \circ \exp$ *is concave on* $\bigtimes_k \left[\ln F_k\left[t_k^*(r_k + 1)\right]; 1\right)$
4. $\varepsilon < \varepsilon^* := 1 - \max F_k(t_k^*(r_k + 1))$

then $X(\varepsilon)$ *is convex.*

- (b) is even improved by VAN ACKOOIJ (2015)

HENRION (2007): complete description of

$$X(\varepsilon) := \left\{x \in X \mid \mathbb{P}\{\xi \in \Xi \mid \xi^T g(x) \leq h\} \geq 1 - \varepsilon\right\}$$

Theorem 14

1. *ξ is elliptically distributed with $(\mu, \Sigma \succ 0)$*
2. *$g(x)$ is*
   - (a) *either affine linear (cf.* KATAOKA (1963), VAN DE PANNE, POPP(1963)*),*
   - (b) *or with nonnegative convex components, $\mu \geq 0$, Σ with nonnegative elements.*

Then $X(\varepsilon)$ is convex for all $\varepsilon < \frac{1}{2}$. If ξ has a (strictly) positive density, then the above works also for $\varepsilon = \frac{1}{2}$.

Also negative result given: $X(\varepsilon)$ is nonconvex if

1. $h < 0$ and $\varepsilon > \frac{1}{2}$, or
2. $h \geq 0$ and $\varepsilon \in \left(\frac{1}{2}; \Phi\left(\sqrt{\mu^T \Sigma^{(-1)} \mu}\right)\right)$

Theorem 15 (Prékopa (1974))

If T has independent normally distributed rows such that their covariance matrices are constant multiples of each other, and $\varepsilon \leq \frac{1}{2}$ then

$$G(x) = \mathbb{P}\{Tx \leq h\}$$

is quasi-concave on $\{G(x) \geq \frac{1}{2}\}$ thus X_ε is convex.

- extended by Prékopa, Yoda, Subasi (2011) (uniformly quasi-concavity)

$$X(\varepsilon) = \mathbb{P}\{\xi \in \Xi \mid \xi_k^T x \le h_k \; \forall k\} \ge 1 - \varepsilon, \;\; x \in X$$

Theorem 16 (Henrion, Strugarek (2008), with an improved threshold by Cheng, Houda, Lisser (2014)))

If ξ_k are pairwise independent normally distributed rows with (μ_k, Σ_k), and

$$\varepsilon < \Phi\left(-\frac{1}{2}\max\left\{\frac{||\mu_k||}{\sqrt{\lambda_{\min}^{(k)}}} + \sqrt{8 + \frac{||\mu_k||^2}{\lambda_{\min}^{(k)}}}\right\}\right)$$

where $\lambda_{\min}^{(k)}$ are the smallest eigenvalues of Σ_k, then $X(\varepsilon)$ is convex.

- extension to elliptical distributions straightforward
- further extension to dependent rows is possible but a problem with the dependence of the copula on the decision vector exists (currently investigated)

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