General conception of derivative

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Differentiability of a function

Consider a function $f : \mathbb{D} \to \mathbb{R}^m$, where $\mathbb{D} \subset \mathbb{R}^n$.
Then, the derivative of f at a point $x \in \mathbb{D}$ in a direction $h \in \mathbb{R}^n$ is defined by

$$f'(x;h) \quad = \quad \lim_{t \to 0} \frac{1}{t} \left(f(x+th) - f(x) \right), \text{ whenever the limit exists.}$$

Let us denote $\mathbb{D}'_x = \{ h \in \mathbb{R}^n \ : \ f'(x;h) \ \text{exists} \}$. Hence,

- $\mathbb{D}'_x$ is a double-cone, i.e. $\alpha h \in \mathbb{D}'_x$ whenever $\alpha \in \mathbb{R}$ and $h \in \mathbb{D}'_x$.
- The function $f'(x;\cdot)$ is homogeneous on $\mathbb{D}'_x$ (a double-cone function), i.e. $f'(x;\alpha h) = \alpha f'(x;h)$ for each $\alpha \in \mathbb{R}$, $h \in \mathbb{D}'_x$.
- For each $h \in \mathbb{D}'_x$ there is an $\delta > 0$ such that $x + th \in \mathbb{D}$ for all $t \in [-\delta, \delta]$.

Differentiability of a function

- Function f is called differentiable at a point $x \in \mathbb{D}$ if $f'(x; h)$ exists for each $h \in \mathbb{R}^n$.
- Function f is called Gâteaux differentiable at $x \in \mathbb{D}$ if f is differentiable at x and $f'(x; \cdot)$ is a continuous linear function on $\mathbb{R}^n$.
- Function f is called Hadamard differentiable at $x \in \mathbb{D}$ if f is differentiable at x and $f'(x; \cdot)$ is a linear function on $\mathbb{R}^n$ fulfilling

$$\lim_{n \to +\infty} \frac{1}{t_n} \left(f(x + t_n h_n) - f(x) \right) = f'(x; h)$$

for all sequences $t_n \in \mathbb{R}$, $t_n \neq 0$, $h_n \in \mathbb{R}^n$, $x + t_n h_n \in \mathbb{D}$, $\lim_{n \to +\infty} t_n = 0$ and $\lim_{n \to +\infty} h_n = h \in \mathbb{R}^n$.

Differentiability of a function

- Function f is called boundedly differentiable at $x \in \mathbb{D}$ being differentiable at x and fulfilling

$$\lim_{t \to 0} \sup_{\substack{h \in \mathbb{R}^n, \|h\|=1 \\ x+th \in \mathbb{D}}} \left\| \frac{1}{t} \left(f(x+th) - f(x) \right) - f'(x;h) \right\| = 0.$$

- Function f is called Fréchet differentiable at $x \in \mathbb{D}$ being Gâteaux differentiable at x and fulfilling

$$\lim_{t \to 0} \sup_{\substack{h \in \mathbb{R}^n, \|h\|=1 \\ x+th \in \mathbb{D}}} \left\| \frac{1}{t} \left(f(x+th) - f(x) \right) - f'(x;h) \right\| = 0.$$

Counterexample

Consider a homogeneous function g on $\mathbb{R}^n$. Then,

$$\forall t \neq 0 \ \forall h \in \mathbb{R}^n \quad \frac{1}{t}\left(g(th) - g(0)\right) = \frac{1}{t} g(th) = g(h)$$

Hence, $g'(0; h) = g(h)$ for each $h \in \mathbb{R}^n$. Thus, g is boundedly differentiable at the origin.
Particularly,

$$\begin{aligned} F(x_1, x_2) &= 0 \quad \text{whenever } x_2 \neq 0, \\ &= x_1 \quad \text{whenever } x_2 = 0. \end{aligned}$$

F is homogenous, therefore, it is boundedly differentiable at the origin. But, F is discontinuous and, hence, its derivative at origin $F'(0; \cdot) = F$ is discontinuous.

Bounded, Fréchet and Hadamard differentiability possess useful equivalent descriptions.

- Function f is boundedly differentiable at $x \in \mathbb{D}$ if and only if it is differentiable at x and fulfills

$$\lim_{t\longrightarrow 0} \sup_{\substack{h\in B\\ x+th\in\mathbb{D}}} \left\| \frac{1}{t}\left(f(x+th)-f(x)\right) - f'(x;h) \right\| = 0$$

  for each bounded set $B \subset \mathbb{R}^n$.

- Function f is Fréchet differentiable at $x \in \mathbb{D}$ if and only if it is Gâteaux differentiable at x and fulfills

$$\lim_{t\longrightarrow 0} \sup_{\substack{h\in B\\ x+th\in\mathbb{D}}} \left\| \frac{1}{t}\left(f(x+th)-f(x)\right) - f'(x;h) \right\| = 0$$

  for each bounded set $B \subset \mathbb{R}^n$.

- Function f is Hadamard differentiable at $x \in \mathbb{D}$ if and only if it is Gâteaux differentiable at x and fulfills

$$\lim_{t \longrightarrow 0} \sup_{\substack{h \in K \\ x + th \in \mathbb{D}}} \left\| \frac{1}{t} \left(f(x + th) - f(x) \right) - f'(x; h) \right\| = 0$$

for each compact $K \subset \mathbb{R}^n$.

Differentiability of a function

In finite dimension we have the following scheme:

$$\begin{array}{ccccc} \text{Hadamard diff.} & \Longrightarrow & \text{Gâteaux diff.} & \Longrightarrow & \text{differentiability} \\ \Updownarrow & & & & \\ \text{Fréchet diff.} & \Longrightarrow & \text{boundedly diff.} & \nearrow\!\!\!\nearrow & \end{array}$$

If f is Gâteaux, Hadamard or Fréchet differentiable then the gradient

$$\nabla f(x) = \begin{pmatrix} \frac{\partial f_1}{\partial x_1}(x) & \cdots & \frac{\partial f_m}{\partial x_1}(x) \\ \vdots & \ddots & \vdots \\ \frac{\partial f_1}{\partial x_n}(x) & \cdots & \frac{\partial f_m}{\partial x_n}(x) \end{pmatrix}$$

possesses the property

$$f'(x; h) = \nabla f(x)^\top h.$$

Directional differentiability of a function

Consider again a function $f : \mathbb{D} \to \mathbb{R}^m$, where $\mathbb{D} \subset \mathbb{R}^n$. Then, the directional derivative of f at a point $x \in \mathbb{D}$ in a direction $h \in \mathbb{R}^n$ is defined by

$$f'_+(x; h) = \lim_{t \to 0+} \frac{1}{t} \left(f(x + th) - f(x) \right), \text{ whenever the limit exists.}$$

Let $\mathbb{D}'_{x+} = \left\{ h \in \mathbb{R}^n \ : \ f'_+(x; h) \text{ exists} \right\}$ denote the definition region of $f'_+(x; \cdot)$.

Hence,

- $\mathbb{D}'_{x+}$ is a cone, i.e. $\alpha h \in \mathbb{D}'_{x+}$ whenever $\alpha \geq 0$ and $h \in \mathbb{D}'_{x+}$.
- The function $f'_+(x; \cdot)$ is positively homogeneous on $\mathbb{D}'_{x+}$ (a cone function), i.e. $f'_+(x; \alpha h) = \alpha f'_+(x; h)$ for each $\alpha \geq 0$, $h \in \mathbb{D}'_{x+}$.
- For each $h \in \mathbb{D}'_x$ there is an $\delta > 0$ such that $x + th \in \mathbb{D}$ for all $t \in [0, \delta]$.

Directional differentiability of a function

- Function f is called Gâteaux directionally differentiable at a point $x \in \mathbb{D}$ if $f'_+(x;h)$ exists for each $h \in \mathbb{R}^n$.
- Function f is called Hadamard directionally differentiable at $x \in \mathbb{D}$ if f is Gâteaux directionally differentiable at x and

$$\lim_{n\to+\infty} \frac{1}{t_n}\left(f(x+t_n h_n) - f(x)\right) = f'_+(x;h)$$

for all sequences $t_n > 0$, $h_n \in \mathbb{R}^n$, $x + t_n h_n \in \mathbb{D}$, $\lim_{n\to+\infty} t_n = 0$ and $\lim_{n\to+\infty} h_n = h \in \mathbb{R}^n$.

- Function f is called Fréchet directionally differentiable at $x \in \mathbb{D}$ being Gâteaux directionally differentiable at x and fulfilling

$$\lim_{t\to 0+} \sup_{\substack{h\in\mathbb{R}^n,\ \|h\|=1 \\ x+th\in\mathbb{D}}} \left\| \frac{1}{t}\left(f(x+th) - f(x)\right) - f'_+(x;h) \right\| = 0.$$

Requirement of continuous linearity is removed from the definition of Gâteaux directional differentiability.

Proposition

If a function f *is Hadamard directionally differentiable at* x *then* $f'_+(x;\cdot)$ *is a continuous function.*

Counterexample

Consider a positively homogeneous function g on $\mathbb{R}^n$. Then,

$$\forall t > 0\ \forall h \in \mathbb{R}^n \quad \frac{1}{t}\left(g(th) - g(0)\right) = \frac{1}{t} g(th) = g(h)$$

Hence, $g'_+(0; h) = g(h)$ for each $h \in \mathbb{R}^n$. Thus, g is Fréchet directionally differentiable at the origin.
Particularly,

$$\begin{aligned} F(x_1, x_2) &= 0 \quad \text{whenever } x_2 \neq 0, \\ &= |x_1| \quad \text{whenever } x_2 = 0. \end{aligned}$$

F is positively homogenous, therefore, it is Fréchet directionally differentiable at the origin. But, F is discontinuous and, hence, its directional derivative at origin $F'_+(0; \cdot) = F$ is discontinuous.

Fréchet and Hadamard directional differentiability possess useful equivalent definitions.

- Function f is Fréchet directionally differentiable at $x \in \mathbb{D}$ if and only if it is Gâteaux directionally differentiable at x and fulfills

$$\lim_{t \to 0+} \sup_{\substack{h \in B \\ x+th \in \mathbb{D}}} \left\| \frac{1}{t} \left(f(x+th) - f(x) \right) - f'_{+}(x;h) \right\| = 0$$

  for each bounded set $B \subset \mathbb{R}^n$.

- Function f is Hadamard directionally differentiable at $x \in \mathbb{D}$ if and only if it is Gâteaux directionally differentiable at x, $f'_{+}(x;\cdot)$ is a continuous function and fulfills

$$\lim_{t \to 0+} \sup_{\substack{h \in K \\ x+th \in \mathbb{D}}} \left\| \frac{1}{t} \left(f(x+th) - f(x) \right) - f'_{+}(x;h) \right\| = 0$$

  for each compact $K \subset \mathbb{R}^n$.

Directional differentiability of a function

In finite dimension we have the following scheme:

$$\begin{array}{ccc} \text{Hadamard dir. diff.} & \Longrightarrow & \text{Gâteaux dir. diff.} \\ \Downarrow & & \\ \text{Fréchet dir. diff.} & \nearrow\!\!\!\nearrow & \end{array}$$

Hadamard dir. diff. possesses continuous directional derivative.

There is no linearity. Thus, we have no reasonable equivalent to gradient.

Hadamard differentiability tangentially to a set

The concept of directional differentiability is limited by the assumption that $f'_+(x; h)$ must exist for all $h \in \mathbb{R}^n$. With intention to relax the requirement, the concept of Hadamard directional differentiability tangentially to a set was developed, cf. [5], [3]. Unfortunately, the definition slightly differs due book to another. Therefore, we present a definition covering both of them.

Hadamard differentiability tangentially to a set

We define the Bouligand tangent cone
(also contingent cone, Bouligand contingent cone)
to $A \subset \mathbb{R}^n$ at $x \in \mathbb{R}^n$ by

$$\mathbb{T}(x\,;A) = \left\{ h \in \mathbb{R}^n \ : \ \begin{array}{c} \text{there is a sequence } a_n \in A, t_n > 0, \\ \lim\limits_{n\to+\infty} t_n = 0, \lim\limits_{n\to+\infty} \frac{1}{t_n}(a_n - x) = h \end{array} \right\}.$$

The contingent cone is the limsup in Kuratowski sense

$$\mathbb{T}(x\,;A) = K\!-\!\limsup_{t\to 0+} \frac{1}{t}(A - x).$$

Hadamard differentiability tangentially to a set

$\mathbb{G} \subset \mathbb{D} \subset \mathbb{R}^n$, $\mathbb{H} \subset \mathbb{R}^n$ be sets and $f : \mathbb{D} \to \mathbb{R}^m$ be a function.

- Function f is called Hadamard directionally differentiable at $x \in \mathbb{D}$ tangentially to $(\mathbb{G}, \mathbb{H})$ if there is a function $f'_H(x, \cdot\,; \mathbb{G}, \mathbb{H}) : \mathbb{T}(x\,; \mathbb{G}) \cap \mathbb{T}(x\,; \mathbb{H}) \to \mathbb{R}^m$ fulfilling

$$\lim_{n \to +\infty} \frac{1}{t_n} \left(f(x + t_n h_n) - f(x) \right) = f'_H(x, h\,; \mathbb{G}, \mathbb{H})$$

  for all sequences $t_n \in \mathbb{R}$, $t_n > 0$, $h_n \in \mathbb{R}^n$, $x + t_n h_n \in \mathbb{G}$, $\lim_{n \to +\infty} t_n = 0$ and $\lim_{n \to +\infty} h_n = h \in \mathbb{T}(x\,; \mathbb{H})$. (Of course, $h \in \mathbb{T}(x\,; \mathbb{G})$ according to the definition of the contingent cone.)
- Function f is called Hadamard differentiable at $x \in \mathbb{D}$ tangentially to $(\mathbb{G}, \mathbb{H})$ being Hadamard directionally differentiable at $x \in \mathbb{D}$ tangentially to $(\mathbb{G}, \mathbb{H})$ and if $\mathbb{T}(x\,; \mathbb{G}) \cap \mathbb{T}(x\,; \mathbb{H})$ is a double-cone and $f'_H(x, -h\,; \mathbb{G}, \mathbb{H}) = -f'_H(x, h\,; \mathbb{G}, \mathbb{H})$ for all $h \in \mathbb{T}(x\,; \mathbb{G}) \cap \mathbb{T}(x\,; \mathbb{H})$.

Hadamard differentiability tangentially to a set

Connection between Hadamard directional differentiability tangentially to $(\mathbb{G}, \mathbb{H})$ and directional differentiability.

Proposition

Let $f : \mathbb{D} \to \mathbb{R}^m$ be Hadamard directionally differentiable at x tangentially to $(\mathbb{G}, \mathbb{H})$ and $h \in \mathbb{T}(x\,; \mathbb{G}) \cap \mathbb{T}(x\,; \mathbb{H})$.
Then, the function f is directionally differentiable at x in the direction h if and only if there is a $\varepsilon_h > 0$ such that $x + th \in A$ for all $0 < t < \varepsilon_h$.
In such a case, we have $f'_+(x; h) = f'_H(x, h\,; \mathbb{G}, \mathbb{H})$.

Proposition

If a function f is Hadamard directionally differentiable at x tangentially to $(\mathbb{G}, \mathbb{H})$ then $f'_H(x, \cdot\,; \mathbb{G}, \mathbb{H})$ is a continuous function on $\mathbb{T}(x\,; \mathbb{G}) \cap \mathbb{T}(x\,; \mathbb{H})$.

Hadamard differentiability tangentially to a set

We possess an equivalent definition.

- Function f is Hadamard directionally differentiable at $x \in \mathbb{D}$ tangentially to $(\mathbb{G}, \mathbb{H})$ if and only if $f'_H(x, h\,;\,\mathbb{G}, \mathbb{H})$ exists for all $h \in \mathbb{T}(x\,;\mathbb{G}) \cap \mathbb{T}(x\,;\mathbb{H})$ and

$$\lim_{\substack{t \to 0+ \\ \varepsilon \to 0+}} \sup_{\substack{h \in K,\, \|\xi\| \leq \varepsilon \\ x + t(h+\xi) \in \mathbb{G}}} \left\| \frac{1}{t} \left(f(x + t(h + \xi)) - f(x) \right) - f'_H(x, h\,;\,\mathbb{G}, \mathbb{H}) \right\| = 0$$

  for all compacts $K \subset \mathbb{T}(x\,;\mathbb{G}) \cap \mathbb{T}(x\,;\mathbb{H})$.

Contingent derivative of a set-valued mapping

Another difficulties arise, if f is a set-valued mapping between two topological vector spaces. Such a case appears very naturally, for instance in a stochastic optimization theory.
Let us consider an optimization problem

$$\varphi(x) = \max \{f(u; x) : u \in \mathcal{U}_x\}$$

depending on a parameter $x \in \mathcal{X} \subset \mathbb{R}^p$, $\mathcal{U}_x \subset \mathbb{R}^n$. The set of all ε-optimal solutions

$$\psi(x; \varepsilon) = \{u \in \mathcal{U}_x : f(u; x) \geq \varphi(x) - \varepsilon\}, \ \varepsilon > 0$$

is of our interest. The mappings is naturally set-valued.
Considering set-valued mappings for purpose of Delta Theorem, one needs a generalization of Hadamard derivative. A convenient one is called contingent derivative and its definition can be found in any monograph on set-valued functions; e.g. in [1], [2].

Contingent derivative of a set-valued mapping

$f : \mathbb{R}^n \to 2^{\mathbb{R}^m}$ be a set-valued function, $x \in \mathbb{R}^n$ and $y \in \mathbb{R}^m$.

- Contingent derivative of f at (x, y) is defined as a set-valued function $\nabla f(x, y; \cdot) : \mathbb{R}^n \to 2^{\mathbb{R}^m}$ with the property:

$$z \in \nabla f(x, y; h) \iff (0, h, z) \in \operatorname{clo}\left(Df(x, y)\right),$$

where $Df(x, y) = \{(t, h, z) \ : \ t > 0, h \in \mathbb{R}^n, y + tz \in f(x + th)\}$.

Contingent derivative of a set-valued mapping

The contingent derivative is connected with Hadamard directional differentiability tangentially to a set.

Proposition

$\mathbb{G} \subset \mathbb{D} \subset \mathbb{R}^n$, $\mathbb{H} \subset \mathbb{R}^n$ and $x \in \mathbb{D}$. Let a function $f : \mathbb{D} \to \mathbb{R}^m$ be Hadamard directionally differentiable at x tangentially to $(\mathbb{G}, \mathbb{H})$. Define a set-valued function $F : \mathbb{R}^n \to 2^{\mathbb{R}^m}$ such that

$$\begin{aligned} F(v) &= \{f(v)\} \text{ if } v \in \mathbb{G}, \\ &= \emptyset \quad \text{if } v \notin \mathbb{G}. \end{aligned}$$

Hence,

$$\begin{aligned} \nabla F(x, f(x); h) &= \{f'_H(x, h\,; \mathbb{G}, \mathbb{H})\} \text{ if } h \in \mathbb{T}(x\,; \mathbb{G}) \cap \mathbb{T}(x\,; \mathbb{H}), \\ &= \emptyset \text{ whenever } y \neq f(x), \\ &= \emptyset \text{ whenever } h \notin \mathbb{T}(x\,; \mathbb{G}), \ y \in \mathbb{R}^m. \end{aligned}$$

Limits in Kuratowski sense

For indexed sets $A_t \subset \mathbb{R}^n$, $t > 0$, 'lim', 'limsup' and 'liminf' in Kuratowski sense are

$$K-\limsup_{t \to 0+} A_t = \left\{ x \in \mathbb{R}^n : \begin{array}{l} \text{there is a sequence } a_{t_n} \in A_{t_n},\ t_n > 0, \\ \lim\limits_{n \to +\infty} t_n = 0,\ \lim\limits_{n \to +\infty} a_{t_n} = x \end{array} \right\},$$

$$K-\liminf_{t \to 0+} A_t = \left\{ x \in \mathbb{R}^n : \begin{array}{l} \text{there is a selection } a_t \in A_t,\ t > 0, \\ \lim\limits_{t \to 0+} a_t = x \end{array} \right\}.$$

If both 'limits' coincide we speak about the limit in Kuratowski sense and set

$$K-\lim_{t \to 0+} A_t = K-\limsup_{t \to 0+} A_t = K-\liminf_{t \to 0+} A_t .$$

Let us mention that all limits in Kuratowski sense are closed sets.

An agreement:

$$
\begin{aligned}
y \to x & \quad \text{means that } y \text{ tends to } x, \\
y \to x+ & \quad \text{means that } y \text{ tends to } x \text{ and } y > x, \\
y \to x- & \quad \text{means that } y \text{ tends to } x \text{ and } y < x, \\
y \to x* & \quad \text{means that } y \text{ tends to } x \text{ and } y \neq x.
\end{aligned}
$$

Tangent cones

Bouligand tangent cone
(also contingent cone, Bouligand contingent cone)
to $A \subset \mathbb{R}^n$ at $x \in \mathbb{R}^n$ is

$$\mathbb{T}(x\,;A) = \left\{ h \in \mathbb{R}^n \ : \ \begin{array}{c} \text{there is a sequence } a_n \in A, t_n > 0, \\ \lim\limits_{n\to+\infty} t_n = 0, \ \lim\limits_{n\to+\infty} \frac{1}{t_n}(a_n - x) = h \end{array} \right\}.$$

The Bouligand tangent cone is the limsup in Kuratowski sense

$$\mathbb{T}(x\,;A) = K\!-\!\limsup_{t\to 0+} \frac{1}{t}(A - x).$$

Clarke tangent cone to $A \subset \mathbb{R}^n$ at $x \in \mathbb{R}^n$ is

$$\mathbb{T}_C(x\,;A) = K\!-\!\liminf_{\substack{y\to x, y\in A \\ t\to 0+}} \frac{1}{t}(A - y).$$

Normal cones

Prenormal cone (also Regular normal cone, Fréchet normals) to $A \subset \mathbb{R}^n$ at $x \in \mathbb{R}^n$ is

$$\widehat{\mathbb{N}}(x\,;A) = \left\{ h \in \mathbb{R}^n \;:\; \begin{array}{l} \text{for each sequence } a_n \in A,\, a_n \neq x,\, a_n \to x, \\ \limsup_{n\to+\infty} \frac{h^\top(a_n - x)}{\|a_n - x\|} \leq 0 \end{array} \right\}.$$

For $\varepsilon \geq 0$,
Set of ε-normals to $A \subset \mathbb{R}^n$ at $x \in \mathbb{R}^n$ is

$$\widehat{\mathbb{N}}_\varepsilon(x\,;A) = \left\{ h \in \mathbb{R}^n \;:\; \begin{array}{l} \text{for each sequence } a_n \in A,\, a_n \to x, \\ \limsup_{n\to+\infty} \frac{h^\top(a_n - x)}{\|a_n - x\|} \leq \varepsilon \end{array} \right\}.$$

Of course, $\widehat{\mathbb{N}}_0(x\,;A) = \widehat{\mathbb{N}}(x\,;A)$.

Normal cones

Normal cone to $A \subset \mathbb{R}^{n}$ at $x \in \mathbb{R}^{n}$ is

$$\mathbb{N}(x\,;A) \quad = \quad K\!-\!\limsup_{\substack{y \longrightarrow x \\ \varepsilon \longrightarrow 0+}} \widehat{\mathbb{N}}_{\varepsilon}(y\,;A).$$

If A is closed then

$$\mathbb{N}(x\,;A) \quad = \quad K\!-\!\limsup_{y \longrightarrow x} \widehat{\mathbb{N}}(y\,;A).$$

Tangent and Normal cones

Always,

$$\begin{aligned}
\mathbb{T}_C(x; A) &\subset \mathbb{T}(x; A), \\
\mathbb{N}(x; A) &\supset \widehat{\mathbb{N}}(x; A), \\
v \in \widehat{\mathbb{N}}(x; A) &\iff \forall w \in \mathbb{T}(x; A) \quad v^\top w \le 0, \\
\mathbb{N}(x; A) &= \mathbb{T}(x; A)^*, \\
\mathbb{N}(x; A)^* &\supset \mathbb{T}(x; A).
\end{aligned}$$

Hence analogically, Clarke normal cone is defined by

$$\mathbb{N}_C(x; A) = \mathbb{T}_C(x; A)^*.$$

Clarke normal cone fulfills

$$\mathbb{N}_C(x; A)^* = \mathbb{T}_C(x; A).$$

Tangent and Normal cones

For a convex set $C \subset \mathbb{R}^n$

$$\begin{aligned} \mathbb{T}(x; C) &= \operatorname{clo}\left(\{w \in \mathbb{R}^n : \exists \lambda > 0 \quad \text{such that} \quad x + \lambda w \in C\}\right), \\ \mathbb{N}(x; C) = \widehat{\mathbb{N}}(x; C) &= \left\{v \in \mathbb{R}^n : \forall c \in C \quad v^\top (c - x) \le 0\right\}, \\ \widehat{\mathbb{N}}_\varepsilon(x; C) &= \left\{v \in \mathbb{R}^n : \forall c \in C \quad v^\top (c - x) \le \varepsilon \|c - x\|\right\}. \end{aligned}$$

Tangent and Normal cones

Let a function $f : \mathbb{R}^n \to \mathbb{R}^m$ be Hadamard differentiable at $x \in \mathbb{R}^n$. Then,

$$\mathbb{T}(x; \operatorname{graph}(f)) = \left\{ \begin{pmatrix} \nabla f(x)^\top h \\ h \end{pmatrix} : h \in \mathbb{R}^n \right\},$$

$$\widehat{\mathbb{N}}(x; \operatorname{graph}(f)) = \left\{ \begin{pmatrix} w \\ -\nabla f(x)\, w \end{pmatrix} : w \in \mathbb{R}^m \right\},$$

$$\mathbb{T}(x; \operatorname{epi}(f)) = \left\{ \begin{pmatrix} y \\ h \end{pmatrix} : h \in \mathbb{R}^n,\ y \geq \nabla f(x)^\top h \right\},$$

$$\widehat{\mathbb{N}}(x; \operatorname{epi}(f)) = \left\{ \begin{pmatrix} w \\ -\nabla f(x)\, w \end{pmatrix} : w \in \mathbb{R}^m,\ w \leq 0 \right\}.$$

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