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Integral Geometry on the Octonionic Plane

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Gutachter

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Tag der öffentlichen Verteidigung:

To my parents.

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Abstract

A basis of the space of $\mathrm{Spin}(9)$-invariant translation-invariant continuous valuations on the octonionic plane is presented in terms of invariant differential forms. Furthermore, the canonical algebra structure on this space is determined in terms of generators and explicit relations. As a result, the Principal kinematic formula on the octonionic plane is computed by means of our basis. A bi-product of the notion of octonion-valued forms, upon which our construction heavily depends, is a new simple algebraic formula for the canonical $\mathrm{Spin}(9)$-invariant 8-form. Finally, motivated by our results, we conjecture a version of the Hodge-Riemann bilinear relations for smooth translation-invariant valuations.

Zusammenfassung

Eine Basis des Raums aller $\mathrm{Spin}(9)$-invarianten translationsinvarianten stetigen Bewertungen in der oktonionischen Ebene wird durch invariante Formen konstruiert. Gleichzeitig ist das kanonische Produkt von solchen Bewertungen durch die Menge seiner Erzeuger und explizite Relationen beschrieben. Anschließend wird die kinematische Hauptformel in der oktonionischen Ebene bewiesen. Ein Nebenprodukt unserer Konstruktion ist eine neue Formel für die kanonische $\mathrm{Spin}(9)$-invariante 8-Form. Schließlich wird eine Vermutung für die Hodge-Riemannschen bilinearen Relationen für glatte translationsinvariante Bewertungen aufgestellt.

Contents

Introduction

Classical Integral Geometry

Of central importance to convex and integral geometry is the notion of *intrinsic volumes*. The study of these fundamental functionals goes back to ancient mathematics. In fact, they generalize the natural quantities of *volume*, *surface area* and *mean width*, assigned in an intuitive way to any solid geometric figure, to a compact convex set in an arbitrary spatial dimension.

The intrinsic volumes can be defined in several equivalent ways. First, according to the famous Steiner formula, the volume of an ε-neighbourhood of a compact convex set in the n-dimensional Euclidean space is a polynomial in ε of degree at most n. Its coefficients are proportional to the intrinsic volumes. Second, the k-th intrinsic volume is the average (in a precise sense) k-dimensional volume of the orthogonal projection to a generic k-dimensional subspace. Third, for sets with sufficiently smooth boundaries, the intrinsic volumes localize to certain curvature integrals. Finally, as we shall discuss below, the intrinsic volumes can be characterized implicitly, collecting their essential properties. Notice that it follows from either of these descriptions that, appropriately normalized, μ_n is just the Lebesgue measure while $\mu_0 = \chi$ is the Euler characteristic.

Already the equivalence of the different definitions is highly non-trivial and makes the intrinsic volumes worth attention and deeper study. Over the years, this led to the discovery of striking relations these fundamental objects satisfy among each other. They are basically of two kinds: equalities and inequalities. The former are called *kinematic formulas* and are of central importance to numerous disciplines of both theoretical and applied mathematics. If $\mu_0, \ldots, \mu_n$ are the intrinsic volumes and $\overline{\mathrm{SO}(n)}$ is the group of rigid motions of the n-dimensional Euclidean space equipped with the Haar measure $\mathrm{d}\bar{g}$, they can be stated as follows:

$$\int_{\overline{\mathrm{SO}(n)}} \mu_k(K \cap gL)\, \mathrm{d}\bar{g} = \sum_{i+j=n+k} c_{i,j}^k\, \mu_i(K)\mu_j(L), \tag{1}$$

for any two convex bodies (i.e. compact convex sets) K, L and some (explicitly known) constants $c_{i,j}^k$. Of particular interest is the special case $k = 0$, the so-called *Principal kinematic formula*. The kinematic formulas were studied in various degrees of generality by Santaló, Blaschke, Federer, and Chern in the first half of the twentieth century. In fact, many other important integral-geometric formulas, such as those of Crofton, Poincaré, Steiner, and Weyl, can be obtained from (1). As for the latter kind of relations, they include the *isoperimetric inequalities*, bounding (weighted) ratios of intrinsic volumes by their values on the Euclidean unit ball, as well as the (in fact much more stronger) *Aleksandrov–Fenchel inequalities*. Also their importance extends far beyond convexity.

From the convex-geometrical point of view it is important that the k-th intrinsic volume is a continuous $\overline{\mathrm{SO}(n)}$-invariant *valuation*, i.e. it fulfils

$$\mu_k(K) + \mu_k(L) = \mu_k(K \cup L) + \mu_k(K \cap L) \tag{2}$$

for any convex bodies K, L such that $K \cup L$ and $K \cap L$ are also convex. Famously, these properties already characterize μ_k up to a normalizing constant. Moreover, according to the remarkable theorem of Hadwiger, any continuous $\overline{\mathrm{SO}(n)}$-invariant valuation is in fact a linear combination of the intrinsic volumes. Hadwiger's result already implies (1), leaving, however, the constants $c_{i,j}^k$ unspecified. Luckily, they can be determined using the so-called *template method*, i.e. by plugging in balls of variable radii.

The aforedescribed connection between the description of the vector space of invariant continuous valuations and the kinematic formulas gives us the first hint that algebra of valuations may be important to integral geometry in general. In fact, its role in temporary integral geometry is crucial.

Alesker Theory and Algebraic Integral Geometry

A major aim of integral geometry is to extend the validity of the kinematic formulas (1) to a more general setting. This problem was treated many times during the second half of the twentieth century and reasonable progress has been achieved. Nonetheless, a more conceptual approach was missing. Things changed dramatically at the turn of the millennium with the work of Semyon Alesker. His revolutionary algebraic approach to the theory of valuations proved to be the key to understanding integral geometry. In a series of seminal articles [3–12], Alesker literally turned the subject of valuation theory upside down, first, introducing a natural product of valuations (in fact this was just one of the whole array of algebraic structures that eventually emerged), second, extending the notion of valuations from Euclidean spaces to smooth manifolds.

At the very heart of Alesker's breakthrough lies his solution of McMullen's conjecture on the structure of the space of translation-invariant continuous valuations, namely, the salient *Irreducibility theorem*. This deep result turned out to have extremely broad consequences, far beyond the conjecture of McMullen.

One of the most important implications Alesker's theory has on integral geometry is its structuralization. Namely, let $G \subset \mathrm{SO}(n)$ be a compact subgroup acting *transitively* on the sphere S^{n-1}. Then the Alesker product turns the space Val^G of G-invariant translation-invariant continuous valuations into an associative, commutative, unital algebra of *finite dimension*. The strength of this result, usually referred to as the *Abstract Hadwiger-type theorem*, is fully revealed once the classification of such groups is recalled: There are six infinite series

$$\mathrm{SO}(n),$$
$$\mathrm{U}\left(\frac{n}{2}\right), \quad \mathrm{SU}\left(\frac{n}{2}\right),$$
$$\mathrm{Sp}\left(\frac{n}{4}\right), \quad \mathrm{Sp}\left(\frac{n}{4}\right)\mathrm{U}(1), \quad \mathrm{Sp}\left(\frac{n}{4}\right)\mathrm{Sp}(1),$$

and three exceptions

$$\mathrm{G}_2 \subset \mathrm{SO}(7), \quad \mathrm{Spin}(7) \subset \mathrm{SO}(8), \quad \mathrm{Spin}(9) \subset \mathrm{SO}(16).$$

Remarkably, the four rows correspond to the four normed division algebras of *reals* $\mathbb{R}$, *complex numbers* $\mathbb{C}$, *quaternions* $\mathbb{H}$, and *octonions* $\mathbb{O}$, respectively. Therefore, a finite basis of Val^G and, consequently, the kinematic formulas analogous to (1) *exist* not only in the case $G = \mathrm{SO}(n)$ but for any G from this list. To put it differently, the classical integral geometry of intrinsic volumes is in fact only an element of a much broader picture.

It quickly turned out that the spaces of invariant valuations in the non-classical cases may be in general truly complicated and, in particular, the template method is no longer sufficient to unfold the unknown constants in the kinematic formulas. Remarkably, a more sophisticated tool is available by the multiplicative structure on valuations. A quarter-century prior to Alesker's discovery of the product, Nijenhuis [107] made the following (non-trivial) observation: If the intrinsic volumes are renormalized properly, then all the constants in (1) become 1. He speculated that there might exist some algebraic explanation of this fact. However, it took more than three decades until Fu [63], having Alesker's perspective of valuations in his arsenal, clarified this phenomenon in an elegant and illuminating way indeed. Bernig and Fu [30] then promptly generalized this result into their striking *Fundamental theorem of algebraic integral geometry*, asserting that the knowledge of the algebra of invariant valuations is essentially equivalent to the knowledge of the constants appearing in the respective kinematic formulas.

Nonetheless, the task of describing the algebra of invariant valuations turns out to be notably difficult and involved in general and the same holds true for actually transforming the algebra structure into the kinematic formulas. In spite of a great effort and a variety of deep and beautiful results, the problem of understanding the valuation algebras remains widely open. Let us briefly review what has been achieved. First, various bases of the space of $\mathrm{U}(n)$-invariant valuations on $\mathbb{C}^n = \mathbb{R}^{2n}$ were introduced and kinematic formulas were proven in certain special cases by Park [109], Tasaki [133, 134], and Alesker [5]. A real breakthrough came with the fundamental paper [31] of Bernig and Fu who, using Fu's previous elegant description [63] of the algebra $\mathrm{Val}^{\mathrm{U}(n)}$, managed to prove kinematic formulas in the n-dimensional Hermitian space *in* their *full generality*. Second, the modification of the algebra structure and kinematic formulas to the special unitary group $\mathrm{SU}(n)$ were found by Bernig [24] who later also fully resolved two of the exceptional cases, G_2 and $\mathrm{Spin}(7)$ in [26]. Third, the dimensions of the valuation algebras in all the three quternionic series were computed by Bernig in [28]. However, an explicit description of the algebra and the kinematic formulas are known only in the first non-trivial case of the quaternionic plane $\mathbb{H}^2$ and $\mathrm{Sp}(2)\mathrm{Sp}(1)$, thanks to Bernig and Solanes [33, 34]. Finally, it is the goal of our thesis to describe the algebra of $\mathrm{Spin}(9)$-invariant valuations and to compute the Principal kinematic formula on the octonionic plane $\mathbb{O}^2$.

Octonion-Valued Forms and the $\mathrm{Spin}(9)$-invariant 8-Form

There are multiple ways of representing valuations. In fact, most of the commonly used pictures generalize, in some way, one of the equivalent definitions of the intrinsic volumes we recalled. For our purpose, it will be particularly convenient to regard the $\mathrm{Spin}(9)$-invariant valuations as certain invariant smooth differential forms that are then 'evaluated' on a convex body by integrating over the collection of its outer normals. Notice that this extends the curvature-integral definition of the intrinsic volumes.

A general guiding principle when working with the group $\mathrm{Spin}(9)$ and related objects, such as subgroups, representations, invariants, etc., turns out to be to keep in mind its close relationship to the octonions. In fact, much of the complexity of various expressions can be then 'wrapped up' in the underlying octonionic structure. From this reason, it is perhaps not surprising that the $\mathrm{Spin}(9)$-invariant differential forms representing $\mathrm{Spin}(9)$-invariant valuations are best to be understood, although being *real* in the end, as being composed of forms with values in $\mathbb{O}$. As anticipated, this point of view simplifies the resulting expressions drastically, however, one has to be very careful here since the octonions are not commutative and, even worse, not associative. And of course, neither are octonion-valued forms. In particular, special attention has to be paid to particular ordering of various products. Still, we shall see that these obstructions are very often worth to struggle with.

On our way to the description of $\mathrm{Spin}(9)$-invariant valuations, we found another interesting application of the aforedescribed principle. Namely, it is a well-known and remarkable fact, first observed by Brown and Gray [43], that there exists a unique (up to scaling) $\mathrm{Spin}(9)$-invariant form of degree 8 on the octonionic plane. The following elegant description of this canonical invariant is due to Berger [23]:

$$\Psi = \int_{\mathbb{O}P^1} \pi_\ell^* \, \nu_\ell \, \mathrm{d}\ell, \tag{3}$$

where $\mathbb{O}P^1$ is the *octonionic projective line*, i.e. the set of certain distinguished 8-planes in $\mathbb{O}^2$ that is preserved by the group $\mathrm{Spin}(9)$, $\pi_\ell : \mathbb{O}^2 \to \ell$ is the orthogonal projection, ν_ℓ is the volume form on $\ell \in \mathbb{O}P^1$, and $\mathrm{d}\ell$ is the canonical $\mathrm{Spin}(9)$-invariant measure. Interestingly, algebraic formulas for Ψ appeared only recently in works of Castrillón López et al. [45, 46], and Parton and Piccini [110]. They show that the 8-form is a complicated object indeed: In the standard basis, it possesses 702 terms that were only computed explicitly with computer assistance.

Using the notion of octonion-valued forms, we were able to prove a new explicit algebraic formula for the $\mathrm{Spin}(9)$-invariant 8-form. Namely, considering the octonionic coordinate 1-forms $\mathrm{d}x, \mathrm{d}y$ on $\mathbb{O}^2$ and their (octonionic) conjugates $\overline{\mathrm{d}x}, \overline{\mathrm{d}y}$, we first put

$$\begin{aligned} \Psi_{40} &= ((\overline{\mathrm{d}x} \wedge \mathrm{d}x) \wedge \overline{\mathrm{d}x}) \wedge \mathrm{d}x, & \Psi_{13} &= ((\overline{\mathrm{d}x} \wedge \mathrm{d}y) \wedge \overline{\mathrm{d}y}) \wedge \mathrm{d}y, \\ \Psi_{31} &= ((\overline{\mathrm{d}y} \wedge \mathrm{d}x) \wedge \overline{\mathrm{d}x}) \wedge \mathrm{d}x, & \Psi_{04} &= ((\overline{\mathrm{d}y} \wedge \mathrm{d}y) \wedge \overline{\mathrm{d}y}) \wedge \mathrm{d}y. \end{aligned}$$

Then it is the first main result of our thesis that

Theorem A (Published in [91])**.** *The form*

$$\Psi_8 = \Psi_{40} \wedge \overline{\Psi_{40}} + 4\,\Psi_{31} \wedge \overline{\Psi_{31}} - 5\left(\Psi_{31} \wedge \Psi_{13} + \overline{\Psi_{13}} \wedge \overline{\Psi_{31}}\right) + 4\,\Psi_{13} \wedge \overline{\Psi_{13}} + \Psi_{04} \wedge \overline{\Psi_{04}}$$

is a non-trivial real multiple of the $\mathrm{Spin}(9)$*-invariant 8-form* Ψ *on* $\mathbb{O}^2$.

Let us emphasize that the proof of the previous statement is only based on rather elementary octonion-algebraic considerations, in particular, the role of combinatorics is eliminated significantly. Moreover, the proposed description of the 8-form Ψ allows us to explicitly determine its 702 terms in the standard basis easily *by hand*.

$\mathrm{Spin}(9)$-Invariant Valuations

Let us now turn back to $\mathrm{Spin}(9)$-invariant valuations on the octonionic plane. The first attempt to study the space $\mathrm{Val}^{\mathrm{Spin}(9)}$ was made by Alesker [13]. First of all, he discussed several natural examples of such valuations, in particular:

$$T_k(K) = \int_{\mathbb{O}P^1} \mu_k(\pi_\ell K) \, \mathrm{d}\ell, \quad 0 \leq k \leq 8, \tag{4}$$

where the notation is the same as above. Second, he found a much less trivial example of a 2-homogeneous valuation, the so-called *octonionic pseudovolume*. However, as Alesker pointed out, neither a classification of $\mathrm{Spin}(9)$-invariant valuations nor the dimension of $\mathrm{Val}^{\mathrm{Spin}(9)}$ was known to him. The latter question was recently answered by Bernig and Voide [35]. In fact, they showed not only

$$\dim \mathrm{Val}^{\mathrm{Spin}(9)} = 143 \tag{5}$$

but also they computed dimensions of all homogeneous subspaces. The authors further constructed yet another natural example of a 2-homogeneous invariant valuation and established its relation to the octonionic pseudovolume. Let us point out that (5) was remarkably achieved without the explicit knowledge of the space.

The aim of our thesis is to compute an explicit basis of $\mathrm{Val}^{\mathrm{Spin}(9)}$ and to determine the algebra structure on this space. As anticipated in the previous paragraph, invariant differential forms will be made use of to this end. Let us describe our method in more detail. It is in fact another extremely useful implication of the Irreducibility theorem that each Spin(9)-invariant valuation is of the form

$$K \mapsto a \operatorname{vol}_n(K) + \int_{\mathrm{nc}(K)} \omega, \tag{6}$$

where a is a constant, vol_n is the Lebesgue measure, ω is a Spin(9)-invariant form on the sphere bundle $\mathbb{O}^2 \times S^{15}$ and $\mathrm{nc}(K)$ is its (Lipschitz) submanifold consisting of unit normals to the body K. This thus reduces the problem of describing invariant valuations to describing invariant differential forms. However, this correspondence is far from being one-to-one and therefore to describe a basis, more work needs to be done. Luckily, a powerful theoretical tool for this purpose was found by Bernig and Bröcker [29] to be based on applying certain second-order differential operator.

The aforedescribed approach has yet another crucial advantage. Dual to the Alesker product is the so-called Bernig-Fu convolution. Both these products are commutative, associative, distributive and graded with respect to the degree of homogeneity. In fact, $\mathrm{Val}^{\mathrm{Spin}(9)}$ equipped with the former is isomorphic to the same space equipped with the latter. But there still is a difference: In general, the convolution is much easier to compute. In particular, there is a truly simple formula for the Bernig-Fu convolution in the representation (6) that does not involve any more information than needed for computing the basis. Moreover, although the Fundamental theorem of algebraic integral geometry relates the kinematic formulas of type (1) to the Alesker product in general, rather than convolution, the particular case of Principal kinematic formula can be still deduced from the knowledge of the convolution entirely.

Following these guidelines, we first find the algebra of Spin(9)-invariant forms on the sphere bundle. Due to the fact that Spin(9) is transitive on spheres, this boils down to describing alternating forms on a single tangent space that inherit invariance under a smaller group, namely Spin(7). This first pillar of our construction is completed by extending invariant theory of this group which allows us to describe the invariant forms in terms of octonion-valued forms, much in the spirit of Theorem A. Then, as outlined above, one needs to differentiate these forms in order to describe a basis for the respective valuations. To this end, Cartan's apparatus of moving frames allows us to stick to our local picture and to compute differentials in a single point only. This completes the theoretical part of the construction and the rest is then achieved by computation in coordinates. The outcome is the second main result of our thesis:

Theorem B. *As a graded algebra,*

$$\mathrm{Val}^{\mathrm{Spin}(9)} \cong \mathbb{R}[t, s, v, u_1, u_2, w_1, w_2, w_3, x_1, x_2, y, z]/\mathcal{I},$$

where the generators are of the following degrees

1	2	3	4	5	6	7	8
t	s	v	u_1, u_2	w_1, w_2, w_3	x_1, x_2	y	z

and $\mathcal{I}$ is an explicitly known ideal generated by 93 independent elements.

Furthermore, explicit knowledge of the forms representing the generators allows us to compute an explicit (monomial) *basis of the algebra* $\mathrm{Val}^{\mathrm{Spin}(9)}$. According to the Fundamental theorem of algebraic integral geometry, we can consequently determine the *Principal kinematic formula* on the octonionic plane with respect to this basis. Finally, the valuations (4) are expressed in terms of our basis as an application.

Hodge-Riemann Bilinear Relations

Our results strongly confirm what is anticipated by (5): The algebra $\mathrm{Val}^{\mathrm{Spin}(9)}$ is indeed a complicated object and equally complicated is the integral geometry on the octonionic plane. This fact, however, should be viewed as an advantage as it is certainly reasonable to hope that there might be some structures involved which, having been hidden behind the simplicity of the other cases, might be revealed here. We shall see why there is a good reason to believe that this is indeed the case.

The Alesker product on smooth translation-invariant valuations, i.e. those that can be expressed as (6), has in fact many more, beautiful and fundamental, properties than we have so far listed. Interestingly, many of them have counterparts in cohomology of compact Kähler manifolds. This is a fascinating phenomenon that has, nonetheless, never been explicitly explained.

First of all, it is a classical result of Hadwiger that the only n-homogeneous smooth translation-invariant valuations of degree n are multiples of the Lebesgue measure. Consequently, induced by the Alesker product is a *non-degenerate* pairing $(\phi \cdot \psi)_n$ that returns this proportionality factor. Furthermore, the multiplication by the first intrinsic volume μ_1 is the *Lefschetz map*, i.e. its appropriate powers are bijections. Like in the theory of Kähler manifolds, such properties are of central importance for valuations.

There is one more important result of cohomology on Kähler manifolds which, we believe, admits a counterpart for valuations, namely *Hodge-Riemann bilinear relations*. In analogy to the Kähler original, let us call a valuation ϕ of degree *k primitive* if

$$\phi \cdot \mu_1^{n-2k+1} = 0. \tag{7}$$

Our computations for Spin(9)-invariant valuations show that the induced pairing

$$Q(\phi, \psi) = (\phi \cdot \psi \cdot \mu_1^{n-2k})_n, \tag{8}$$

when restricted to primitive k-homogeneous valuations, is positive or negative definite, depending on the parity of k. This is precisely in analogy with the Kähler Hodge-Riemann relations. However, a closer look at a recent work [32] of Bernig and Hug, which allows us to explicitly compute (8) for $k = 1$, shows that there must be also a dependence on the parity of the valuation ϕ.

All in all, let even valuations have parity 0 and odd valuations parity 1, and let us also consider complex valued valuations. Then it is the third main result of our thesis that we propose the following

Conjecture C. *For any non-zero primitive smooth k-homogeneous valuation ϕ of parity s,*

$$(-1)^{k+s}\, Q(\phi, \overline{\phi}) > 0. \tag{9}$$

Chapter 1

Algebraic Integral Geometry

In the opening chapter, both classical and modern aspects of integral geometry and the theory of valuations on convex bodies will be discussed. Albeit roughly, we aim to follow the historical development in order to capture an increasing significance of various algebraic constructions to these areas of mathematics.

Throughout the whole chapter, we shall assume that V is a finite-dimensional (real) Euclidean vector space with $\dim V = n$.

1.1 Valuations on Convex Bodies

To begin with, let us review the very basics of valuations on convex bodies. A particular emphasis will be placed on an important collection of examples, the so-called *intrinsic volumes*. Our general references are standard: [72, 88, 121].

1.1.1 Convex Bodies

Definition 1.1. A non-empty compact convex subset $K \subset V$ is called a *convex body*. The set of all convex bodies in V is denoted $\mathcal{K}(V)$ or simply by $\mathcal{K}$.

Example 1.2. The following sets belong to $\mathcal{K}$:
(a) the closed unit ball B,
(b) any (convex) polytope, i.e. convex hull of finitely many points $\{x_1, \ldots, x_N\} \subset V$, in particular any one-point set $\{x\}$, $x \in V$.

The set $\mathcal{K}$ is naturally equipped with a binary operation, the so-called *Minkowski addition*, defined for $K, L \in \mathcal{K}$ as

$$K + L = \{x + y\, ;\, x \in K, y \in L\}. \tag{1.1}$$

Notice that Minkowski addition is clearly associative as well as commutative, in other words, it makes $\mathcal{K}$ into an abelian semigroup. Similarly, one defines *scaling* of a convex body K by $\lambda \in \mathbb{R}$ as

$$\lambda K = \{\lambda x\, ;\, x \in K\}. \tag{1.2}$$

Again $\lambda K \in \mathcal{K}$ clearly. We write $-K = (-1)K$. Further, we denote $K + x = K + \{x\}$, the *translate* of $K \in \mathcal{K}$ by $x \in V$. Observe that

$$K + L = \bigcup_{x \in L} (K + x) \tag{1.3}$$

and so, for $\varepsilon \geq 0$,

$$K_\varepsilon = K + \varepsilon B = \bigcup_{x \in K} (\varepsilon B + x) \tag{1.4}$$

is the (closed) ε-neighbourhood of $K \in \mathcal{K}$.

There is a natural topology on $\mathcal{K}$, induced by the so-called *Hausdorff metric* that is defined for $K, L \in \mathcal{K}$ as follows:

$$d_H(K, L) = \inf\{\varepsilon > 0\,;\, K \subset L_\varepsilon, L \subset K_\varepsilon\}. \tag{1.5}$$

An important property of the metric is expressed by the *Blaschke selection theorem*. For a contemporary proof as well as for a discussion on other topological aspects of the space $(\mathcal{K}, d_H)$ we refer to [121], §1.8.

Theorem 1.3 (Blaschke [37], §18.I)**.** *Each bounded sequence in $\mathcal{K}$ has a subsequence that converges to an element of $\mathcal{K}$.*

Remark 1.4. It is well known that although the metric d_H depends a priori on the choice of the Euclidean structure on V, the resulting topology does not (see [103]). And in fact, there are even more equivalent metrics on $\mathcal{K}$ (see [130]).

1.1.2 Valuations

Definition 1.5. A functional $\mu : \mathcal{K} \to \mathbb{R}$ is called a *valuation* if

$$\mu(K) + \mu(L) = \mu(K \cup L) + \mu(K \cap L) \tag{1.6}$$

holds for any $K, L \in \mathcal{K}$ whenever $K \cup L \in \mathcal{K}$.

Remark 1.6. Notice that if $K \cup L \in \mathcal{K}$, then $K \cap L \neq \emptyset$ and hence $K \cap L \in \mathcal{K}$ as well.

In other words, valuations are *finitely additive measures* on convex bodies. The notion of a valuation dates back to Dehn's solution [52] of *Hilbert's third problem* [81]. Namely, constructing a scissors-congruence-invariant valuation on polytopes that takes distinct values on a cube and a tetrahedron of equal volume, Dehn showed that these solids are *not* scissors congruent (see also §8.6 of [88] and references therein). Before we list a first couple of examples, let us mention certain generalizations of this concept.

First, the notion of valuations on convex bodies can be extended by replacing $\mathbb{R}$ in Definition 1.5 with a general abelian semigroup A. In this setting, the plus signs in (1.6) stand for the semigroup multiplication. The most important non-scalar cases are *Minkowski valuations* where $A = \mathcal{K}$ (see e.g. [1, 95, 96, 108, 122–126, 138]), or *tensor valuations* where A is a quotient of the tensor algebra $\mathfrak{T}(V)$, typically the symmetric algebra (see the collection [135] as well as numerous references therein).

Second, other domains than $\mathcal{K}$ have been considered, with (1.6) being appropriately modified. Examples include *function spaces* [16, 48, 97, 106], *lattice polytopes* [39, 98, 101], or other sets more resembling a convex body, in particular *smooth polyhedra* (touched upon briefly in §1.2.3 below).

Example 1.7. The following functionals are valuations (in the sense of Definition 1.5):

(a) restriction of any Borel measure on V to $\mathcal{K}$, in particular the *Lebesgue measure* vol_n;

(b) the *Euler characteristic* defined by $\chi(K) = 1$, $K \in \mathcal{K}$;

(c) $K \mapsto \#(K \cap \Gamma)$, where Γ is a $\mathbb{Z}$-lattice in V and # stands for the cardinality.

Definition 1.8. A valuation μ is said to be
(a) *translation invariant* if $\mu(K+x) = \mu(K)$ for any $K \in \mathcal{K}$ and $x \in V$;
(b) *continuous* if it is so with respect to the Hausdorff metric.
The set of all translation-invariant continuous valuations is denoted by $\mathrm{Val}(V)$ or Val.

It is readily verified (see e.g. [121], Theorem 1.8.20) that vol_n is continuous and thus an element of Val. $\chi \in \mathrm{Val}$ is obvious. On the contrary, the valuation defined in Example 1.7 (c) is clearly neither translation invariant nor continuous. In what follows, we shall deal entirely with valuations that *do* enjoy both of these properties.

It follows at once from the linear nature of the defining conditions that Val carries a natural vector-space structure. As for its dimension, one has

$$\dim \mathrm{Val}(V) = \begin{cases} 2 & \text{if } \dim V = 1, \\ \infty & \text{if } \dim V \geq 2. \end{cases} \tag{1.7}$$

The former is an easy exercise (see e.g. [15], Proposition 3.0.1), the latter follows from a certain more involved characterisation result discussed below.

It turns out that, in spite of being infinite-dimensional in general, the space Val is remarkably structured. The basic pillar underlying a full array of further constructions is the *McMullen grading*:

Definition 1.9. $\mu \in \mathrm{Val}$ is said to be *k-homogeneous*, $k \in \mathbb{N}_0$, if $\mu(\lambda K) = \lambda^k \mu(K)$ holds for any $\lambda > 0$ and $K \in \mathcal{K}$. The corresponding subspace of Val is denoted by Val_k.

Theorem 1.10 (McMullen [99]). *Let $n = \dim V$ as usual. Then*

$$\mathrm{Val} = \bigoplus_{k=0}^{n} \mathrm{Val}_k \,. \tag{1.8}$$

Up to the present, only certain classes of valuations have been described explicitly. With respect to the degree of homogeneity, the following three cases are settled: First, since $tK \to \{0\}$ in the Hausdorff topology as $t \to 0$, it is easy to see that

Proposition 1.11. $\mathrm{Val}_0 = \mathrm{span}\{\chi\}$.

Second, we have the deep theorem due to Hadwiger:

Theorem 1.12 (Hadwiger [76]). $\mathrm{Val}_n = \mathrm{span}\{\mathrm{vol}_n\}$.

Remark 1.13. In its original version [76], p. 79, Hadwiger's theorem characterizes vol as the unique (up to scaling) n-homogeneous translation-invariant valuation on *polytopes*. This is, however, clearly equivalent to the statement above if we take into account that polytopes are dense in $\mathcal{K}$ (see e.g. [121], Theorem 1.8.16).

Third, Val_{n-1} is in a certain (precisely described) one-to-one correspondence with the set of classes of continuous functions on the sphere S^{n-1} modulo adding a linear functional. This was first shown for $n = 2$ by Hadwiger [74, 75] and later on generalized to any dimension by McMullen [100]. In particular, provided $n \geq 2$, we therefore have $\dim \mathrm{Val} = \dim \mathrm{Val}_{n-1} = \infty$.

Other classification results all require additional assumptions. *Simple* valuations, vanishing on convex bodies of dimension less then n (the dimension of a convex body is defined to be the dimension of its affine hull), are an important instance resolved by Klain [86] and Schneider [120]. Last but not least, finite-dimensional subspaces of Val consisting of valuations fulfilling an extra invariant property will be, rather extensively, discussed below.

1.1.3 The Intrinsic Volumes

Besides continuity and translation invariance, the Lebesgue measure has one more remarkable and fundamental feature. It is a notorious fact that vol_n is actually invariant under *all* rigid motions of the Euclidean space V, i.e. under the Lie group

$$\overline{\mathrm{SO}(V)} = \mathrm{SO}(V) \ltimes V, \tag{1.9}$$

where V is regarded as the abelian group of translations. The same is obviously true for the constant valuation χ but there are more elements of Val with this property, interpolating thus, in certain sense, χ and vol_n. Historically, the role such valuations have played in convex and integral geometry has been crucial.

One possible starting point and perhaps the most illustrative way to define these quantities is via the so-called *Steiner formula* (see e.g. [88], §9.2, or [121], §4.2). See also the monograph [69] for a more general context of *tube formulas*.

Theorem 1.14 (Steiner formula). *For any $K \in \mathcal{K}$ and $\varepsilon > 0$,*

$$\mathrm{vol}_n(K_\varepsilon) = \sum_{k=0}^{n} \omega_k \, \mu_{n-k}(K) \, \varepsilon^k. \tag{1.10}$$

Remark 1.15. The spirit of the Steiner formula is well illuminated in a simple particular case $n = 2$ when K is a triangle (see [72], Fig. 6.2).

Definition 1.16. The via (1.10) defined functionals $\mu_k : \mathcal{K} \to \mathbb{R}$, $0 \leq k \leq n$, are called the *intrinsic volumes*.

It is almost immediate (see also [72], p. 105) that the intrinsic volumes inherit from vol_n its valuation property, continuity and rigid-motion invariance, as well as that μ_k is k-homogeneous. Further, one can easily see that $\mu_0 = \chi$ and $\mu_n = \mathrm{vol}_n$: just set $K = \{0\}$ or send $\varepsilon \to 0$, respectively, in (1.10).

To manifest the attribute *intrinsic*, let us recall that if $\iota : V \to W$ is an isometric embedding into a Euclidean space, $\dim W = N$, and $\tilde{\mu}_k$ are the intrinsic volumes on W, then $\iota^* \tilde{\mu}_k = \mu_k$, $0 \leq k \leq n$, and $\iota^* \tilde{\mu}_k = 0$, $n+1 \leq k \leq N$ (see e.g. [72], Proposition 6.7). In particular, for any k-dimensional convex body K one has $\mu_k(K) = \mathrm{vol}_k(K)$. This also shows that none of the functionals μ_k, $0 \leq k \leq n$, vanishes identically.

As anticipated, the intrinsic volumes can be in fact defined in a number of other equivalent ways. First, recall the so-called *Kubota formulas*:

$$\mu_k(K) = \begin{bmatrix} n \\ k \end{bmatrix} \int_{\mathrm{Gr}_k(V)} \mathrm{vol}_k(\pi_E K) dE, \tag{1.11}$$

π_E is the orthogonal projection to $E \in \mathrm{Gr}_k(V)$, vol_k is the Lebesgue measure on E, and $\mathrm{d}E$ is the unique $\mathrm{SO}(V)$-invariant probability measure on $\mathrm{Gr}_k(V)$. Second, one has the *Crofton formulas*:

$$\mu_k(K) = \begin{bmatrix} n \\ k \end{bmatrix} \int_{\overline{\mathrm{Gr}}_{n-k}(V)} \chi(K \cap \overline{E}) d\overline{E}, \tag{1.12}$$

where $\mathrm{d}\overline{E}$ is the unique $\overline{\mathrm{SO}(V)}$-invariant measure on $\overline{\mathrm{Gr}}_{n-k}(V)$ with

$$\mathrm{d}\overline{E} \left\{ \overline{F} \in \overline{\mathrm{Gr}}_{n-k}(V) \, ; \, \overline{F} \cap B \right\} = \omega_{n-k},$$

where B is the unit ball in V (see [88], §6). Concernig the normalizing constants $\begin{bmatrix} n \\ k \end{bmatrix}$, the so-called *flag coefficients*, let us postpone their precise definition to §4.5.2 below.

Important Remark 1.17. Observe that the argument of χ in (1.12) may be the empty set. Here and everywhere else the standard convention is adhered to: We put

$$\mu(\emptyset) = 0, \quad \mu \in \mathrm{Val}\,. \tag{1.13}$$

Finally, μ_k is characterized as the unique k-homogeneous rigid-motion-invariant continuous valuation that agrees with vol_k on k-dimensional convex bodies ($\mathrm{vol}_0 = \chi$) as expressed by the famous *Hadwiger theorem*:

Theorem 1.18 (Hadwiger [76], §6.1.10)**.** *Let μ be a continuous $\overline{\mathrm{SO}(V)}$-invariant valuation on $\mathcal{K}$. Then there are constants $\alpha_0, \ldots, \alpha_n \in \mathbb{R}$ such that $\mu = \sum_{k=0}^{n} \alpha_k \mu_k$.*

For a modern proof of this classical result as well as for (1.11) and (1.12) derived as its consequences, see §9 of [88]. The Hadwiger theorem has in fact very strong implications on integral geometry, going far beyond the Kubota and the Crofton formulas. This will be discussed in the following section.

1.1.4 Kinematic Formulas

Remarkable integral relations are well known to exist among the intrinsic volumes. Namely, of central importance to numerous disciplines of both theoretical and applied mathematics are the so-called *kinematic formulas*, studied in various settings and degrees of generality by Blaschke, Chern, Federer, or Santaló (see [88], §10, and [121], §4.4):

Theorem 1.19 (Blaschke kinematic formulas)**.** *For any $0 \leq k \leq n$ and $K, L \in \mathcal{K}$,*

$$\int_{\overline{\mathrm{SO}(V)}} \mu_k(K \cap \overline{g}L)\,\mathrm{d}\overline{g} = \sum_{i,j=1}^{n} c_{i,j}^k \mu_i(K)\mu_j(L). \tag{1.14}$$

Theorem 1.20 (Additive kinematic formulas)**.** *For any $0 \leq k \leq n$ and $K, L \in \mathcal{K}$,*

$$\int_{\mathrm{SO}(V)} \mu_k(K + gL)\,\mathrm{d}g = \sum_{i,j=1}^{n} d_{i,j}^k \mu_i(K)\mu_j(L). \tag{1.15}$$

Here $\mathrm{d}g$ is the Haar probability measure on $\mathrm{SO}(V)$ and $\mathrm{d}\overline{g}$ is the product measure of the Haar probability measure and the Lebesgue measure on $\mathrm{SO}(V) \ltimes V$.

The existence of kinematic formulas is a consequence of the Hadwiger characterization theorem as follows: It is not difficult to verify that the left-hand side of both (1.14) and (1.15) is a continuous $\overline{\mathrm{SO}(V)}$-invariant valuation in both arguments K and L. The constants $c_{i,j}^k$ and $d_{i,j}^k$ appearing in the sums on the right can be then determined explicitly (in terms of the flag coefficients, see §2.3 in [27]) using the so-called *template method*, i.e. by plugging in origin-centered balls of variable radii.

1.2 Smooth Valuations

Let us now proceed to study the important class of valuations that are in certain precise sense *smooth*. Later on, we shall see that such an assumption allows them to carry remarkable algebraic structures. Also, it will turn out that in fact all the valuations we come into contact with are smooth. As usual, let V be an n-dimensional Euclidean space.

1.2.1 The Klain Embedding

Clearly, any $\mu \in \mathrm{Val}$ admits a (unique) decomposition into even and odd parts:

$$\mu(K) = \frac{1}{2}\left[\mu(K) + \mu(-K)\right] + \frac{1}{2}\left[\mu(K) - \mu(-K)\right].$$

In other words, the grading (1.8) can be refined as follows:

$$\mathrm{Val} = \mathrm{Val}^+ \oplus \mathrm{Val}^- = \bigoplus_{\substack{\sigma=\pm \\ 0\leq k\leq n}} \mathrm{Val}_k^\sigma, \tag{1.16}$$

where $\mathrm{Val}^\pm = \{\mu \in \mathrm{Val}\,;\, \mu(-K) = \pm\mu(K) \text{ for all } K \in \mathcal{K}\}$ is the subspace of *even* / *odd* valuations, and $\mathrm{Val}_k^\pm = \mathrm{Val}_k \cap \mathrm{Val}^\pm$.

An important and useful description of *even* valuations was given by Klain [87]. Let $\mu \in \mathrm{Val}_k(V)$ and $E \in \mathrm{Gr}_k(V)$. By Theorem 1.12, the restriction $\mu|_E \in \mathrm{Val}_k(E)$ is a multiple of the (k-dimensional) Lebesgue measure on E. Denote the proportionality factor by $\mathrm{Kl}_\mu(E)$. Continuity of μ then clearly implies continuity of the so-called *Klain function*

$$\mathrm{Kl}_\mu : \mathrm{Gr}_k(V) \to \mathbb{R} : E \mapsto \mathrm{Kl}_\mu(E) \tag{1.17}$$

of μ. The induced linear mapping $\mu \mapsto \mathrm{Kl}_\mu$, when restricted to Val_k^+, is an embedding:

Theorem 1.21 (Klain [87]). *Let $\mu \in \mathrm{Val}_k^+$. If $\mathrm{Kl}_\mu = 0$, then $\mu = 0$.*

There is a counterpart theorem for *odd* valuations proven by Schneider [120]. However, for the construction of the *Schneider embedding* is slightly more technical and, as we shall see later, all valuations we shall work with are in fact even, we do not go into details here. Instead, we refer to §3.3 of [27] for a lucid exposition.

Example 1.22. Since the Lebesgue measure is even, it follows at once that $\mu_k \in \mathrm{Val}_k^+$. Further, it is immediate that $\mathrm{Kl}_{\mu_k} \equiv 1$ on $\mathrm{Gr}_k(V)$.

1.2.2 Alesker's Irreducibility Theorem

For coherence of our review, let us recall that it is an easy consequence of the Blaschke selection theorem 1.3 and the McMullen decomposition (1.8) that Val is a Banach space with respect to

$$\|\mu\|_{\mathrm{Val}} = \sup\{|\mu(K)|\,;\, K \in \mathcal{K}, K \subset B\}. \tag{1.18}$$

It is clear that (1.16) is then a decomposition of Val into closed subspaces. Now, since $\mathrm{GL}(V)$ maps line segments to line segments, the defining action of the Lie group $\mathrm{GL}(V)$ clearly extends from V to $\mathcal{K}$. This extension then naturally induces a continuous representation of $\mathrm{GL}(V)$ on the Banach space Val as follows:

$$(g\cdot\mu)(K) = \mu(g^{-1}K), \quad g \in \mathrm{GL}(V), \mu \in \mathrm{Val}, K \in \mathcal{K}. \tag{1.19}$$

Notice that $g \cdot \mu \in \mathrm{Val}$ is readily verified. It is also immediate that this action preserves both degree and parity of a valuation. And, in fact,

Theorem 1.23 (Alesker [4])**.** *For each k and σ, the* $\mathrm{GL}(V)$*-module* Val_k^σ *is irreducible.*

Remark 1.24. Recall that in this context *irreducibility* means that the (possibly infinite-dimensional) Banach spaces Val_k^σ do not admit any proper *closed* invariant subspaces (see also [136], §1.1.1).

The importance of the *Irreducibility theorem* 1.23 is extraordinary. In fact, Alesker's achievement was one of the milestones, if not the starting point itself, of modern valuation theory and it relatively quickly crystallized into a whole variety of algebraic tools that changed the view of valuations and integral geometry once and for all. The first consequence (and in fact Alesker's original motivation, see also [3]) was an affirmative answer to the conjecture of P. McMullen:

Corollary 1.25 (Alesker's solution [4] of McMullen's conjecture [100])**.** *The valuations*

$$K \mapsto \psi_A(K) = \mathrm{vol}_n(K + A), \quad A \in \mathcal{K}, \tag{1.20}$$

span a dense subspace of Val.

Remark 1.26. The original statement of the conjecture mentioned the so-called *mixed volumes* instead. These valuations are, however, expressible as linear combinations of valuations (1.20), and vice versa (see [121], Theorem 5.1.7).

One may also wish to consult Alesker's lecture notes [15].

1.2.3 Smooth Valuations and the Normal Cycle

To illustrate another important implication of the Irreducibility theorem, let us recall the notion of *smooth valuations*.

Definition 1.27 (Alesker [8])**.** A valuation $\mu \in \mathrm{Val}$ is called *smooth* if the mapping $g \mapsto g \cdot \mu$ from the Lie group $\mathrm{GL}(V)$ to the Banach space Val is infinitely differentiable.

It is well known that the $\mathrm{GL}(V)$-invariant subspace Val^∞ of smooth valuations must be dense in Val (see e.g. [136], §1.6). Importantly, there is an equivalent and much more explicit description of Val^∞ based on the following geometric construction: Consider the *sphere bundle* $SV = V \times S^{n-1}$ over V, then

Definition 1.28. The *normal cycle* of a convex body $K \in \mathcal{K}$ is defined as

$$\mathrm{nc}(K) = \{(x, v) \in SV\,;\, \langle v, y - x\rangle \leq 0 \text{ for all } y \in K\}. \tag{1.21}$$

The notion of normal cycle dates back to work of Wintgen [142] and Zähle [143,144], it was also studied extensively by Fu [58–62]. Remarkably, this concept extends far beyond convexity as the class of sets admitting some version of (1.21) is much broader than $\mathcal{K}$, including e.g. sets of positive reach, the so-called WDC sets (a good source of reference here is the recent monograph [113]), or smooth manifolds, spanning hence a bridge to Alesker's groundbreaking *Theory of valuations on manifolds* [9–12,18].

It is well known that $\mathrm{nc}(K) \subset SV$ is a naturally oriented Lipschitz submanifold of dimension $n-1$ (see [59]). It therefore makes sense to regard it as a *current*, acting on $\Omega^{n-1}(SV)$ by integration. Crucially, thus viewed, it has the valuation property (1.5) and, in fact, the Irreducibility theorem implies

Theorem 1.29 (Alesker, Fu [9,18])**.** $\mu \in \mathrm{Val}$ *is smooth if and only if there exist* $a \in \mathbb{R}$ *and* $\omega \in \Omega^{n-1}(SV)$*, the latter invariant under translations in* V*, such that, for any* $K \in \mathcal{K}$*,*

$$\mu(K) = a\,\mathrm{vol}_n(K) + \int_{\mathrm{nc}(K)} \omega. \tag{1.22}$$

We shall indicate the translation invariance by superscript tr (not to be confused with the trace). Further, let us denote the valuation corresponding to the second factor of (1.22) by $[[\omega]]$. This defines the (linear) mapping

$$\Omega^{n-1}(SV)^{\mathrm{tr}} \to \mathrm{Val}^\infty : \omega \mapsto [[\omega]] = \int_{\mathrm{nc}(\,\cdot\,)} \omega, \tag{1.23}$$

that is graded with respect to the natural bi-grading of $\Omega^\bullet(SV)$:

$$[[\omega]] \in \mathrm{Val}_k \quad \text{if} \quad \omega \in \Omega^{k,n-1-k}(SV)^{\mathrm{tr}}, \quad 0 \le k \le n-1. \tag{1.24}$$

1.2.4 The Rumin Differential and the Kernel Theorem

Theorem 1.29 can be also rephrased as follows: *The mapping*

$$\mathbb{R} \times \Omega^{n-1}(SV)^{\mathrm{tr}} \to \mathrm{Val}^\infty : (a, \omega) \mapsto a \,\mathrm{vol}_n + [[\omega]]$$

is well defined and onto. A natural question then is: What is the kernel of this map? This clearly shrinks to: When is $[[\omega]]$ (identically) zero? Let us recall here an elegant and extremely useful answer due to Bernig and Bröcker [29], the so-called *Kernel theorem*.

The sphere bundle carries a natural *contact* structure encoded in the *1-form*

$$\alpha_{(x,v)}(Z) = \langle v, \mathrm{d}\pi(Z) \rangle, \quad Z \in \mathfrak{X}(SV), \tag{1.25}$$

where $\pi : SV \to V$ is the bundle projection, that distinguishes the *vertical forms*

$$\Omega_v^\bullet(SV) = \{\omega \in \Omega^\bullet(SV)\,;\, \omega \wedge \alpha = 0\}. \tag{1.26}$$

The quotient algebra of *horizontal forms* is then given by

$$\Omega_h^\bullet(SV) = \Omega^\bullet(SV)/\Omega_v^\bullet(SV). \tag{1.27}$$

As usual, the notation Ω_v and Ω_h extends also to subspaces of $\Omega^\bullet$.

Lemma 1.30 (Rumin [116]). *For any $\omega \in \Omega^{n-1}(SV)$, there exists a unique $\xi \in \Omega_h^{n-2}(SV)$ such that $\mathrm{d}(\omega + \alpha \wedge \xi) \in \Omega_v^n(SV)$.*

Definition 1.31. Keeping the notation of Lemma 1.30, we define the *Rumin differential*

$$\mathrm{D}\omega = \mathrm{d}(\omega + \alpha \wedge \xi). \tag{1.28}$$

Example 1.32. Clearly, $\mathrm{D}\omega = 0$ if $\omega \in \Omega_v^{n-1}(SV)$.

Theorem 1.33 (Bernig, Bröcker [29]). *Let $0 \le k \le n-1$ and $\omega \in \Omega^{k,n-1-k}(SV)^{\mathrm{tr}}$. Then*
(a) *if $k = 0$, then $[[\omega]] = 0$ if and only if ω is exact;*
(b) *if $k > 0$, then $[[\omega]] = 0$ if and only if $\mathrm{D}\omega = 0$.*

1.3 G-Invariant Valuations

Another striking fact that emerges from Alesker's theory is the fundamental observation that the classical integral geometry of intrinsic volumes, as exposed in §1.1.3 and §1.1.4 above, is in fact only an element of a much broader picture. As Alesker revealed, the key attribute of the group of rotations that may seem unspectacular at a first glance but that causes the classical kinematic formulas to actually exist is its transitive action on the sphere. It is well known that there are more groups with such a property, admitting thus the existence of kinematic formulas. Of course, *to exist* does not necessarily mean *to be known explicitly*. However, the Irreducibility theorem gives rise to a sophisticated algebraic apparatus by means of which general kinematic formulas can be *always* obtained, at least in principle.

1.3.1 Abstract Hadwiger-Type Theorem

For the rest of the chapter, we shall live in the standard Euclidean space $V = \mathbb{R}^n$, i.e. have $\mathcal{K} = \mathcal{K}(\mathbb{R}^n)$ and $\mathrm{Val} = \mathrm{Val}(\mathbb{R}^n)$. Let $G \subset \mathrm{SO}(n)$ be a compact subgroup. By

$$\mathrm{Val}^G = \mathrm{Val}(\mathbb{R}^n)^G = \{\mu \in \mathrm{Val}\,;\ \mu(gK) = \mu(K) \text{ for all } K \in \mathcal{K} \text{ and } g \in G\}, \tag{1.29}$$

we denote the subspace of *G-invariant valuations*. The McMullen grading becomes

$$\mathrm{Val}^G = \bigoplus_{k=0}^{n} \mathrm{Val}_k^G, \quad \text{where } \mathrm{Val}_k^G = \mathrm{Val}^G \cap \mathrm{Val}_k\,. \tag{1.30}$$

Theorem 1.34 (Alesker[1] [3,11])**.** $\dim \mathrm{Val}^G < \infty$ *if and only if G acts transitively on the unit sphere S^{n-1}. If this is the case, then* $\mathrm{Val}^G \subset \mathrm{Val}^\infty$.

Remark 1.35. Groups of this nature are widely known from works of A. Borel [38] and Montgomery and Samelson [104]: There are six infinite series

$$\mathrm{SO}(n),$$
$$\mathrm{U}\Big(\frac{n}{2}\Big), \quad \mathrm{SU}\Big(\frac{n}{2}\Big),$$
$$\mathrm{Sp}\Big(\frac{n}{4}\Big), \quad \mathrm{Sp}\Big(\frac{n}{4}\Big)\mathrm{U}(1), \quad \mathrm{Sp}\Big(\frac{n}{4}\Big)\mathrm{Sp}(1),$$

and three exceptions

$$\mathrm{G}_2 \subset \mathrm{SO}(7), \quad \mathrm{Spin}(7) \subset \mathrm{SO}(8), \quad \mathrm{Spin}(9) \subset \mathrm{SO}(16).$$

Although the notation we use is standard, definitions of all the listed groups will be recalled in the sequel (see §2.2 and §3.1). For now, let us only mention that the groups are divided such that the four rows correspond to the four *normed division algebras*: the *reals* $\mathbb{R}$, the *complex numbers* $\mathbb{C}$, the *quaternions* $\mathbb{H}$, and the *octonions* $\mathbb{O}$, respectively.

Alesker's result is sometimes referred to as *Abstract Hadwiger-type theorem*: For any G from the list, there is a (finite) basis of Val^G and hence kinematic formulas analogous to (1.14) and (1.15) exist for precisely the same reason as in the classical case $G = \mathrm{SO}(n)$. Nonetheless, in order to actually obtain the kinematic formulas, one has to deal with

[1] Alesker proved the *if* part in [3] and announced the *only if* part of the theorem in [11]. To the best of our knowledge, a proof of the latter first appeared in Bernig's survey [27]. An alternative proof of the former is due to Fu [63].

two (in general highly non-trivial) tasks: to find a basis of Val^G, and to determine the unknown constants. The latter will be discussed below. As for the former, let us outline two possible techniques, especially the first one being particularly convenient for our later purpose.

By the second part of Theorem 1.34, any G-invariant valuation of degree $k < n$ is represented by a smooth differential form in the sense of (1.22) – (1.24). Averaging over the (compact) group G, there is no loss of generality in assuming that the differential form is G invariant as well (with respect to the diagonal action on the sphere bundle). In other words, it is enough to consider the forms from $\Omega^\bullet(S\mathbb{R}^n)^{\overline{G}}$, i.e. invariant under the group

$$\overline{G} = G \ltimes \mathbb{R}^n \subset \overline{\mathrm{SO}(n)} = \overline{\mathrm{SO}(\mathbb{R}^n)}, \tag{1.31}$$

acting on the sphere bundle $S\mathbb{R}^n$ as follows:

$$\overline{g} = (g, x) : (y, v) \mapsto (gy + x, gv) \tag{1.32}$$

Observe that the action is clearly transitive.

Proposition 1.36. *$\alpha \in \Omega(S\mathbb{R}^n)^{\overline{G}}$.*

Proof. Since $\mathrm{d}\overline{g} = (g\mathrm{d}y, g\mathrm{d}v)$, this follows at once from (1.25) and $G \subset \mathrm{SO}(n)$. □

Proposition 1.37. *For any $\omega \in \Omega^{n-1}(S\mathbb{R}^n)^{\overline{G}}$ one has $\mathrm{D}\omega \in \Omega^n_v(S\mathbb{R}^n)^{\overline{G}}$. Moreover there is a unique $\xi \in \Omega^{n-2}_h(S\mathbb{R}^n)^{\overline{G}}$ such that $\mathrm{D}\omega = \mathrm{d}(\omega + \alpha \wedge \xi)$.*

Proof. Let $\xi \in \Omega^{n-2}_h(S\mathbb{R}^n)$ be such that $\mathrm{D}\omega = \mathrm{d}(\omega + \alpha \wedge \xi)$ and take any $\overline{g} \in \overline{G}$. By Proposition 1.36, the following form is vertical:

$$\overline{g}^*\mathrm{D}\omega = \overline{g}^*\mathrm{d}(\omega + \alpha \wedge \xi) = \mathrm{d}(\overline{g}^*\omega + \overline{g}^*\alpha \wedge \overline{g}^*\xi) = \mathrm{d}(\omega + \alpha \wedge \overline{g}^*\xi),$$

and hence equal to $\mathrm{D}\omega$. From uniqueness of ξ it then follows that $\overline{g}^*\xi = \xi$. □

If the group G contains the element $-\mathrm{id}$, then any G-invariant valuation is even. This is true for almost all the groups listed above, except $\mathrm{SU}(2m+1)$, $m \in \mathbb{N}$, and G_2. In fact, as Bernig showed in [24] and [26], respectively, the hypothesis $-\mathrm{id} \in G$ is not necessary, as all G-invariant valuations are even also in these two remaining cases. Altogether, one has the following result, whose conceptual understanding is, however, still missing:

Theorem 1.38 (Bernig [26])**.** *If G acts transitively on S^{n-1}, then $\mathrm{Val}^G \subset \mathrm{Val}^+$.*

In particular, any k-homogeneous G-invariant valuation is uniquely represented by its (G-invariant) Klain function on $\mathrm{Gr}_k(\mathbb{R}^n)$.

1.3.2 Algebraic Structures on G-Invariant Valuations

Let us now proceed to review some of the important algebraic structures on the space Val^G whose existence is implied by the Irreducibility theorem 1.23. Besides the original articles, we refer to §3 of the survey [27]. Let us emphasize that although it is enough for us to consider the case of G-invariant valuations, versions of all of the algebraic operations and statements we list below are in fact available in much greater generality, namely on the whole (infinite-dimensional) space Val^∞ (cf. §5.2 below).

First, there is the natural *Alesker product* that turns Val^G into a graded algebra and has in fact all the nice properties one can imagine. It is based on a simple geometric construction. Let ΔK be the diagonal embedding of $K \in \mathcal{K}(\mathbb{R}^n)$ into $\mathbb{R}^{2n} = \mathbb{R}^n \times \mathbb{R}^n$.

Theorem 1.39 (Alesker [8]). *Let $A, B \in \mathcal{K}$ have smooth boundaries with positive curvature. Then*

$$(\psi_A \cdot \psi_B)(K) = \mathrm{vol}_{2n}(\Delta K + A \times B), \quad K \in \mathcal{K}, \tag{1.33}$$

defines a commutative associative distributive continuous graded product on Val^G *with unit* χ.

Second, there is a remarkable duality on Val^G, the so-called *Alesker-Fourier transform*, induced by the operation of taking the orthogonal complement. Recall from Theorem 1.38 that $\mathrm{Val}^G \subset \mathrm{Val}^+$.

Theorem 1.40 (Alesker [5]). *There exists a linear isomorphism* $\mathbb{F} : \mathrm{Val}^G \to \mathrm{Val}^G$ *such that*
(a) $\mathbb{F}^2 = \mathrm{id}$,
(b) $\mathbb{F}\,\mathrm{Val}_k^G = \mathrm{Val}_{n-k}^G$,
(c) $\mathbb{F}(\chi) = \mathrm{vol}_n$.
In terms of Klain functions, $\mathbb{F}$ *of* $\mu \in \mathrm{Val}_k^G$ *is given by*

$$\mathrm{Kl}_{\mathbb{F}\mu}(E) = \mathrm{Kl}_\mu(E^\perp), \quad E \in \mathrm{Gr}_{n-k}(\mathbb{R}^n). \tag{1.34}$$

Third, having a product and a Fourier-type transform at our disposal, it is natural to think of the convolution given by

$$\phi * \psi = \mathbb{F}(\mathbb{F}\phi \cdot \mathbb{F}\psi), \quad \phi, \psi \in \mathrm{Val}^G. \tag{1.35}$$

The question is, however, whether a formula analogous (1.33) is available. Remarkably, Bernig and Fu [30] showed that the answer is indeed affirmative and in fact the geometric meaning of the convolution is particularly simple, and that, moreover, an equally simple formula exists in terms of invariant differential forms. Hence, let us take the liberty to follow Alesker [14] and Wannerer [139, 140] and talk about the *Bernig-Fu convolution*. In order to recall the result here, let us first establish some notation. Namely, we define a linear operator $*_1$ on $\Omega^\bullet(S\mathbb{R}^n)$ as follows: For $\eta_V \in \Omega^k(\mathbb{R}^n)$ and $\eta_S \in \Omega^l(S^{n-1})$, let

$$*_1(\eta_V \wedge \eta_S) = (-1)^{\binom{n-k}{2}}(*_V \eta_V) \wedge \eta_S, \tag{1.36}$$

where $*_V$ is the standard Hodge star operator on $\Omega^\bullet(\mathbb{R}^n)$. Then

Theorem 1.41 (Bernig, Fu [30]). *Let $A, B \in \mathcal{K}$ be as in Theorem 1.39. Then*

$$\psi_A * \psi_B = \psi_{A+B} \tag{1.37}$$

defines the convolution on Val^G, *i.e. a commutative associative distributive continuous graded product on* Val^G *with unit* vol_n *that satisfies* (1.35). *Furthermore, for* $\omega, \tau \in \Omega^{n-1}(S\mathbb{R}^n)^{\overline{G}}$,

$$[[\omega]] * [[\tau]] = \left[\left[*_1^{-1} (*_1\omega \wedge *_1 \mathrm{D}\tau)\right]\right]. \tag{1.38}$$

Remark 1.42.
(a) The Bernig-Fu convolution is graded by the *codegree* of a valuation, i.e.

$$\mathrm{Val}_{n-k}^G * \mathrm{Val}_{n-l}^G \subset \mathrm{Val}_{n-(k+l)}^G. \tag{1.39}$$

(b) A formula for the Alesker product analogous to (1.38) was proven by Alesker and Bernig [17]. However, at the top of the operations of (1.38), it involves fibre integration which makes it much more difficult to use in practise, in comparison with (1.38).

We shall also need the following related statement that is certainly well known:

Proposition 1.43. *For any $\omega, \tau \in \Omega^{n-1}(S\mathbb{R}^n)^{\overline{G}}$,*

$$\mathrm{D}\left(*_1^{-1}(*_1\omega \wedge *_1\mathrm{D}\tau)\right) = *_1^{-1}(*_1\mathrm{D}\omega \wedge *_1\mathrm{D}\tau). \tag{1.40}$$

Proof. First, as $\mathrm{d} *_1 = (-1)^n *_1 \mathrm{d}$ (see [29], Proposition 4.1) and $\mathrm{d}(\mathrm{D}\tau) = 0$, one has

$$\mathrm{d}\left(*_1^{-1}(*_1\omega \wedge *_1\mathrm{D}\tau)\right) = *_1^{-1}(*_1\mathrm{d}\omega \wedge *_1\mathrm{D}\tau).$$

Let $\xi \in \Omega^{n-2}(S\mathbb{R}^n)$ be such that $\mathrm{D}\omega = \mathrm{d}(\omega + \alpha \wedge \xi)$. Then $*_1^{-1}(*_1(\alpha \wedge \xi) \wedge *_1\mathrm{D}\tau)$ is clearly vertical and since

$$\begin{aligned} &\mathrm{d}\left(*_1^{-1}(*_1\omega \wedge *_1\mathrm{D}\tau) + *_1^{-1}(*_1(\alpha \wedge \xi) \wedge *_1\mathrm{D}\tau)\right) \\ &\quad = *_1^{-1}(*_1\mathrm{d}\omega \wedge *_1\mathrm{D}\tau) + *_1^{-1}(*_1\mathrm{d}(\alpha \wedge \xi) \wedge *_1\mathrm{D}\tau) \\ &\quad = *_1^{-1}(*_1\mathrm{D}\omega \wedge *_1\mathrm{D}\tau) \end{aligned}$$

is vertical as well, the proof is finished. □

Fourth, the algebra Val^G equipped with either of the two multiplicative structures we discussed (clearly, $(\mathrm{Val}^G, \cdot, \chi)$ and $(\mathrm{Val}^G, *, \mathrm{vol}_n)$ are isomorphic as unital algebras) satisfies the so-called *Alesker-Poincaré duality*. Namely, let us recall the definition of the *Alesker-Poincaré pairing* on Val^G:

$$\mathrm{pd} : \mathrm{Val}^G \times \mathrm{Val}^G \to \mathbb{R} : (\phi, \psi) \mapsto (\phi \cdot \psi)_n, \tag{1.41}$$

where for $\mu \in \mathrm{Val}^G$, $(\mu)_n \,\mathrm{vol}_n$ is its n-homogeneous component. Then

Theorem 1.44 (Alesker [8])**.** *The pairing* pd *is perfect, i.e. non-degenerate, on* Val^G.

Important for us is the following observation:

Lemma 1.45 (Bernig, Fu [30])**.** *For any $\phi, \psi \in \mathrm{Val}_k^G$,*

$$(\mathbb{F}\phi) \cdot \psi = \phi \cdot (\mathbb{F}\psi). \tag{1.42}$$

As a consequence, one has

Proposition 1.46. *For any $\phi, \psi \in \mathrm{Val}^G$,*

$$\mathrm{pd}(\phi, \psi) = (\phi * \psi)_0, \tag{1.43}$$

where for $\mu \in \mathrm{Val}^G$, $(\mu)_0\, \chi$ is its 0-homogeneous component.

Proof. Consider $\phi = \sum_{i=0}^n \phi_i$ and $\psi = \sum_{j=0}^n \psi_j$ with $\phi_i, \psi_i \in \mathrm{Val}_i$. According to (1.42),

$$\begin{aligned} (\phi \cdot \psi)_n &= \sum_{i=0}^n (\phi_i \cdot \psi_{n-i})_n = \sum_{i=0}^n (\phi_i \cdot \mathbb{F}^2\psi_{n-i})_n = \sum_{i=0}^n (\mathbb{F}\phi_i \cdot \mathbb{F}\psi_{n-i})_n = \sum_{i=0}^n \left(\mathbb{F}(\phi_i * \psi_{n-i})\right)_n \\ &= \sum_{i=0}^n (\phi_i * \psi_{n-i})_0 = (\phi * \psi)_0. \end{aligned}$$

□

Finally, Val^G satisfies two versions of the *hard Lefschetz property*. Let L and Λ be the linear operators on Val^G given by

$$L : \mathrm{Val}_k^G \to \mathrm{Val}_{k+1}^G : \phi \mapsto \phi \cdot \mu_1, \tag{1.44}$$

and

$$\Lambda : \mathrm{Val}_k^G \to \mathrm{Val}_{k-1}^G : \phi \mapsto \phi * \mu_{n-1}. \tag{1.45}$$

Remark 1.47. Clearly, $G \subset \mathrm{SO}(n)$ yields $\mathrm{Val}^{\mathrm{SO}(n)} \subset \mathrm{Val}^G$, in particular, $\mu_1, \mu_{n-1} \in \mathrm{Val}^G$.

Theorem 1.48 (Alesker [5,6], Bernig and Fu [30])**.**
(a) *For $0 \le k \le \frac{n}{2}$, the following map is an isomorphism:*

$$L^{n-2k} : \mathrm{Val}_k^G \to \mathrm{Val}_{n-k}^G. \tag{1.46}$$

In particular, $L : \mathrm{Val}_k^G \to \mathrm{Val}_{k+1}^G$ is injective if $k < \frac{n}{2}$ and surjective if $k > \frac{n}{2} - 1$.
(b) *For $\frac{n}{2} \le k \le n$, the following map is an isomorphism:*

$$\Lambda^{2k-n} : \mathrm{Val}_k^G \to \mathrm{Val}_{n-k}^G. \tag{1.47}$$

In particular, $\Lambda : \mathrm{Val}_k^G \to \mathrm{Val}_{k-1}^G$ is injective if $k > \frac{n}{2}$ and surjective if $k < \frac{n}{2} + 1$.

Remark 1.49. The two parts of the previous theorem are obviously equivalent to each other via the Alesker-Fourier transform.

Corollary 1.50. *Let $b_k = \dim \mathrm{Val}_k^G$. Then $b_k = b_{n-k}$ and*

$$1 = b_0 \le b_1 \le \cdots \le b_{\lfloor \frac{n}{2} \rfloor} = b_{\lceil \frac{n}{2} \rceil} \ge \cdots \ge b_{n-1} \ge b_n = 1. \tag{1.48}$$

In fact, one always has $b_1 = b_{n-1} = 1$:

Proposition 1.51 (Alesker [8])**.** $\mathrm{Val}_1^G = \mathrm{span}\{\mu_1\}$ *and* $\mathrm{Val}_{n-1}^G = \mathrm{span}\{\mu_{n-1}\}$.

1.3.3 Fundamental Theorem of Algebraic Integral Geometry

Importance of the multiplicative structures introduced in the previous section was fully revealed when it turned out that a beautiful and intimate relation, usually referred to as the *Fundamental theorem of algebraic integral geometry* (FTAIG), exist between them and the kinematic formulas:

Theorem 1.52 (Bernig, Fu [30])**.** *Let $\phi_1, \ldots, \phi_N$ be a basis of* Val^G *and let M be the matrix of the Alesker-Poincaré pairing in this basis, i.e.*

$$M_{i,j} = \mathrm{pd}(\phi_i, \phi_j), \quad 1 \le i, j \le N. \tag{1.49}$$

Then for any $1 \le i \le N$ and $K, L \in \mathcal{K}$, one has

$$\int_{\overline{G}} \phi_i(K \cap \overline{g}L)\, \mathrm{d}\overline{g} = \sum_{j,k=1}^{N} (M^{-1})_{j,k}\, (\phi_i \cdot \phi_j)(K)\, \phi_k(L), \tag{1.50}$$

and

$$\int_G \phi_i(K + gL)\, \mathrm{d}g = \sum_{j,k=1}^{N} (M^{-1})_{j,k}\, (\phi_i * \phi_j)(K)\, \phi_k(L), \tag{1.51}$$

where $\mathrm{d}g$ is the Haar probability measure on G and $\mathrm{d}\overline{g}$ is the product measure of the Haar probability measure and the Lebesgue measure on $\overline{G} = G \ltimes \mathbb{R}^n$.

According to the FTAIG, knowledge of the Alesker product on Val^G is equivalent to knowledge of the *Blaschke kinematic formulas* (1.50) while the Bernig-Fu convolution is related, in precisely the same manner, to the *additive kinematic formulas* (1.51). In this sense, the two collections of formulas are *dual* to each other, the duality being induced by the Alesker-Fourier transform.

To conclude, let us point out that an important instance of (1.50), the *Principal kinematic formula*

$$\int_{\overline{G}} \chi(K \cap \overline{g}L)\, \mathrm{d}\overline{g} = \sum_{j,k=1}^{N} (M^{-1})_{j,k}\, \phi_j(K)\, \phi_k(L), \tag{1.52}$$

can be also achieved solely by means of the Bernig-Fu convolution as knowledge of the convolution is sufficient to determine the Alesker-Poincaré pairing by virtue of (1.43).

1.3.4 A Review of Achieved Results and Open Problems

The contents of the preceding three sections together establish a program whose ideal outcome would be the complete set of explicit kinematic formulas corresponding to all of the listed groups with transitive action on a sphere. However, in spite of the whole array of deep and beautiful results achieved over the past decade, this ultimate goal remains far from being completely understood and solved. To be more precise, let us review the most important results and highlight the questions that remain open.

As we have seen, a crucial step in each case is to unfold at least one of the two reincarnations of the canonical multiplicative structure on the space of invariant valuations. Discussing the other aspects of the surveyed results as well, let us particularly focus on the algebra structure, as this will establish a suitable context to put the results of our thesis in.

First, the prototypical case. According to classical Theorem 1.18, the space $\mathrm{Val}^{\mathrm{SO}(n)}$ is spanned by the intrinsic volumes. As for the algebra structure, it follows at once from Theorem 1.48 that each intrinsic volume must be in fact a non-trivial multiple of a suitable power of either μ_1 or μ_{n-1}, depending whether with respect to the product or the convolution. I.e.,

Theorem 1.53 (Hadwiger [76], Alesker [8])**.**

$$\mathrm{Val}^{\mathrm{SO}(n)} = \mathrm{span}\{\mu_0, \ldots, \mu_n\} \cong \mathbb{R}[t]/(t^{n+1}), \tag{1.53}$$

It is then only a matter of careful treatment of scaling factors to reconstruct the classical kinematic formulas (1.14) and (1.15) by means of the FTAIG (see e.g. §2.3.4 of [19]).

Second, the Hermitian case $G = \mathrm{U}(n)$ has been studied extensively and turned out to be much more complicated and (thus) interesting. First of all, the inequalities (1.48) are no longer trivial:

Theorem 1.54 (Alesker [4])**.** *For* $0 \le k \le 2n$, $\dim \mathrm{Val}_k^{\mathrm{U}(n)} = 1 + \left\lfloor \frac{\min\{k, 2n-k\}}{2} \right\rfloor$.

Further, various bases of $\mathrm{Val}^{\mathrm{U}(n)}$ were introduced and kinematic formulas were proven in certain special cases by Park [109], Tasaki [133, 134], and Alesker [5]. A crucial step towards systematic understanding of Hermitian integral geometry was then made by

Theorem 1.55 (Fu [63])**.** *Let* t, s *be of formal degree* $1, 2$*, respectively. As graded algebras,*

$$\mathrm{Val}^{\mathrm{U}(n)} \cong \mathbb{R}[t, s]/(f_{n+1}, f_{n+2}), \tag{1.54}$$

where $f_k = f_k(t,s)$ *are the polynomials formally given by*

$$\log(1+tx+sx^2) = \sum_{k=0}^{\infty} f_k(t,s)x^k. \tag{1.55}$$

The general effort culminated in a seminal article [31], where Bernig and Fu introduced Hermitian analogue of the intrinsic volumes and performed a remarkable synthesis of all previously known fragments of knowledge, resulting in determination of *explicit* kinematic formulas in an n-dimensional Hermitian space *in* their *full generality*.

Fourth, the case of the special unitary group turned out not to be very different: As shown by Bernig [24], in addition to the $\mathrm{U}(n)$-invariant valuations there are two or four extra generators in degree n, depending on parity of n, responsible for rather cosmetic changes in the resulting kinematic formulas. See also [7,25] for earlier results on $n = 2$.

Fifth, Bernig [26] also fully resolved two of the exceptional cases, G_2 and $\mathrm{Spin}(7)$. For both groups, a geometric basis was introduced, and kinematic formulas as well as the algebra structure were determined explicitly, the latter shown to be as follows:

Theorem 1.56 (Bernig [26])**.** *Let* t, v, u *be of formal degree* $1, 3, 4$*, respectively. Then*

$$\mathrm{Val}^{\mathrm{G}_2} \cong \mathbb{R}[t,v]/(t^8, t^2v, v^2+4t^6) \tag{1.56}$$

and

$$\mathrm{Val}^{\mathrm{Spin}(7)} \cong \mathbb{R}[t,u]/(t^9, u^2-t^8, ut). \tag{1.57}$$

Sixth, integral geometry of the first non-trivial symplectic group $G = \mathrm{Sp}(2)\mathrm{Sp}(1)$ is well explored thanks to Bernig and Solanes [33,34]. It is in contrast with the preceding cases that the algebra of $\mathrm{Sp}(2)\mathrm{Sp}(1)$-invariant valuations is truly complicated: One has

Theorem 1.57 (Bernig, Solanes [34])**.** *Let* t, s, v, u *be of degree* $1, 2, 3, 4$*, respectively. Then*

$$\mathrm{Val}^{\mathrm{Sp}(2)\mathrm{Sp}(1)} \cong \mathbb{R}[t,s,v,u]/\mathcal{I}, \tag{1.58}$$

where $\mathcal{I}$ *is the ideal generated by*

$$\begin{aligned}
&tk_4, \quad tn_4, \quad k_2n_3 - \frac{63}{32}\pi^2t^3k_2 + \frac{743}{24}\pi t^2n_3,\\
&t^3n_3, \quad k_2k_4 - \frac{49}{4}\pi^2t^4k_2, \quad k_2n_4, \quad n_3^2 + \frac{27}{14}\pi^4t^6 - \frac{33435}{896}\pi^3t^4k_2,\\
&t^5k_2, \quad k_4n_3, \quad n_3n_4,\\
&k_4^2 - \pi^4t^8, \quad k_4n_4, \quad n_4^2 - 864\pi^4t^8,
\end{aligned}$$

with

$$\begin{aligned}
k_2 &= -12\pi t^2 + 56s,\\
k_4 &= -\frac{5}{2}\pi^2t^4 - 16\pi t^2s + 160s^2 - \frac{105}{2}tv,\\
n_3 &= -63\pi^2t^3 + 378\pi ts - 630v,\\
n_4 &= -2340\pi^2t^4 + 17280\pi t^2s - 11520s^2 - 31500tv + 10080u.
\end{aligned}$$

In higher ranks and also in the cases of $\mathrm{Sp}(n)\mathrm{U}(1)$ and $\mathrm{Sp}(n)$, however, the problem of determining kinematic formulas remains almost completely open for the 'only' piece of information currently available here are Bernig's combinatorial formulas [28] for the Betti numbers, i.e. the dimensions of Val_k^G. A glimpse at the first few cases (see Table 2 in [28] for $n \leq 5$) foreshadows that it will presumably be a challenge to understand quaternionic integral geometry. In particular it might be a long way towards a closed description of the valuation algebras à la Theorem 1.55.

Finally, the last exceptional octonionic case $G = \mathrm{Spin}(9)$ is the subject of this thesis and will be discussed thoroughly in Chapter 4.

Chapter 2

Octonionic Geometry

2.1 The Algebra of Octonions

The mathematics of octonions is truly essential to our thesis and underlies, literally, all the problems we discuss. Let us, therefore, begin the second chapter with a careful review of their basic algebraic properties. For our purpose, we believe, this is best to be done in a more general context of normed division algebras as this approach naturally provides us with room for discussing the genealogy of the octonions as well.

2.1.1 Normed Division Algebras

What follows is classical and very clearly explained e.g. in §6 of [79] where the reader is referred for a reference.

Definition 2.1. A *normed division algebra* is a Euclidean vector space $\mathcal{A}$ equipped with a bilinear product that admits a unit $1 \in \mathcal{A}$ and satisfies, for any $x, y \in \mathcal{A}$,

$$|xy|^2 = |x|^2 \, |y|^2 . \tag{2.1}$$

Remark 2.2. An obvious consequence of (2.1) is that $\mathcal{A}$ has *no zero divisors*:

$$\text{if } xy = 0, \text{ then } x = 0 \text{ or } y = 0. \tag{2.2}$$

As usual for unital algebras, we naturally identify $\mathbb{R}$ with the subalgebra $\mathbb{R} \cdot 1 \subset \mathcal{A}$. Further, we denote $\operatorname{Im} \mathcal{A} = 1^{\perp}$. Then we have $\mathcal{A} = \mathbb{R} \oplus \operatorname{Im} \mathcal{A}$ and with this respect we define the *real* and *imaginary part*, and the *conjugation* of $x \in \mathcal{A}$ respectively as

$$\operatorname{Re}(x) = \langle x, 1 \rangle, \tag{2.3}$$

$$\operatorname{Im}(x) = x - \operatorname{Re}(x), \tag{2.4}$$

$$\overline{x} = \operatorname{Re}(x) - \operatorname{Im}(x). \tag{2.5}$$

(2.4) and (2.5) can be easily inverted as follows:

$$\operatorname{Re}(x) = \frac{1}{2}(x + \overline{x}), \tag{2.6}$$

$$\operatorname{Im}(x) = \frac{1}{2}(x - \overline{x}), \tag{2.7}$$

and so $x \in \mathbb{R}$ if and only if $\overline{x} = x$ and similarly $x \in \operatorname{Im} \mathcal{A}$ if and only if $\overline{x} = -x$. It is also obvious that the conjugation is a linear involution on $\mathcal{A}$.

For $w \in \mathcal{A}$, consider the linear operators

$$R_w : x \mapsto xw \quad \text{and} \quad L_w : x \mapsto wx \tag{2.8}$$

of *right* and *left multiplication* on $\mathcal{A}$ (notice that the product may not be commutative). Polarizing the central identity (2.1), it is not difficult to conclude (see [79], p. 103) that

$$R_w^* = R_{\overline{w}} \quad \text{and} \quad L_w^* = L_{\overline{w}}, \tag{2.9}$$

as well as

$$R_{\overline{w}}R_z + R_{\overline{z}}R_w = L_{\overline{w}}L_z + L_{\overline{z}}L_w = 2\langle w, z\rangle \,\mathrm{id}\,. \tag{2.10}$$

As an immediate consequence of (2.9), one has

$$\langle x, y\rangle = \mathrm{Re}(\overline{x}y) = \mathrm{Re}(x\overline{y}) = \langle \overline{x}, \overline{y}\rangle \tag{2.11}$$

and, since $\langle \overline{xy}, z\rangle = \langle xy, \overline{z}\rangle = \langle y, \overline{x}\,\overline{z}\rangle = \langle yz, \overline{x}\rangle = \langle z, \overline{y}\,\overline{x}\rangle$ holds for any $z \in \mathcal{A}$, also

$$\overline{xy} = \overline{y}\,\overline{x}. \tag{2.12}$$

In particular,

$$|x|^2 = \overline{x}x = x\overline{x} \tag{2.13}$$

and therefore each non-zero $x \in \mathcal{A}$ has a (unique) *multiplicative inverse* $x^{-1} = \frac{1}{|x|^2}\overline{x}$.

Recall that Definition 2.1 does not require $\mathcal{A}$ to be associative either. Still, a weaker form of associativity is always guaranteed in a normed division algebra. Consider the *associator* $\mathcal{A} \times \mathcal{A} \times \mathcal{A} \to \mathcal{A}$ given by

$$[x, y, z] = (xy)z - x(yz). \tag{2.14}$$

It is easily verified (in [79], Lemma 6.11, for instance) that this is an alternating trilinear map. Consequently, the important *Moufang identities* hold:

Theorem 2.3 (Moufang [105]). *Any elements x, y, z of a normed division algebra $\mathcal{A}$ satisfy*

$$x(y(xz)) = (xyx)z, \tag{2.15}$$
$$((zx)y)x = z(xyx), \tag{2.16}$$
$$(xy)(zx) = x(yz)x. \tag{2.17}$$

Remark 2.4. Notice that no more additional brackets are needed in the expressions xyx and $x(yz)x$ as the corresponding associators vanish.

Clearly, the associator is also trivial when (at least) one of its variables is real. This fact, first, has the following consequence:

$$[x, y, z] = [\mathrm{Im}(x), y, z] = -[\overline{x}, y, z], \tag{2.18}$$

second, can be in fact strengthened by induction into

Theorem 2.5 (Artin[1] [145]). *In a normed division algebra, any subalgebra generated by two elements is associative.*

Remark 2.6. Observe that such a subalgebra is in fact generated by imaginary parts of the respective elements: for any $x \in \mathcal{A}$ one has

$$x = \mathrm{Re}(x)\,\mathrm{Im}(x)^0 + \mathrm{Im}(x), \tag{2.19}$$

where we put $0^0 = 1$ if necessary. Therefore, it also equals to the subalgebra generated by the two elements *and* their *conjugates*.

Remark 2.7. For a clear exposition of the previous two theorems see also [119], §III.1.

[1] Emil Artin was given credit for this result by his student Max Zorn who published it in [145].

2.1.2 The Hurwitz Theorem

We have seen that the compatibility condition (2.1), without any further assumptions, impose quite non-trivial restrictions on the algebraic structure of a normed division algebra. It turns out that such a feature is indeed exclusive and in fact, there are essentially just four spaces it is innate to. More precisely,

Theorem 2.8 (Hurwitz [82])**.** *If $\mathcal{A}$ is a normed division algebra, then* $\dim \mathcal{A} \in \{1, 2, 4, 8\}$.

Theorem 2.9 (Robert [115], Hurwitz [83])**.** *Any two normed division algebras of the same dimension are isomorphic.*

Remark 2.10. These two results together are usually referred to as the *Hurwitz theorem*.[2]

Representatives of the four (thus non-empty) isomorphism classes were very well known already prior to the Hurwitz theorem. First and trivially, one has the *reals* $\mathbb{R}$. Second, there are the *complex numbers* $\mathbb{C}$, whose two-dimensional algebraic representation goes back independently to Hamilton [77] and Gauss (see [114], §I.8). Third, after almost a decade of unsuccessful struggle towards a normed division algebra in three dimensions, Hamilton eventually realized that his efforts can only meet with success when 'admitting, in some sense, a fourth dimension' (see [78], p. 108), discovering thus the algebra $\mathbb{H}$ of *quaternions* (see also [90]). Finally, not long after, the eight-dimensional *octonions* $\mathbb{O}$ appeared in works of Graves [68] and Cayley (see [47], p. 127).

Let us remark that both parts of the Hurwitz theorem were, nonetheless, originally stated in terms of existence and uniqueness, respectively, of *n-square formulas*. That these could be interpreted concerning the norms of the algebras $\mathbb{R}$, $\mathbb{C}$, $\mathbb{H}$ and $\mathbb{O}$ was only realized by Dickson [54].

Historically, the Hurwitz theorem has had a number of relatives. Although they are not, strictly speaking, directly relevant to our work, let us mention at least some of these results here. First, as for an ancestor, it was shown independently by Frobenius [57] and C. S. Pierce[3] [111] that any associative unital algebra without zero divisors must be isomorphic to one of $\mathbb{R}$, $\mathbb{C}$ and $\mathbb{H}$. Proceeding to descendants, Albert [2] strengthened the Hurwitz theorem remarkably by showing that it in fact holds true even when the norm (2.1) does not necessarily come from an inner product. Finally, from a deep and fundamental topological result of Bott [40], the so-called *Periodicity theorem*, Bott and Milnor [41] and independently Kervaire [85] were able to deduce that general algebras with no zero divisors can (and as we have seen they really do) only exist in dimensions 1, 2, 4 and 8. Nota bene, there are in fact much more of them than just the four normed division ones (see [21, 22]). For an excellent systematic account on the aforementioned as well as many related developments, see Part B of the Collection [55].

2.1.3 The Octonions

We conclude the first part of this chapter by presenting an explicit model for the four normed division algebra. As we shall see, this very much resembles Matryoshka. Let us begin with the 'outer doll' - the octonions.

As a Euclidean space, $\mathbb{O}$ is just $\mathbb{R}^8$ equipped with the standard inner product. Let us denote the standard orthonormal basis by $1, e_1, e_2, \ldots, e_7$ and let us define the algebra structure on $\mathbb{O}$ as follows:

---

[2] Eugène Robert, a student of Adolf Hurwitz, proved the assertion of Theorem 2.9 in his dissertation [115]. Hurwitz' Article [83] was published a decade later, containing an explicit reference to the thesis.

[3] Charles Sanders Pierce published this result as an Appendix to his father Benjamin Pierce's work [111].

(a) 1 is the unit: $1^2 = 1$ and $1e_i = e_i 1 = e_i$, $1 \leq i \leq 7$,
(b) e_i's are imaginary units: $e_i^2 = -1$, $1 \leq i \leq 7$,
(c) $e_i e_j = -e_j e_i$, $1 \leq i < j \leq 7$, and
(d) $e_{1+i}e_{2+i} = e_{4+i}$, $e_{2+i}e_{4+i} = e_{1+i}$, $e_{4+i}e_{1+i} = e_{2+i}$, $1 \leq i \leq 7$.

In (d) the indices must be read modulo 7, i.e. e.g. $e_1 e_2 = e_4$, $e_7 e_2 = e_6$, et cetera. Observe that $\operatorname{Im}\mathbb{O} = \operatorname{span}\{e_1, \ldots, e_7\}$. Further, it will be sometimes convenient to denote $e_0 = 1$.

Remark 2.11. A useful mnemonic for the array of rules (d) is the following table

$$\begin{array}{ccccccc} 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ 2 & 3 & 4 & 5 & 6 & 7 & 1 \\ 4 & 5 & 6 & 7 & 1 & 2 & 3 \end{array}$$

that can be easily completed once we remember the first column. Others may prefer the so-called Fano plane (see e.g. [20], p. 152) or the circle diagram that is (to the best of our knowledge) due to Günaydin and Gürsey (see [73], Figure 1).

It is straightforward to check that such a product is indeed compatible, in the sense of (2.1), with the standard Euclidean structure, recovering thus the famous *Degen eight-square formula* [51] (see also [54]). It is also immediately seen that the algebra is *neither* commutative *nor* associative. As for the latter, consider for instance

$$(e_1 e_2)e_3 = e_4 e_3 = -e_6 = -e_1 e_5 = -e_1(e_2 e_3).$$

Nonetheless, recall that because of being a normed division algebra, $\mathbb{O}$ still possesses numerous non-trivial algebraic structures and properties as listed above. In this connection, let us emphasize the role of equations (2.10) and of the Moufang identities (2.15) - (2.17): they are not only extremely useful when computing within $\mathbb{O}$ but, since they provide one with almost the only tool for doing so, they very often exhibit the power to literally shape the octonionic geometry. We believe that a careful reader may observe this phenomenon multiple times within the text.

2.1.4 The Quaternions, the Complex Numbers, and the Reals

Let us now recognize the remaining three normed division algebras as subalgebras of $\mathbb{O}$. First, consider the 4-dimensional subspace $\operatorname{span}\{1, e_2, e_2, e_4\} \subset \mathbb{O}$. It follows directly from the definition of the octonionic algebra in §2.1.3 that this is actually a subalgebra, it is normed division, and hence the Hurwitz theorem issues the permit to define

$$\mathbb{H} = \operatorname{span}\{1, e_1, e_2, e_4\}. \tag{2.20}$$

Observe that the algebra $\mathbb{H}$ of *quaternions* is generated by two elements and thus it is associative by Theorem 2.5. Clearly, $\mathbb{H}$ is still non-commutative however. Notice also that our choice of the inclusion $\mathbb{H} \subset \mathbb{O}$ is far from unique. Second, one possible choice to define the (commutative) *complex numbers* is $\mathbb{C} = \operatorname{span}\{1, e_1\}$. Finally, one has the *reals* $\mathbb{R} = \operatorname{span}\{1\}$.

Remark 2.12. Later on, the following easy observation will be useful: For any $x \in \mathbb{O}\backslash\mathbb{R}$, the subalgebra $\mathcal{S} \subset \mathbb{O}$ generated by x is isomorphic to $\mathbb{C}$. Indeed, $\mathcal{S}$ is equivalently generated by $\operatorname{Im}(x) \neq 0$ (see also Remark 2.6), and so

$$e_1 \mapsto \frac{1}{|\operatorname{Im}(x)|} \operatorname{Im}(x)$$

defines an isomorphism $\mathbb{C} \to \mathcal{S}$.

2.2 Some Spin Groups Related to the Octonions

2.2.1 Clifford Algebras and Spin Groups

It is well known that the (connected) groups $\mathrm{SO}(n)$ are not simply connected for $n \geq 3$ but rather their fundamental groups are $\pi_1(\mathrm{SO}(n)) \cong \mathbb{Z}_2$. In other words, associated with each of these groups is its universal two-sheeted covering group, commonly labelled $\mathrm{Spin}(n)$. There is a unified construction through which all the Spin groups can be constructed. It mimics the fact that every rotation composes of a finite number of reflections. We shall briefly recall the concept here, following §9 and §10 of [79].

Let $n \geq 3$ be an integer. The *Clifford algebra* is by definition the quotient

$$\mathfrak{C}(n) = \mathfrak{T}(\mathbb{R}^n)/I \tag{2.21}$$

by the ideal $I \subset \mathfrak{T}(\mathbb{R}^n)$ generated by $\{x \otimes x + |x|^2 \,;\, x \in \mathbb{R}^n\}$. This is a unital associative algebra of dimension 2^n. It comes equipped with the *canonical automorphism* $x \mapsto \widetilde{x}$ extended from the involution $x \mapsto -x$ on $\mathbb{R}^n$ by multilinearity.

Consider further the subset $\mathfrak{C}^*(n) \subset \mathfrak{C}(n)$ of invertible elements. Obviously, $\mathfrak{C}^*(n)$ forms a group. Observe that in particular $\mathbb{R}^n \backslash \{0\} \subset \mathfrak{C}^*(n)$ since $x^2 = -|x|^2$ yields

$$x^{-1} = -\frac{1}{|x|^2}x, \quad 0 \neq x \in \mathbb{R}^n. \tag{2.22}$$

Definition 2.13. One defines

$$\mathrm{Spin}(n) = \{a \in \mathfrak{C}^*(n)\,;\, a = x_1 x_2 \cdots x_{2r}, r \in \mathbb{N}_0, x_i \in \mathbb{R}^n, |x_i| = 1\}. \tag{2.23}$$

Remark 2.14. The empty product ($r = 0$) in (2.23) stands for $a = 1$, the unit.

One has $x \otimes y + y \otimes x + 2\langle x, y\rangle \in I$ and thus the Clifford product xy is symmetric or skew-symmetric if $x, y \in \mathbb{R}^n$ are collinear or perpendicular, respectively. Consequently, if $0 \neq y \in \mathbb{R}^n$,

$$x \mapsto -yxy^{-1}, \quad x \in \mathbb{R}^n, \tag{2.24}$$

is nothing else but the reflection in the hyperplane $y^\perp$. (2.24) then extends to the *twisted adjoint representation* $\widetilde{\mathrm{Ad}} : \mathfrak{C}^*(n) \to \mathrm{GL}(\mathfrak{C}(n)) : a \mapsto \widetilde{\mathrm{Ad}}_a$ defined by

$$\widetilde{\mathrm{Ad}}_a(x) = \widetilde{a}xa^{-1}, \quad x \in \mathfrak{C}(n). \tag{2.25}$$

It is easily seen that for any $a = x_1 \cdots x_{2r} \in \mathrm{Spin}(n)$, $\widetilde{\mathrm{Ad}}_a(\mathbb{R}^n) \subset \mathbb{R}^n$ and $\widetilde{\mathrm{Ad}}_a|_{\mathbb{R}_n} \in \mathrm{O}(n)$ with $\det(\widetilde{\mathrm{Ad}}_a|_{\mathbb{R}_n}) = (-1)^{2r} = 1$. In fact, one can show with a little effort that $\widetilde{\mathrm{Ad}}$, thus viewed, is the covering homomorphism, i.e. the sequence

$$1 \to \mathbb{Z}_2 \to \mathrm{Spin}(n) \xrightarrow{\widetilde{\mathrm{Ad}}} \mathrm{SO}(n) \to 1 \tag{2.26}$$

is exact. In this context, $\widetilde{\mathrm{Ad}}$ is also referred to as the *vector representation* of $\mathrm{Spin}(n)$.

It turns out that each Clifford algebra is isomorphic to one or two copies of a matrix algebra over either $\mathbb{R}$, $\mathbb{C}$, or $\mathbb{H}$. This isomorphism then descends to another canonical representation of $\mathrm{Spin}(n)$, the *spin representation*. Depending on the anatomy of the maternal Clifford algebra, this representation is either irreducible or a sum of two irreducible ones, and it is always *faithful*. See [79], §11, [20], §2.3 and §2.4, and [56], §1.5.

Remarkably, while the spin groups for $n \leq 6$ are just classical matrix groups (cf. *Cartan's* or *exceptional isomorphisms*), in higher dimensions they are intimately related to the octonions. The particular cases $n = 7, 8, 9$ relevant to us were worked out (carefully indeed) in §14 of [79] and will be recalled in the following sections.

2.2.2 The Group $\mathrm{Spin}(9)$

Let us begin with $n = 9$. For it will be very convenient for our purpose, let us adopt the image of $\mathrm{Spin}(9)$ under its (faithful) spin representation as the definition of the group. To see that this description fits into the general framework outlined in the previous section, we refer the reader to Lemma 14.77 of [79]. As anticipated, it is natural to identify the 16-dimensional spin module with the *octonionic plane* $\mathbb{O}^2$.

Definition 2.15. We define $\mathrm{Spin}(9)$ to be the subgroup of $\mathrm{GL}(\mathbb{O}^2)$ generated by

$$\left\{ \begin{pmatrix} r & R_x \\ R_{\overline{x}} & -r \end{pmatrix} ; r \in \mathbb{R}, x \in \mathbb{O}, r^2 + |x|^2 = 1 \right\}. \tag{2.27}$$

Notice that the generators act on $\mathbb{O}^2$ blockwise and from the left, as 2-by-2 block operators. Formulas for determinants of such operators are well known. Namely, since the two blocks $R_{\overline{x}}$ and $-r$ in the lower row commute, according to [128], Theorem 3,

$$\det \begin{pmatrix} r & R_x \\ R_{\overline{x}} & -r \end{pmatrix} = \det(-r^2 - R_x R_{\overline{x}}) = \det(-\,\mathrm{id}) = 1. \tag{2.28}$$

Thus, as $\mathrm{Spin}(9)$ obviously preserves the standard inner product

$$\left\langle \begin{pmatrix} x_0 \\ x_1 \end{pmatrix}, \begin{pmatrix} y_0 \\ y_1 \end{pmatrix} \right\rangle = \langle x_0, y_0 \rangle + \langle x_1, y_1 \rangle \tag{2.29}$$

on $\mathbb{O}^2$, one has $\mathrm{Spin}(9) \subset \mathrm{SO}(\mathbb{O}^2) = \mathrm{SO}(16)$ in fact. Let us now show what we anticipated in the opening chapter:

Proposition 2.16. $\mathrm{Spin}(9)$ *acts transitively on* $S^{15} \subset \mathbb{O}^2$.

Proof. First, for any $x \in \mathbb{O}$, $|x| = 1$ we have

$$\begin{pmatrix} x \\ 0 \end{pmatrix} = \begin{pmatrix} 0 & R_x \\ R_{\overline{x}} & 0 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix}.$$

Second, if $x, y \in \mathbb{O}$ satisfy $|x|^2 + |y|^2 = 1$ and $y \neq 0$, we can write

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 & R_{\frac{\overline{y}}{|y|}} \\ R_{\frac{y}{|y|}} & 0 \end{pmatrix} \begin{pmatrix} |y| & R_{\frac{\overline{yx}}{|y|}} \\ R_{\frac{xy}{|y|}} & -|y| \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} \qquad \square$$

It may perhaps illuminate more of the structure of $\mathrm{Spin}(9)$ and it will be useful for us to introduce a different generating set as follows:

Lemma 2.17. $\mathrm{Spin}(9)$ *is generated by*

$$\left\{ \begin{pmatrix} R_z & 0 \\ 0 & R_{\overline{z}} \end{pmatrix} ; z \in \mathbb{O}, |z| = 1 \right\} \cup \left\{ \begin{pmatrix} \cos(t) & \sin(t) \\ \sin(t) & -\cos(t) \end{pmatrix} ; t \in [0, 2\pi) \right\}. \tag{2.30}$$

Proof. First, the generators (2.30) can be obviously expressed in terms of (2.27): observe

$$\begin{pmatrix} R_z & 0 \\ 0 & R_{\overline{z}} \end{pmatrix} = \begin{pmatrix} 0 & R_z \\ R_{\overline{z}} & 0 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

As for the other direction, assume we are given $r \in \mathbb{R}$ and $x \in \mathbb{O}$ with $r^2 + |x|^2 = 1$. The case $x \in \mathbb{R}$ is trivial so assume otherwise. Choose $t \in [0, 2\pi)$ such that $r = \cos(t)$ and

$z \in \mathbb{O}$, $|z| = 1$ such that $x = \sin(t) z^2$. Notice that the latter is possible since, according to Remark 2.12, the subalgebra of $\mathbb{O}$ generated by x is isomorphic to $\mathbb{C}$. Then

$$\begin{aligned} \begin{pmatrix} R_z & 0 \\ 0 & R_{\overline{z}} \end{pmatrix} \begin{pmatrix} \cos(t) & \sin(t) \\ \sin(t) & -\cos(t) \end{pmatrix} \begin{pmatrix} R_{\overline{z}} & 0 \\ 0 & R_z \end{pmatrix} &= \begin{pmatrix} \cos(t) R_z R_{\overline{z}} & \sin(t)(R_z)^2 \\ \sin(t)(R_{\overline{z}})^2 & -\cos(t) R_{\overline{z}} R_z \end{pmatrix} \\ &= \begin{pmatrix} \cos(t) & \sin(t) R_{z^2} \\ \sin(t) R_{\overline{z}^2} & -\cos(t) \end{pmatrix} \\ &= \begin{pmatrix} r & R_x \\ R_{\overline{x}} & -r \end{pmatrix}. \end{aligned}$$

□

The following description of the *Lie algebra* of $\mathrm{Spin}(9)$ is well known (see [45], §2.1). Consider the following nine elements of the generating set (2.27):

$$\mathcal{I}_i = \begin{pmatrix} 0 & R_{e_i} \\ R_{\overline{e_i}} & 0 \end{pmatrix}, \quad 0 \le i \le 7, \quad \text{and} \quad \mathcal{I}_8 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \tag{2.31}$$

where $e_0, \dots, e_7$ is the standard orthonormal basis of $\mathbb{O}$, and denote $\mathcal{I}_{i,j} = \mathcal{I}_i \mathcal{I}_j$. Then

Lemma 2.18. *The set $\{\mathcal{I}_{i,j}\,;\, 0 \le i < j \le 8\}$ is a basis for the Lie algebra $\mathfrak{spin}(9)$ of* $\mathrm{Spin}(9)$.

Proof. First of all, the relations $\mathcal{I}_i^2 = \mathrm{id}$ and $\mathcal{I}_{i,j} = -\mathcal{I}_{j,i}$, $i \neq j$, are easily shown, the latter using (2.10). Consequently, as $\mathcal{I}_i^* = \mathcal{I}_i^{-1} \in \mathrm{SO}(16)$, one has $\mathcal{I}_i = \mathcal{I}_i^*$ and, for $i \neq j$,

$$\mathcal{I}_{i,j} = -(\mathcal{I}_{i,j})^{-1} = -(\mathcal{I}_{i,j})^* \in \mathfrak{so}(16).$$

Observe that the set in question is linearly independent as it can be orthonormalized with respect to the Frobenius inner product (see [110], Proposition 8): First, for $i < j$,

$$\mathrm{tr}(\mathcal{I}_{i,j} \mathcal{I}_{i,j}^*) = \mathrm{tr}(\mathcal{I}_i \mathcal{I}_j \mathcal{I}_j \mathcal{I}_i) = \mathrm{tr}(\mathrm{id}) = 16.$$

Second, for $i < j < k$, the inner product is

$$\mathrm{tr}(\mathcal{I}_{i,j} \mathcal{I}_{i,k}^*) = \mathrm{tr}(\mathcal{I}_i \mathcal{I}_j \mathcal{I}_k \mathcal{I}_i) = \mathrm{tr}(\mathcal{I}_i \mathcal{I}_i \mathcal{I}_j \mathcal{I}_k) = \mathrm{tr}(\mathcal{I}_j \mathcal{I}_k) = -\mathrm{tr}(\mathcal{I}_k \mathcal{I}_j) = -\mathrm{tr}(\mathcal{I}_j \mathcal{I}_k)$$

and hence trivial. Similarly, for i, j, k, l distinct, one has

$$\mathrm{tr}(\mathcal{I}_{i,j} \mathcal{I}_{k,l}^*) = \mathrm{tr}(\mathcal{I}_i \mathcal{I}_j \mathcal{I}_l \mathcal{I}_k) = \mathrm{tr}(\mathcal{I}_j \mathcal{I}_l \mathcal{I}_k \mathcal{I}_i) = -\mathrm{tr}(\mathcal{I}_i \mathcal{I}_j \mathcal{I}_l \mathcal{I}_k) = 0.$$

Further, it is immediate to verify

$$[\mathcal{I}_{i,j}, \mathcal{I}_{k,l}] = \begin{cases} 0 & \text{if } \{i,j\} \cap \{k,l\} = \emptyset, \\ -2\,\mathcal{I}_{j,l} & \text{if } i = k, j \neq l, \\ 2\,\mathcal{I}_{j,k} & \text{if } i = l, j \neq k. \end{cases} \tag{2.32}$$

Therefore, $\mathrm{span}\{\mathcal{I}_{i,j}\,;\, 0 \le i < j \le 8\}$ is a 36-dimensional subalgebra of $\mathfrak{so}(16)$.

To finish the proof is to show that the corresponding one-parameter subgroups

$$g_{i,j}(t) = \exp(t\,\mathcal{I}_{i,j}) = \cos(t)\,\mathrm{id} + \sin(t) \mathcal{I}_{i,j}$$

lie in $\mathrm{Spin}(9)$. First, for $0 \le i \le 7$, since

$$\mathcal{I}_{i,8} = \begin{pmatrix} 0 & R_{e_i} \\ R_{\overline{e_i}} & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = \begin{pmatrix} 0 & -R_{e_i} \\ R_{\overline{e_i}} & 0 \end{pmatrix},$$

one has

$$g_{i,8}(t) = \begin{pmatrix} \cos(t) & -\sin(t)R_{e_i} \\ \sin(t)R_{\overline{e_i}} & \cos(t) \end{pmatrix} = \begin{pmatrix} \cos(t) & R_{\sin(t)e_i} \\ R_{\overline{\sin(t)e_i}} & -\cos(t) \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \in \mathrm{Spin}(9)$$

by (2.27). Second, for $0 \le i < j \le 7$, since

$$\mathcal{I}_{i,j} = \begin{pmatrix} 0 & R_{e_i} \\ R_{\overline{e_i}} & 0 \end{pmatrix} \begin{pmatrix} 0 & R_{e_j} \\ R_{\overline{e_j}} & 0 \end{pmatrix} = \begin{pmatrix} R_{e_i}R_{\overline{e_j}} & 0 \\ 0 & R_{\overline{e_i}}R_{e_j} \end{pmatrix},$$

the one-parameter subgroups are

$$g_{i,j}(t) = \begin{pmatrix} \cos(t) + \sin(t)R_{e_i}R_{\overline{e_j}} & 0 \\ 0 & \cos(t) + \sin(t)R_{\overline{e_i}}R_{e_j} \end{pmatrix} = \begin{pmatrix} R_{e_i} & 0 \\ 0 & R_{\overline{e_i}} \end{pmatrix} \begin{pmatrix} R_{\overline{x_{i,j}(t)}} & 0 \\ 0 & R_{x_{i,j}(t)} \end{pmatrix}$$

where we denoted $x_{i,j}(t) = \cos(t)e_i + \sin(t)e_j \in \mathbb{O}$. Observe that $|e_i| = |x_{i,j}(t)| = 1$. Then $g_{i,j}(t) \in \mathrm{Spin}(9)$ by (2.30). $\square$

Let us give one more equivalent definition of the group $\mathrm{Spin}(9)$, this time by more geometric means. We refer to [20] for details. In analogy to the other normed division algebras, we define the *octonionic projective line* $\mathbb{O}P^1$ to be the set of *octonionic lines*

$$\begin{aligned} \ell_a &= \left\{ \begin{pmatrix} x \\ xa \end{pmatrix} \in \mathbb{O}^2 \, ; \, x \in \mathbb{O} \right\}, \quad a \in \mathbb{O}, \\ \ell_\infty &= \left\{ \begin{pmatrix} 0 \\ x \end{pmatrix} \in \mathbb{O}^2 \, ; \, x \in \mathbb{O} \right\}. \end{aligned} \tag{2.33}$$

$\mathbb{O}P^1$ is an 8-dimensional submanifold of $\mathrm{Gr}_8(\mathbb{O}^2)$, naturally diffeomorphic to S^8. Over this base, the *octonionic Hopf fibration* $S^7 \hookrightarrow S^{15} \to S^8$ is modelled as follows: a point of the total space $S^{15} \subset \mathbb{O}^2$ is projected to the octonionic line it belongs (observe that there is always such a line as well as two distinct lines meet only at the origin). Clearly, the fibre over $\ell \in \mathbb{O}P^1$ then equals $S^{15} \cap \ell = S^7$. Now,

Proposition 2.19. $\mathrm{Spin}(9)$ *maps octonionic lines to octonionic lines.*

Proof. This is easy to verify once we have the generating set (2.30). First, according to the Moufang identity (2.16), for $z \in \mathbb{O}$, $|z| = 1$, we have

$$\begin{pmatrix} R_z & 0 \\ 0 & R_{\overline{z}} \end{pmatrix} \begin{pmatrix} x \\ xa \end{pmatrix} = \begin{pmatrix} xz \\ (xa)\overline{z} \end{pmatrix} = \begin{pmatrix} xz \\ (((xz)\overline{z})a)\overline{z} \end{pmatrix} = \begin{pmatrix} xz \\ (xz)(\overline{z}a\overline{z}) \end{pmatrix}.$$

Second, let us abbreviate $c = \cos(t)$ and $s = \sin(t)$, and observe that

$$\begin{pmatrix} c & s \\ s & -t \end{pmatrix} \begin{pmatrix} x \\ xa \end{pmatrix} = \begin{pmatrix} cx + sxa \\ sx - cxa \end{pmatrix} = \begin{pmatrix} x(c+sa) \\ x(c+sa)(c+sa)^{-1}(s-ca) \end{pmatrix},$$

assuming $c + sa \neq 0$. The remaining cases of ℓ_∞ or $c + sa = 0$ are obvious. $\square$

At the same time, $\mathrm{Spin}(9)$ preserves the total space $S^{15} \subset \mathbb{O}^2$. Hence the group elements may be viewed as symmetries of the octonionic Hopf fibration. In fact,

Theorem 2.20 (Gluck, Warner, Ziller [66])**.** *The group of symmetries of the octonionic Hopf fibration is preciely* $\mathrm{Spin}(9)$*, i.e.*

$$\mathrm{Spin}(9) = \{g \in \mathrm{O}(\mathbb{O}^2) \, ; \, g\ell \in \mathbb{O}P^1 \textit{ for any } \ell \in \mathbb{O}P^1\}. \tag{2.34}$$

2.2.3 The Group $\mathrm{Spin}(8)$

It is clear from the abstract point of view of §2.2.1 that the spin groups are naturally embedded into each other. In this connection it turns out that $\mathbb{O}^2$ is the spin module for $\mathrm{Spin}(8) \subset \mathrm{Spin}(9)$ as well. However, while being irreducible under the latter, it decomposes into two irreducible components under the action of the former. Let us again identify $\mathrm{Spin}(8)$ with its image under the spin representation. See Theorem 14.19 in [79] for consistency with (2.23). Also, it will soon become apparent that the natural inclusion $\mathrm{Spin}(8) \subset \mathrm{Spin}(9)$ is preserved in this picture.

Definition 2.21. We define

$$\mathrm{Spin}(8) = \left\{ \begin{pmatrix} g_+ & 0 \\ 0 & g_- \end{pmatrix} ; g_\pm \in \mathrm{O}(\mathbb{O}), g_+(xy) = g_-(x) g_0(y) \text{ for all } x, y \in \mathbb{O} \right\}, \tag{2.35}$$

where we denote

$$g_0(y) = \overline{g_-(1)}\, g_+(y). \tag{2.36}$$

The *vector representation* ρ_0, *positive spin representation* ρ_+, and *negative spin representation* ρ_- of $\mathrm{Spin}(8)$ are defined as follows:

$$\rho_\sigma : \begin{pmatrix} g_+ & 0 \\ 0 & g_- \end{pmatrix} \mapsto g_\sigma, \quad \sigma = 0, +, -. \tag{2.37}$$

Observe that these irreducible representations are all 8-dimensional, however mutually non-equivalent (see [79], Theorem 14.3). Remarkably, they are still related to each other by means of the so-called *triality principle*. We shall return to this in §2.2.4 below.

Proposition 2.22. *For any* $z \in \mathbb{O}$ *with* $|z| = 1$, $\begin{pmatrix} R_z & 0 \\ 0 & R_{\overline{z}} \end{pmatrix} \in \mathrm{Spin}(8)$.

Proof. Let $x, y, z \in \mathbb{O}$. By (2.16) one has

$$R_{\overline{z}}(x) \left[\overline{R_{\overline{z}}(1)} R_z(y) \right] = (x\overline{z})(zyz) = (((x\overline{z})z)y)z = (xy)z = R_z(xy). \qquad \square$$

As one may observe from (2.32), $\mathrm{span}\{\mathcal{I}_{i,j} ; 0 \le i < j \le 7\}$ is a (28-dimensional) subalgebra of $\mathfrak{spin}(9)$. We have seen in the previous section that the corresponding one-parameter subgroups $g_{i,j}(t)$, $0 \le i < j \le 7$, all lie in the intersection of the generating set (2.30) with $\mathrm{Spin}(8)$ (see Proposition 2.22). Therefore, the subalgebra is in fact $\mathfrak{spin}(8)$ and the one-parameter subgroups generate $\mathrm{Spin}(8)$ which is thus seen to be a subgroup of $\mathrm{Spin}(9)$. Also, the following version of Lemma 2.17 holds:

Lemma 2.23. $\mathrm{Spin}(8)$ *is generated by*

$$\left\{ \begin{pmatrix} R_z & 0 \\ 0 & R_{\overline{z}} \end{pmatrix} ; z \in \mathbb{O}, |z| = 1 \right\}. \tag{2.38}$$

Remark 2.24. It is now easily seen what would require some thought using (2.36) only:

$$\rho_0 : \begin{pmatrix} R_z & 0 \\ 0 & R_{\overline{z}} \end{pmatrix} \mapsto L_z R_z, \tag{2.39}$$

and so it follows from (2.17) that ρ_0 is a representation indeed.

To conclude, recall that $\mathrm{Spin}(9)$ acts transitively also on $\mathbb{O}P^1$. In this connection,

Theorem 2.25. *One has* $\mathrm{Spin}(8) = \mathrm{Stab}_{\ell_0}\mathrm{Spin}(9)$*. Equivalently* $\mathrm{Spin}(9)/\mathrm{Spin}(8) \cong \mathbb{O}P^1$*.*

Proof. Clearly, $\mathrm{Spin}(8) \subset \mathrm{Stab}_{\ell_0}\mathrm{Spin}(9)$ is a closed subgroup. Therefore the projection $\pi : \mathrm{Spin}(9)/\mathrm{Spin}(8) \to \mathbb{O}P^1 : g\,\mathrm{Spin}(8) \mapsto g\ell_0$ is a surjective smooth map of smooth manifolds that moreover commutes with the action of $\mathrm{Spin}(9)$ which is clearly transitive on the source space. According to the Equivariant rank theorem (see [94], Theorem 7.25), π is a submersion. Ehresmann's lemma (see e.g. [42], p. 84) then implies that π is a fibration. Since

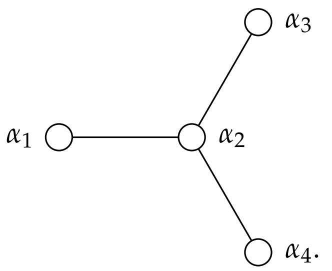
$$\dim \mathrm{Spin}(9) - \dim \mathrm{Spin}(8) = 36 - 28 = 8 = \dim \mathbb{O}P^1,$$

it must be in fact a covering map. Now the claim follows from the fact that $\mathbb{O}P^1 \cong S^8$ is simply connected and π is thus necessarily a diffeomorphism. $\square$

2.2.4 The Triality Principle

Let us now briefly explain one possible view of *triality*, an important phenomenon that is fundamental and unique to the Lie group $\mathrm{Spin}(8)$. For general reference see [20], §2.4, and [65], §20.3.

First of all, associated to $\mathrm{Spin}(8)$ is the following symmetric Dynkin diagram:



In particular, there is a 'rotational' symmetry fixing the root α_2 and sending $\alpha_1, \alpha_3, \alpha_4$ to $\alpha_3, \alpha_4, \alpha_1$, respectively. This transformation induces clearly an automorphism of the corresponding Cartan subalgebra which then extends to an outer automorphism of the whole $\mathfrak{spin}(8)$ (see [65], pp. 338 and 498) and it lifts, finally, to an outer automorphism τ of $\mathrm{Spin}(8)$.

By means of the inverse Cartan matrix, the fundamental weights λ_i for $\mathrm{Spin}(8)$ are

$$\begin{pmatrix} \lambda_1 \\ \lambda_2 \\ \lambda_3 \\ \lambda_4 \end{pmatrix} = \frac{1}{2} \begin{pmatrix} 2 & 2 & 1 & 1 \\ 2 & 4 & 2 & 2 \\ 1 & 2 & 2 & 1 \\ 1 & 2 & 1 & 2 \end{pmatrix} \begin{pmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \\ \alpha_4 \end{pmatrix}.$$

Therefore, if ρ is an irreducible representation with highest weight $\sum_{i=1}^4 k_i\lambda_i$, for some $k_i \in \mathbb{N}_0$, then $k_4\lambda_1 + k_2\lambda_2 + k_1\lambda_3 + k_3\lambda_4$ is the highest weight of the (irreducible) representation $\rho \circ \tau$.

Finally, it is well known that the fundamental weights are the highest weights for the vector representation ρ_0, adjoint representation Ad, and positive and negative spin representations $\rho_\pm$ of $\mathrm{Spin}(8)$, respectively. Thus, in particular, the *triality automorphism* τ rotates ρ_0, ρ_+, ρ_- and fixes Ad in the following sense:

$$\rho_0 \circ \tau \cong \rho_+, \quad \rho_+ \circ \tau \cong \rho_-, \quad \rho_- \circ \tau \cong \rho_0, \quad \text{and} \quad \mathrm{Ad} \circ \tau \cong \mathrm{Ad}. \tag{2.40}$$

This can be made fairly explicit in terms of the octonions - see (14.27) in [79]. Namely, in the language of the previous section, for $(g_0, g_+, g_-) \in \mathrm{Spin}(8)$, meaning that

$$\begin{pmatrix} g_+ & 0 \\ 0 & g_- \end{pmatrix} \in \mathrm{Spin}(8)$$

and g_0 is as in (2.36), one has

$$\tau : (g_0, g_+, g_-) \mapsto (g'_+, g'_-, g_0), \tag{2.41}$$

where for $g \in \mathrm{SO}(\mathbb{O})$ we denote

$$g'(x) = \overline{g(\overline{x})}, \quad x \in \mathbb{O}. \tag{2.42}$$

2.2.5 The Group $\mathrm{Spin}(7)$

As in the two previous cases, we define the group $\mathrm{Spin}(7)$ as the image under its spin representation, this time an 8-dimensional one. This choice at first distorts the natural inclusion $\mathrm{Spin}(7) \subset \mathrm{Spin}(8)$, later on, however, we shall identify a copy of $\mathrm{Spin}(7)$ in the groups discussed above. Lemma 14.61 of [79] justifies the following

Definition 2.26. We define

$$\mathrm{Spin}(7) = \{g \in \mathrm{O}(\mathbb{O})\,;\, g(xy) = \chi_g(x)g(y) \text{ for all } x, y \in \mathbb{O}\}, \tag{2.43}$$

where we denote

$$\chi_g(x) = g\Big(x \cdot g^{-1}(1)\Big). \tag{2.44}$$

Remark 2.27. Strictly speaking, our definition of $\mathrm{Spin}(7)$ differs from the conventions used in Harvey's monograph [79]. Namely, Harvey requires $g(xy) = g(x)\chi_g(y)$ with $\chi_g(y) = g(g^{-1}(1) \cdot y)$ in place of (2.43). However, it is not difficult to see that the two resulting (spin) modules are equivalent via the isomorphism (2.42).

Repeating, essentially, the proof of Proposition 2.22 above, it is easily seen that left multiplication by an imaginary octonion of unit length is an element of $\mathrm{Spin}(7)$. In fact,

Lemma 2.28 ([79], Lemma 14.66)**.** $\mathrm{Spin}(7)$ *is generated by*

$$\{L_u\,;\, u \in \mathrm{Im}\,\mathbb{O}, |u| = 1\}. \tag{2.45}$$

Remark 2.29. In the conventions of [79], one gets R_u in (2.45) instead.

Clearly, $\det(R_1) = \det(L_1) = \det(\mathrm{id}) = 1$. In fact, since $S^7 \subset \mathbb{O}$ is connected, one has $\det(R_x) = \det(L_x) = 1$ for all $x \in \mathbb{O}$ with $|x| = 1$. In particular, $\mathrm{Spin}(7) \subset \mathrm{SO}(\mathbb{O})$. Further, for $u \in \mathrm{Im}\,\mathbb{O}$ with $|u| = 1$,

$$\chi_{L_u} = -L_u R_u \in \mathrm{SO}(\mathbb{O}) \quad \text{and} \quad \chi_{L_u}(1) = 1. \tag{2.46}$$

Therefore, (2.44) defines a $\mathrm{Spin}(7)$-representation on $\mathbb{O}$ that has two irreducible factors: the trivial representation $\mathbb{R}$ and the *vector representation* $\mathrm{Im}\,\mathbb{O}$.

Let us conclude this section by explaining why the group $\mathrm{Spin}(7)$ is relevant to our work at all (see [79], Theorem 14.79):

Theorem 2.30. *One has* $\mathrm{Spin}(9)/\mathrm{Spin}(7) \cong S^{15}$*. More precisely, for* $p = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \in S^{15} \subset \mathbb{O}^2$,

$$\mathrm{Stab}_p\,\mathrm{Spin}(9) = \left\{ \begin{pmatrix} \chi_g & 0 \\ 0 & g \end{pmatrix} ; g \in \mathrm{Spin}(7) \right\}. \tag{2.47}$$

Proof. Let $H \cong \mathrm{Spin}(7)$ be the group on the right-hand side of (2.47). By (2.46), we have $H\{p\} = \{p\}$. It only remains to show $H \subset \mathrm{Spin}(9)$. Indeed, if so, then, repeating the argument of the proof of Theorem 2.25, we obtain that the projection $\mathrm{Spin}(9)/H \to S^{15}$ is actually a diffeomorphism. We prove a stronger statement, namely, that $H \subset \mathrm{Spin}(8)$ in fact. By Lemma 2.28, H is generated by

$$\left\{ \begin{pmatrix} -L_u R_u & 0 \\ 0 & L_u \end{pmatrix} ; u \in \mathrm{Im}\,\mathbb{O}, |u| = 1 \right\}.$$

The proof will be finished once we show that each generator fulfils the condition (2.35). To this end, take $x, y \in \mathbb{O}$ and observe that, by (2.17),

$$L_u(x) \left[\overline{L_u(1)} \big(- L_u R_u(y) \big) \right] = -(ux)(\overline{u}uyu) = -(ux)(yu) = -L_u R_u(xy). \qquad \square$$

Corollary 2.31. *As a decomposition into irreducible* $\mathrm{Spin}(7)$*-modules,*

$$\mathbb{O}^2 \cong \mathbb{R} \oplus \mathrm{Im}\,\mathbb{O} \oplus \mathbb{O}. \tag{2.48}$$

Corollary 2.32. *The isotropy representation of* $\mathrm{Stab}_p \mathrm{Spin}(9) \cong \mathrm{Spin}(7)$ *at* $p = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \in S^{15}$ *decomposes into irreducible modules as*

$$T_p S^{15} \cong \mathrm{Im}\,\mathbb{O} \oplus \mathbb{O}. \tag{2.49}$$

2.3 Invariant Theory of $\mathrm{Spin}(7)$

Later on, Spin(9)-invariant valuations will be studied by means of certain invariant differential forms (see §1.2.3 and §1.3.1). As we shall see, they are in fact determined in a single point and the Spin(9)-invariance thus descends to that under the stabilizer. To this end, the goal of the section that follows is to study invariants of both spin and vector representation of the group Spin(7).

2.3.1 The Cayley Calibration

We begin by introducing an invariant object that is fundamental to the group Spin(7), in that sense that the amount of information it carries is equivalent to knowledge the group itself. We refer to [80], §IV.1.C.

Definition 2.33. Let $w, x, y, z \in \mathbb{O}$. The *triple cross product* is defined as

$$x \times y \times z = \frac{1}{2} \big[(x\overline{y})z - (z\overline{y})x \big]. \tag{2.50}$$

Then we define the *Cayley calibration* as

$$\Phi(w, x, y, z) = \langle w, x \times y \times z \rangle. \tag{2.51}$$

Remark 2.34. In [80], slightly different conventions are used. Namely,

$$\frac{1}{2} \big[x(\overline{y}z) - z(\overline{y}x) \big]$$

appears on the right-hand side of (2.50) instead, modifying the definition (2.51) of Φ accordingly. In fact, this choice is that of the *anti-self-dual* Cayley calibration while in our case Φ is *self-dual* with respect to the standard Hodge star operator (see also [117], Remark 5.29).

An alternative description will be often useful:

Proposition 2.35. *For any $w, x, y, z \in \mathbb{O}$ one has*

$$\Phi(w, x, y, z) = \langle w, (x\overline{y})z\rangle - \langle w, x\rangle\langle y, z\rangle + \langle w, y\rangle\langle x, z\rangle - \langle w, z\rangle\langle x, y\rangle. \tag{2.52}$$

Proof. Observe that an easy consequence of (2.10) and (2.11) is

$$w\overline{z} + z\overline{w} = R_w R_{\overline{z}}(1) + R_z R_{\overline{w}}(1) = 2\langle \overline{w}, \overline{z}\rangle 1 = 2\langle w, z\rangle.$$

Using this and (2.10) in its original version, one gets

$$\begin{aligned} 2x \times y \times z - (x\overline{y})z &= -(z\overline{y})x \\ &= (y\overline{z})x - 2\langle y, z\rangle x \\ &= -(y\overline{x})z - 2\langle y, z\rangle x + 2\langle x, z\rangle y \\ &= (x\overline{y})z - 2\langle y, z\rangle x + 2\langle x, z\rangle y - 2\langle x, y\rangle z, \end{aligned}$$

hence

$$x \times y \times z = (x\overline{y})z - \langle y, z\rangle x + \langle x, z\rangle y - \langle x, y\rangle z,$$

and (2.52) follows by taking the inner product with w. □

Proposition 2.36. *The Cayley calibration is multilinear, alternating, and* Spin(7) *invariant:*

$$\Phi \in \left[\bigwedge^4(\mathbb{O})^*\right]^{\mathrm{Spin}(7)}. \tag{2.53}$$

Proof. Multilinearity is obvious. As for the alternating property, observe at first that the triple cross product is itself alternating:

$$\begin{aligned} x \times y \times x &= 0, \\ x \times y \times y &= \frac{1}{2}\left(x\,|y|^2 - |y|^2\,x\right) = 0, \\ x \times x \times z &= -x \times z \times x = 0. \end{aligned}$$

Then it remains to show $\Phi(x, x, y, z) = \langle x, x \times y \times z\rangle = 0$ where mutual orthogonality of x, y, z may be assumed. To this end, by (2.52),

$$\Phi(x, x, y, z) = \langle x, (x\overline{y})z\rangle = \langle x\overline{z}, x\overline{y}\rangle = |x|^2\,\langle \overline{z}, \overline{y}\rangle = 0.$$

Finally, to prove Spin(7)-invariance, according to Lemma 2.28 and (2.52), it is enough to show the following holds if $u \in \operatorname{Im}\mathbb{O}$, $|u| = 1$:

$$\begin{aligned} \langle uw, [(ux)(\overline{uy})](uz)\rangle &= -\langle uw, [u(x\overline{y})u](uz)\rangle \\ &= -\langle L_u(w), L_u L_{x\overline{y}} L_u L_u(z)\rangle \\ &= \langle L_u(w), L_u L_{x\overline{y}}(z)\rangle \\ &= \langle w, (x\overline{y})z\rangle, \end{aligned}$$

the first two equalities following (2.17) and (2.15), respectively. □

In fact, as anticipated above and shown e.g. in [117], §9,

Theorem 2.37.

$$\mathrm{Spin}(7) = \{g \in \mathrm{GL}(\mathbb{O})\,;\, g^*\Phi = \Phi\}. \tag{2.54}$$

Remark 2.38. In the sense of Proposition 2.36 and Theorem 2.37, the version of Spin(7) discussed in Remark 2.27 is compatible with the anti-self-dual Cayley calibration.

2.3.2 Two Classical First Fundamental Theorems

After we introduced a particular yet important instance, let us now proceed to more systematic study of Spin(7)-invariants. We shall adopt the language of the standard references [92] and [112] (in the latter, see in particular §3.2, §9.1, and §11). It is again assumed throughout that V is a finite-dimensional real vector space.

Let $\mathbb{R}[V]$ be the ring of polynomials on V, i.e. functions $V \to \mathbb{R}$ that are polynomial in coordinates with respect to a basis of V. Observe that the notion is independent of a particular choice of basis. Let G be a group. If V is also a G-module, we denote

$$\mathbb{R}[V]^G = \{p \in \mathbb{R}[V]\,;\, g^*p = p \text{ for any } g \in G\}, \tag{2.55}$$

the subring of *G-invariants*. Naturally, $\mathbb{R}[V]^G = \bigoplus_{d\geq 0} \mathbb{R}[V]^G_d$ is graded by the degree of homogeneity. Further, via the common procedures of polarization and restitution, respectively, knowledge of $\bigoplus_{d\geq 0} \mathbb{R}[V]^G_d$ is equivalent to that of $\mathbb{R}[V^d]^G_{\mathrm{multi}}$, the space of multilinear G-invariant polynomials on $V^d = V \oplus \cdots \oplus V$ (d-times). More generally, if $V = V_1 \oplus \cdots \oplus V_n$ is a sum of submodules, $\mathbb{R}[V]^G$ determines and is determined by $\mathbb{R}[V_1^{d_1} \oplus \cdots \oplus V_n^{d_n}]^G_{\mathrm{multi}}$, $d_i \geq 0$.

According to the terminology of H. Weyl [141], a result describing a set of generating elements for either $\mathbb{R}[V]^G$ or its multilinear equivalents is usually referred to as the *First fundamental theorem* (FFT). The *Second fundamental theorem* (SFT) then specifies which relations the generators satisfy among each other.

Let us turn our attention to the case $G = \mathrm{Spin}(7)$. First, consider the spin module $V = \mathbb{O}$. As $\mathrm{Spin}(7) \subset \mathrm{SO}(8)$, the inner product on $\mathbb{O}$ is clearly a Spin(7)-invariant. Another invariant we have encountered is the Cayley calibration that was studied in the previous section. Remarkably, there are no others:

Theorem 2.39 (Schwarz [127])**.** *Let $m \geq 0$. $\mathbb{R}[\mathbb{O}^m]^{\mathrm{Spin}(7)}_{\mathrm{multi}}$ is spanned by products of*

$$\begin{aligned} &\langle x_{k_1}, x_{k_2}\rangle, && 1 \leq k_1 < k_2 \leq m, \\ &\Phi(x_{k_1}, x_{k_2}, x_{k_3}, x_{k_4}), && 1 \leq k_1 < \cdots < k_4 \leq m. \end{aligned}$$

Second, consider the vector module $V = \mathrm{Im}\,\mathbb{O}$. Since Spin(7) acts here just like $\mathrm{SO}(\mathrm{Im}\,\mathbb{O}) = \mathrm{SO}(7)$, the FFT is classical in this case (see e.g. [112], §11.2.1):

Theorem 2.40. *Let $l \geq 0$. $\mathbb{R}\big[(\mathrm{Im}\,\mathbb{O})^l\big]^{\mathrm{Spin}(7)}_{\mathrm{multi}} = \mathbb{R}\big[(\mathrm{Im}\,\mathbb{O})^l\big]^{\mathrm{SO}(7)}_{\mathrm{multi}}$ is spanned by products of*

$$\begin{aligned} &\langle u_{j_1}, u_{j_2}\rangle, && 1 \leq j_1 < j_2 \leq l, \\ &\det(u_{j_1}, \ldots, u_{j_7}), && 1 \leq j_1 < \cdots < j_7 \leq l, \end{aligned}$$

Remark 2.41. Let us normalize the determinant such that

$$\det(e_1, \ldots, e_7) = 1 \tag{2.56}$$

holds for the standard basis of $\mathrm{Im}\,\mathbb{O}$ introduced in §2.1.3.

2.3.3 The First Fundamental Theorem for the Isotropy Representation

For our purpose, however, the two FFTs alone are not completely sufficient. What we shall need is their generalization (one may also say interpolation), namely the FFT for the Spin(7)-module $\mathrm{Im}\,\mathbb{O} \oplus \mathbb{O}$. Later on, this fact will be explained carefully, for now, the decompositions (2.48) and (2.49) may perhaps serve as a rough argument. For we are not aware that such a result has appeared in the literature, we shall prove the FFT here. The following two simple but important observations are in the background of our construction:

Proposition 2.42. *The linear mapping* $\operatorname{Im}\mathbb{O} \to \operatorname{End}(\mathbb{O}) : u \mapsto L_u$ *is* $\mathrm{Spin}(7)$ *equivariant.*

Proof. This follows at once from Definition 2.26. In fact, let $u \in \operatorname{Im}\mathbb{O}$ and $g \in \mathrm{Spin}(7)$. Then for any $x \in \mathbb{O}$ one has

$$L_{\chi_g(u)}(x) = \chi_g(u) \cdot x = \chi_g(u) \cdot g(g^{-1}(x)) = g\left(u \cdot g^{-1}(x)\right) = g \circ L_u \circ g^{-1}(x),$$

and thus $L_{\chi_g(u)} = g \circ L_u \circ g^{-1}$. □

Proposition 2.43. *The linear map* $\Pi : \mathbb{O} \otimes \mathbb{O} \to \operatorname{Im}\mathbb{O}$ *given by*

$$\Pi(x \otimes y) = x\overline{y} - y\overline{x} \tag{2.57}$$

is $\mathrm{Spin}(7)$ *equivariant.*

Proof. According to Lemma 2.28, it is enough to show equivariance under L_u, $u \in \operatorname{Im}\mathbb{O}$, $|u| = 1$. Using the Moufang identity (2.17), this is straightforward:

$$\Pi(ux \otimes uy) = (ux)\overline{(uy)} - (uy)\overline{(ux)} = -u(x\overline{y} - y\overline{x})u = -u\left[\Pi(x \otimes y)\right]u.$$

Thus, according to (2.46), $\Pi \circ (L_u \otimes L_u) = \chi_{L_u} \circ \Pi$, as desired. □

Remark 2.44. Together with the isomorphisms $\operatorname{End}(\mathbb{O}) \cong \mathbb{O} \otimes \mathbb{O}^* \cong \mathbb{O} \otimes \mathbb{O}$, the former being canonical, the latter induced by the (standard) inner product, the previous two propositions describe a $\mathrm{Spin}(7)$-equivariant embedding of the vector module $\operatorname{Im}\mathbb{O}$ into a tensor power of the (defining) spin module $\mathbb{O}$. This is the initial step of the general strategy towards invariant theory of a general representation outlined in §6.8 of [112].

For brevity, let us denote

$$P_{l,m} = \mathbb{R}\left[(\operatorname{Im}\mathbb{O})^l \oplus \mathbb{O}^m\right]^{\mathrm{Spin}(7)}_{\mathrm{multi}} \tag{2.58}$$

for the rest of this section. Also, we hope it may increase readability that we adhere to the following rule: the letter u will always refer (within this section) to an element of the first, while x to an element of the second factor of $\operatorname{Im}\mathbb{O} \oplus \mathbb{O}$.

Lemma 2.45. *The map* $\mathcal{F}_{l,m} : P_{l-1,m+2} \to P_{l,m}$ *given by*

$$(\mathcal{F}_{l,m}p)(u_1, \ldots, u_l, x_1, \ldots, x_m) = \sum_{i=0}^{7} p(u_1, \ldots, u_{l-1}, u_l e_i, e_i, x_1, \ldots, x_m), \tag{2.59}$$

for some positively oriented orthonormal basis $e_0, \ldots, e_7$ *of* $\mathbb{O}$*, is well defined, linear, and onto.*

Proof. First, let $f_0, \ldots, f_7$ be another positively oriented basis of $\mathbb{O}$, i.e. $f_i = \sum_{j=0}^{7} A^{i,j} e_j$ for some $(A^{i,j})_{i,j=0}^{7} \in \mathrm{SO}(8)$. Then

$$\begin{aligned}
\sum_{i=0}^{7} p(u_1, \ldots, u_{l-1}, u_l f_i, f_i, x_1, \ldots, x_m) &= \sum_{i,j,k=0}^{7} A^{i,j} A^{i,k} p(u_1, \ldots, u_{l-1}, u_l e_j, e_k, x_1, \ldots, x_m) \\
&= \sum_{j,k=0}^{7} (A^T A)^{j,k} p(u_1, \ldots, u_{l-1}, u_l e_j, e_k, x_1, \ldots, x_m) \\
&= \sum_{j,k=0}^{7} \delta_{j,k} p(u_1, \ldots, u_{l-1}, u_l e_j, e_k, x_1, \ldots, x_m)
\end{aligned}$$

$$= \sum_{j=0}^{7} p(u_1, \dots, u_{l-1}, u_l e_j, e_j, x_1, \dots, x_m)$$

and so (2.59) is independent of the choice of basis. Second, $\mathcal{F}_{l,m}p$ is clearly multilinear. Third, let us show that it is Spin(7) invariant. For any $g \in \text{Spin}(7) \subset \text{SO}(8)$ we have

$$\begin{aligned}
&(\mathcal{F}_{l,m}p)\left(\chi_g(u_1), \dots, \chi_g(u_l), g(x_1), \dots, g(x_m)\right) \\
&\quad = \sum_{i=0}^{7} p\left(\chi_g(u_1), \dots, \chi_g(u_{l-1}), \chi_g(u_l) g(g^{-1}(e_i)), g(g^{-1}(e_i)), g(x_1), \dots, g(x_m)\right) \\
&\quad = \sum_{i=0}^{7} p\left(\chi_g(u_1), \dots, \chi_g(u_{l-1}), g(u_l g^{-1}(e_i)), g(g^{-1}(e_i)), g(x_1), \dots, g(x_m)\right) \\
&\quad = \sum_{i=0}^{7} p\left(u_1, \dots, u_{l-1}, u_l g^{-1}(e_i), g^{-1}(e_i), x_1, \dots, x_m\right) \\
&\quad = (\mathcal{F}_{l,m}p)(u_1, \dots, u_l, x_1, \dots, x_m),
\end{aligned}$$

since $g^{-1}(e_0), \dots, g^{-1}(e_7)$ is a positively oriented orthonormal basis of $\mathbb{O}$. Altogether, we showed that $\mathcal{F}_{l,m}$ is a well-defined mapping.

Linearity of $\mathcal{F}_{l,m}$ is obvious, so let us, finally, show $\mathcal{F}_{l,m}$ is onto. For $q \in P_{l,m}$ we put

$$p(u_1, \dots, u_{l-1}, y, z, x_1, \dots, x_m) = \frac{1}{16} q(u_1, \dots, u_{l-1}, y\bar{z} - z\bar{y}, x_1, \dots, x_m),$$

$y, z \in \mathbb{O}$. Clearly, $p \in P_{l-1,m+2}$ and since

$$\begin{aligned}
(\mathcal{F}_{l,m}p)(u_1, \dots, u_l, x_1, \dots, x_m) &= \frac{1}{16} \sum_{i=0}^{7} q\left(u_1, \dots, u_{l-1}, (u_l e_i)\overline{e_i} - e_i \overline{(u_l e_i)}, x_1, \dots, x_m\right) \\
&= \frac{1}{16} \sum_{i=0}^{7} q(u_1, \dots, u_{l-1}, 2u_l, x_1, \dots, x_m) \\
&= \frac{1}{8} \sum_{i=0}^{7} q(u_1, \dots, u_{l-1}, u_l, x_1, \dots, x_m) \\
&= q(u_1, \dots, u_l, x_1, \dots, x_m),
\end{aligned}$$

we in fact have $\mathcal{F}_{l,m}p = q$. □

By induction, one immediately arrives at the following

Corollary 2.46. *The linear map $\mathcal{G}_{l,m} : P_{0,m+2l} \to P_{l,m}$ given by*

$$\mathcal{G}_{l,m} = \mathcal{F}_{l,m} \circ \mathcal{F}_{l-1,m+2} \circ \dots \circ \mathcal{F}_{1,m+2l-2} \tag{2.60}$$

is onto. Explicitly,

$$(\mathcal{G}_{l,m}p)(u_1, \dots, u_l, x_1, \dots, x_m) = \sum_{i_1,\dots,i_l=0}^{7} p(u_1 e_{i_1}, e_{i_1}, \dots, u_l e_{i_l}, e_{i_l}, x_1, \dots, x_m). \tag{2.61}$$

We can finally proceed to the statement and proof of the First fundamental theorem:

Theorem 2.47. *$P_{l,m}$ is spanned by products of appropriate numbers of the following functions:*

$$\begin{array}{ll}
\langle u_{j_1}, u_{j_2}\rangle, & 1 \le j_1 < j_2 \le l, \\
\det(u_{j_1}, \ldots, u_{j_7}), & 1 \le j_1 < \cdots < j_7 \le l, \\
\langle L_{u_{j_1}} \cdots L_{u_{j_r}}(x_{k_1}), x_{k_2}\rangle, & 0 \le r \le 7, \quad 1 \le j_1 < \cdots < j_r \le l, \quad 1 \le k_1 < k_2 \le m, \\
\Phi(x_{k_1}, x_{k_2}, x_{k_3}, x_{k_4}), & 1 \le k_1 < \cdots < k_4 \le m, \\
\Phi(u_j x_{k_1}, x_{k_2}, x_{k_3}, x_{k_4}), & 1 \le k_1 < \cdots < k_4 \le m, \quad 1 \le j \le l,
\end{array}$$

where the usual convention $u_i \in \operatorname{Im} \mathbb{O}$ and $x_i \in \mathbb{O}$ is employed, such that each of the variables $u_1, \ldots, u_l, x_1, \ldots, x_m$ occurs exactly once.

Remark 2.48. It is natural to ask about the relations among these generators, in other words, for the corresponding Second fundamental theorem. In this connection, let us recall that while the classical SFT for the vector Spin(7)-module is relatively simple (see [141], §II.17), things are much more complicated in the case of the spin representation as studied by Schwarz [127]. Either of these invariant theories is included in the invariant theory for the isotropy representation we are interested in and it is therefore logical to expect serious difficulties in obtaining an analogous result in our case. No attempt in this direction has, however, been made.

Proof. Let $p \in P_{0,m+2l}$ be a product of

$$\begin{array}{ll}
\langle x_{k_1}, x_{k_2}\rangle, & 1 \le k_1 < k_2 \le m + 2l, \\
\Phi(x_{k_1}, x_{k_2}, x_{k_3}, x_{k_4}), & 1 \le k_1 < \cdots < k_4 \le m + 2l.
\end{array}$$

According to Theorem 2.39 and Corollary 2.46, $\mathcal{G}_{l,m} p \in P_{l,m}$ and furthermore, the latter space is spanned by elements of this type. It is evident from its definition that the map $\mathcal{G}_{l,m}$ merges the factors of p into (possibly branched) 'chains' by plugging ue_i in some factor, e_i in another, and summing over i (see (2.59)). Observe that the number of x's in any such chain is necessarily even. Let us investigate which chains in general occur in $\mathcal{G}_{l,m} p$. We shall distinguish four cases. All summations are taken from 0 to 7 if not stated otherwise for the rest of the proof.

(a) First, there are chains containing no x, i.e. elements of $P_{r,0}$, for $r \ge 1$. According to Theorem 2.40, these must be polynomials in inner products and determinants on $\operatorname{Im} \mathbb{O}$.

(b) Second, there are chains without Φ that contain two x's:

$$\sum_{i_1,\ldots,i_r} \langle x_{k_1}, e_{i_r}\rangle \langle u_{j_r} e_{i_r}, e_{i_{r-1}}\rangle \cdots \langle u_{j_2} e_{i_2}, e_{i_1}\rangle \langle u_{j_1} e_{i_1}, x_{k_2}\rangle, \quad r \ge 0, \tag{2.62}$$

where for $r = 0$ we have $\langle x_{k_1}, x_{k_2}\rangle$. Using the decomposition $a = \sum_i \langle a, e_i\rangle e_i$ into an orthonormal basis of $\mathbb{O}$, (2.62) can be rewritten as follows:

$$\begin{aligned}
&\sum_{i_1,\ldots,i_r} \langle x_{k_1}, e_{i_r}\rangle \langle u_{j_r} e_{i_r}, e_{i_{r-1}}\rangle \cdots \langle u_{j_2} e_{i_2}, e_{i_1}\rangle \langle u_{j_1} e_{i_1}, x_{k_2}\rangle \\
&\quad = \sum_{i_1,\ldots,i_r} \langle u_{j_r} \langle x_{k_1}, e_{i_r}\rangle e_{i_r}, e_{i_{r-1}}\rangle \cdots \langle u_{j_2} e_{i_2}, e_{i_1}\rangle \langle u_{j_1} e_{i_1}, x_{k_2}\rangle \\
&\quad = \sum_{i_1,\ldots,i_{r-1}} \langle u_{j_r} x_{k_1}, e_{i_{r-1}}\rangle \langle u_{j_{r-1}} e_{i_{r-1}}, e_{i_{r-2}}\rangle \cdots \langle u_{j_2} e_{i_2}, e_{i_1}\rangle \langle u_{j_1} e_{i_1}, x_{k_2}\rangle \\
&\quad = \sum_{i_1,\ldots,i_{r-1}} \langle u_{j_{r-1}} \langle u_{j_r} x_{k_1}, e_{i_{r-1}}\rangle e_{i_{r-1}}, e_{i_{r-2}}\rangle \cdots \langle u_{j_2} e_{i_2}, e_{i_1}\rangle \langle u_{j_1} e_{i_1}, x_{k_2}\rangle
\end{aligned}$$

$$= \sum_{i_1,\dots,i_{r-2}} \langle u_{j_{r-1}}(u_{j_r} x_{k_1}), e_{i_{r-2}} \rangle \langle u_{j_{r-2}} e_{i_{r-2}}, e_{i_{r-3}} \rangle \cdots \langle u_{j_2} e_{i_2}, e_{i_1} \rangle \langle u_{j_1} e_{i_1}, x_{k_2} \rangle$$

$$\vdots$$

$$= \langle L_{u_{j_1}} \cdots L_{u_{j_r}}(x_{k_1}), x_{k_2} \rangle. \tag{2.63}$$

Let us explain why there is no loss of generality in assuming $r \leq 7$. Suppose $r = 8$. Since $\dim \operatorname{Im} \mathbb{O} = 7$, the following skew-symmetrization must be identically trivial:

$$\sum_{\pi \in \mathcal{S}_8} \operatorname{sgn}(\pi)\, L_{u_{j_{\pi(1)}}} \cdots L_{u_{j_{\pi(8)}}} = 0. \tag{2.64}$$

By (2.10) we have $L_u L_v + L_v L_u = -2\langle u, v \rangle \,\mathrm{id}$ for $u, v \in \operatorname{Im} \mathbb{O}$ and thus, since the parity of a permutation equals the parity of the number of transpositions it consists of, for any $\pi \in \mathcal{S}_8$, one has

$$\operatorname{sgn}(\pi)\, L_{u_{j_{\pi(1)}}} \cdots L_{u_{j_{\pi(8)}}} = L_{u_{j_1}} \cdots L_{u_{j_8}} + p_\pi(u_{j_1}, \dots, u_{j_8}), \tag{2.65}$$

where p_π is some $\operatorname{End}(\mathbb{O})$-valued polynomial in inner products and compositions of at most seven (at most six, in fact) left multiplications. Then, summing (2.65) over all permutations and making use of (2.64) gives us

$$0 = \sum_{\pi \in \mathcal{S}_8} \operatorname{sgn}(\pi)\, L_{u_{j_{\pi(1)}}} \cdots L_{u_{j_{\pi(8)}}} = 8!\, L_{u_{j_1}} \cdots L_{u_{j_8}} + \sum_{\pi \in \mathcal{S}_8} \operatorname{sgn}(\pi)\, p_\pi(u_{j_1}, \dots, u_{j_8}),$$

i.e. $L_{u_{j_1}} \cdots L_{u_{j_8}}$ expressed in terms of inner products and at most seven Ls. By induction, the same statement extends to all $r \geq 8$. Similarly we may assume that $j_1 < \cdots < j_r$ and $k_1 < k_2$ in (2.63). Notice also that any chain without Φ that contains more than two x's decomposes into the chains that we have already encountered.

(c) Due to higher complexity of the other cases, let us formalize the reductive method we used in part (b). Namely, let us introduce an equivalence relation on $P_{s,r}$ as follows: we put $p \sim q$ if $p - q$ is expressible in the invariants considered in parts (a) and (b). For instance, (2.52) implies

$$\Phi(x_{k_1}, x_{k_2}, x_{k_3}, x_{k_4}) \sim \langle x_{k_1}, (x_{k_2} \overline{x_{k_3}}) x_{k_4} \rangle. \tag{2.66}$$

Similarly, the following consequence of (2.66) and (2.10) holds:

$$\begin{aligned} \Phi(x_{k_1}, u_j x_{k_2}, x_{k_3}, x_{k_4}) &\sim \langle x_{k_1}, ((u_j x_{k_2}) \overline{x_{k_3}}) x_{k_4} \rangle \\ &\sim -\langle x_{k_1}, ((u_j x_{k_3}) \overline{x_{k_2}}) x_{k_4} \rangle \\ &\sim -\Phi(x_{k_1}, u_j x_{k_3}, x_{k_2}, x_{k_4}) \\ &= \Phi(x_{k_1}, x_{k_2}, u_j x_{k_3}, x_{k_4}), \end{aligned} \tag{2.67}$$

and extends to other pairs of entries by skew-symmetry.

Consider now a chain in $\mathcal{G}_{l,m} p$ with precisely one Cayley calibration and two x's. In this case, however, we have, for some $r \geq 1$,

$$\begin{aligned} &\sum_{i_1,\dots,i_r} \langle u_{j_r} e_{i_r}, e_{i_{r-1}} \rangle \cdots \langle u_{j_2} e_{i_2}, e_{i_1} \rangle \Phi(u_{j_1} e_{i_1}, e_{i_r}, x_{k_1}, x_{k_2}) \\ &\quad = \sum_i \Phi(L_{u_{j_1}} \cdots L_{u_{j_r}}(e_i), e_i, x_{k_1}, x_{k_2}) \\ &\quad \sim \sum_i \Phi(e_i, e_i, L_{u_{j_r}} \cdots L_{u_{j_r}}(x_{k_1}), x_{k_2}) \\ &\quad = 0, \end{aligned}$$

where the same procedure as to obtain (2.63), and the relation (2.67) were used, respectively. As for one Cayley calibration and four x's, one possibility is

$$\sum_{i_1,\dots,i_r} \langle x_{k_1}, e_{i_r}\rangle \langle u_{j_r} e_{i_r}, e_{i_{r-1}}\rangle \cdots \langle u_{j_2} e_{i_2}, e_{i_1}\rangle \Phi(u_{j_1} e_{i_1}, x_{k_2}, x_{k_3}, x_{k_4})$$
$$= \Phi\left(L_{u_{j_1}} \cdots L_{u_{j_r}}(x_{k_1}), x_{k_2}, x_{k_3}, x_{k_4}\right),$$

for $r \geq 0$. In general we could get, through the same mechanism, left multiplications in other entries of Φ as well but this is redundant because of (2.67). Furthermore, if $r = 2$,

$$\begin{aligned}
&2\Phi\left(x_{k_1}, u_{j_1}(u_{j_2}x_{k_2}), x_{k_3}, x_{k_4}\right) \\
&\quad \sim \Phi\left(x_{k_1}, u_{j_1}(u_{j_2}x_{k_2}), x_{k_3}, x_{k_4}\right) + \Phi\left(x_{k_1}, u_{j_2}(u_{j_1}x_{k_2}), x_{k_3}, x_{k_4}\right) \\
&\quad = \Phi\left(x_{k_1}, u_{j_1}(u_{j_2}x_{k_2}) + u_{j_2}(u_{j_1}x_{k_2}) + [u_{j_1}, u_{j_2}, x_{k_2}] + [u_{j_2}, u_{j_1}, x_{k_2}], x_{k_3}, x_{k_4}\right) \\
&\quad = \Phi\left(x_{k_1}, (u_{j_1}u_{j_2})x_{k_2} + (u_{j_2}u_{j_1})x_{k_2}, x_{k_3}, x_{k_4}\right) \\
&\quad = -2\langle u_{j_1}, u_{j_2}\rangle \Phi(x_{k_1}, x_{k_2}, x_{k_3}, x_{k_4}) \\
&\quad \sim 0.
\end{aligned}$$

By induction this easily extends also to $r > 2$ and we may thus assume $r \leq 1$.
(d) Finally, let us redefine the equivalence relation introduced above by including also the two new kinds of invariants obtained in part (c) of the proof. Then, observe that $\Phi(x_{k_1} \times x_{k_2} \times x_{k_3}, x_{k_4}, x_{k_5}, x_{k_6}) \in P_{0,6}$ and so, by Theorem 2.39,

$$\Phi(x_{k_1} \times x_{k_2} \times x_{k_3}, x_{k_4}, x_{k_5}, x_{k_6}) \sim 0.$$

Using this, consider the following chain, admissible in $\mathcal{G}_{l,m}p$, containing two Φ's:

$$\begin{aligned}
&\sum_i \Phi(u_j e_i, x_{k_1}, x_{k_2}, x_{k_3}) \Phi(e_i, x_{k_4}, x_{k_5}, x_{k_6}) \\
&\quad = \sum_i \langle u_j e_i, x_{k_1} \times x_{k_2} \times x_{k_3}\rangle \langle e_i, x_{k_4} \times x_{k_5} \times x_{k_6}\rangle \\
&\quad = -\sum_i \langle e_i, u_j(x_{k_1} \times x_{k_2} \times x_{k_3})\rangle \langle e_i, x_{k_4} \times x_{k_5} \times x_{k_6}\rangle \\
&\quad = -\langle u_j(x_{k_1} \times x_{k_2} \times x_{k_3}), x_{k_4} \times x_{k_5} \times x_{k_6}\rangle \\
&\quad = -\Phi(u_j(x_{k_1} \times x_{k_2} \times x_{k_3}), x_{k_4}, x_{k_5}, x_{k_6}) \\
&\quad \sim -\Phi(x_{k_1} \times x_{k_2} \times x_{k_3}, u_j x_{k_4}, x_{k_5}, x_{k_6}) \\
&\quad \sim 0.
\end{aligned}$$

Now it is easily seen by induction that any chain containing more that one Φ is in fact expressible in terms of the invariants introduced above. This completes the proof. □

2.4 Moving $\mathrm{Spin}(9)$-Frames

2.4.1 La Méthode de Repère Mobil

Later on we shall take advantage of differentiating invariant differential forms via the so-called *Method of moving frames* which is one of the numerous ingenious techniques invented by Élie Cartan. Let us now recall this concept, not in its full generality of the so-called *Cartan geometries*, but rather from the perspective of Klein's *Erlangen program* where a 'geometry' is regarded as a homogeneous space of its symmetry group by the stabilizer of a point. §3 of the monograph [129] will serve us as a reference.

Let G be a Lie group and $\mathfrak{g}$ its Lie algebra. Consider the standard actions L_g, R_g, and $\mathrm{Ad}_g = L_g R_{g^{-1}}$ of G on itself. The key notion here is that of the so-called *Maurer-Cartan form*. This is the canonical $\mathfrak{g}$-valued differential 1-form on G defined as follows:

$$\omega_g(X_g) = (L_{g^{-1}})_*(X_g), \quad g \in G, \quad X \in \mathfrak{X}(G). \tag{2.68}$$

It is easily shown that ω is left G invariant, i.e.

$$L_g^* \omega = \omega, \quad g \in G, \tag{2.69}$$

is right G contravariant in the following sense:

$$R_g^* \omega = (\mathrm{Ad}_{g^{-1}})_* \, \omega = (L_{g^{-1}})_* (R_g)_* \, \omega, \quad g \in G, \tag{2.70}$$

and satisfies the important *Maurer-Cartan equation*

$$\mathrm{d}\omega(X, Y) = -[\omega(X), \omega(Y)], \quad X, Y \in \mathfrak{X}(G). \tag{2.71}$$

In the special case when G is a subgroup of $\mathrm{GL}(m, \mathbb{R})$ and $\mathfrak{g}$ is the corresponding Lie subalgebra of $\mathfrak{gl}(m, \mathbb{R})$, we can write

$$\omega_g = g^{-1} \mathrm{d}g. \tag{2.72}$$

(2.71) then becomes, entrywise,

$$\mathrm{d}\omega_{ij}(X, Y) = -\sum_{k=1}^{m} [\omega_{ik}(X)\omega_{kj}(Y) - \omega_{ik}(Y)\omega_{kj}(X)] = -\sum_{k=1}^{m} \omega_{ik} \wedge \omega_{kj}(X, Y) \tag{2.73}$$

which is often abbreviated as

$$\mathrm{d}\omega = -\omega \wedge \omega. \tag{2.74}$$

Moreover, (2.70) can be rewritten in terms of matrix multiplication as follows:

$$R_g^* \omega = g^{-1} \omega g, \quad g \in G. \tag{2.75}$$

Let us point out that a generic matrix in $\mathfrak{g} \subset \mathfrak{gl}(m, \mathbb{R})$ has $\dim \mathfrak{g}$ independent entries while the other $m^2 - \dim \mathfrak{g}$ are uniquely determined as linear combinations of them. Viewed in this light, the Maurer-Cartan form may be regarded as a collection of $\dim \mathfrak{g}$ left-invariant differential 1-forms on G whose differentials are expressed back in these forms themselves, in accordance to the Maurer-Cartan equation. This is the decisive feature of Cartan's approach turning the Maurer-Cartan form into a powerful tool for studying various geometric aspects of manifolds of the form $M \cong G/H$, where $H \subset G$ is a closed subgroup.

Remark 2.49. In the special case when M is a finite-dimensional vector space and $H \subset \mathrm{GL}(M)$, the group of symmetries $G = H \ltimes M$ can be identified with the subbundle of the frame bundle over M corresponding to the reduction of the structure group to H. So much for the label 'moving frames'.

2.4.2 Invariant Differential Forms on a Homogeneous Space

Let us recall the standard procedure by virtue of which invariant differential forms are constructed on a homogeneous space. For reference see [137], §3, and [94], §21.

Let G be a Lie group with a transitive action on a smooth manifold M. Fix a point $p \in M$ and denote $H = \mathrm{Stab}_p\, G$. Then $M \cong G/H$. Consider further the corresponding projection $\pi : G \to M : g \mapsto g(p)$. Recall that π is a smooth submersion. Clearly, it intertwines the actions of G and H on G and M as follows:

$$\pi L_g = g\pi, \quad g \in G, \tag{2.76}$$
$$\pi \operatorname{Ad}_h = h\pi, \quad h \in H. \tag{2.77}$$

The corresponding pullback and pushforward maps commute accordingly.

Lemma 2.50. *Let $\beta \in \Omega^k(M)^G$. Then the form $\tilde{\beta} = \pi^*\beta \in \Omega^k(G)$ is*

(a) *left G invariant:*

$$(L_g)^*\tilde{\beta} = \tilde{\beta}, \quad g \in G, \tag{2.78}$$

(b) *right H invariant:*

$$(R_h)^*\tilde{\beta} = \tilde{\beta}, \quad h \in H, \tag{2.79}$$

(c) *horizontal:*

$$X \lrcorner \tilde{\beta} = 0, \quad X \in \ker \pi_* \subset \mathfrak{X}(G). \tag{2.80}$$

Conversely, assume $\tilde{\beta} \in \Omega^k(G)$ satisfies (2.78) – (2.80)*. Then there is a unique $\beta \in \Omega^k(M)^G$ with $\tilde{\beta} = \pi^*\beta$.*

Proof. First, take an arbitrary $\beta \in \Omega^k(M)^G$ and put $\tilde{\beta} = \pi^*\beta$. Obviously, $\tilde{\beta} \in \Omega^k(G)$. To show (2.78), according to (2.76) and invariance of β, for any $g \in G$ one has

$$L_g^*\tilde{\beta} = L_g^*\pi^*\beta = \pi^*g^*\beta = \pi^*\beta = \tilde{\beta}.$$

Similarly, to show (2.79), using (2.78), (2.77) and invariance of β, for any $h \in H$,

$$R_{h^{-1}}^*\tilde{\beta} = R_{h^{-1}}^*L_h^*\tilde{\beta} = R_{h^{-1}}^*L_h^*\pi^*\beta = \operatorname{Ad}_h^*\pi^*\beta = \pi^*h^*\beta = \pi^*\beta = \tilde{\beta}.$$

Finally, to see (2.80), take $X \in \ker \pi_*$. Then, for any $X^{(2)}, \ldots, X^{(k)} \in \mathfrak{X}(G)$,

$$\begin{aligned}
(X \lrcorner \tilde{\beta})\left(X^{(2)}, \ldots, X^{(k)}\right) &= \tilde{\beta}\left(X, X^{(2)}, \ldots, X^{(k)}\right) \\
&= (\pi^*\beta)\left(X, X^{(2)}, \ldots, X^{(k)}\right) \\
&= \beta\left(\pi_*X, \pi_*X^{(2)}, \ldots, \pi_*X^{(k)}\right) \\
&= 0.
\end{aligned}$$

Conversely, consider $\tilde{\beta} \in \Omega^k(G)$ with (2.78) – (2.80). Let us construct $\beta_p \in \bigwedge^k(T_pM)^*$ as follows: For any $Y_p^{(1)}, \ldots, Y_p^{(k)} \in T_pM$ we put

$$\beta_p\left(Y_p^{(1)}, \ldots, Y_p^{(k)}\right) = \tilde{\beta}_e\left(X_e^{(1)}, \ldots, X_e^{(k)}\right),$$

where $X_e^{(j)} \in T_eG$ are such that $\pi_*X_e^{(j)} = Y_p^{(j)}$. Notice that there is no ambiguity in the definition thanks to (2.80). Thus, we have $\pi^*\beta_p = \tilde{\beta}_e$. Observe also that for any $h \in H$,

$$\begin{aligned}
(h^*\beta_p)\left(Y_p^{(1)}, \ldots, Y_p^{(k)}\right) &= \beta_p\left(h_*\pi_*X_p^{(1)}, \ldots, h_*\pi_*X_p^{(k)}\right) \\
&= \beta_p\left(\pi_*(\operatorname{Ad}_h)_*X_p^{(1)}, \ldots, \pi_*(\operatorname{Ad}_h)_*X_p^{(k)}\right) \\
&= \tilde{\beta}_e\left((\operatorname{Ad}_h)_*X_p^{(1)}, \ldots, (\operatorname{Ad}_h)_*X_p^{(k)}\right) \\
&= \operatorname{Ad}_h^*\tilde{\beta}_e\left(X_p^{(1)}, \ldots, X_p^{(k)}\right) \\
&= \tilde{\beta}_e\left(X_p^{(1)}, \ldots, X_p^{(k)}\right) \\
&= \beta_p\left(Y_p^{(1)}, \ldots, Y_p^{(k)}\right),
\end{aligned} \tag{2.81}$$

where (2.77) as well as the invariance assumptions (2.78) and (2.79) were made use of. Now we smoothly distribute β_p over the whole M by the action of G. Namely, for any $q \in M$, there is $g_q \in G$ with $g_q(q) = p$. Put

$$\beta_q = g_q^* \beta_p \in \bigwedge^k (T_q M)^*. \tag{2.82}$$

Notice that g_q is not unique: Let $\tilde{g}_q \in G$ with $\tilde{g}_q(q) = p$, then $\tilde{g}_q = h g_q$ for some $h \in H$. However, it is seen at once from (2.81) that there is no ambiguity in (2.82) either:

$$(hg_q)^* \beta_p = g_q^* h^* \beta_p = g_q^* \beta_p.$$

Clearly, $\beta \in \Omega^k(M)^G$ as it is invariant from the construction. Let us, finally, show that β has the desired property $\pi^* \beta = \tilde{\beta}$. According to (2.76) and (2.78) we indeed have

$$\pi^* \beta_q = \pi^* g_q^* \beta_p = L_{g_q}^* \pi^* \beta_p = L_{g_q}^* \tilde{\beta}_e = \tilde{\beta}_{g_q^{-1}}$$

for any $q \in M$ and, as we have seen, for any $g_q^{-1} \in \pi^{(-1)}(q)$. Obviously, β must be unique with this property due to the surjectivity of π_*. □

The previous lemma gives us a one-to-one correspondence between invariant forms on the homogeneous space M and properly invariant horizontal forms on the group G. In other words, a differential form on G built of entries of the Maurer-Cartan form ω (which is automatically left G invariant) descends to an invariant form on M if and only if it is horizontal and invariant under H acting on ω by (2.70). Recall that when G is a matrix group, this action simplifies into (2.75).

2.4.3 Invariant Differential Forms on the Sphere Bundle $S\mathbb{O}^2$

Let us now describe in more detail how the general apparatus recalled in the preceding two sections applies to the case of

$$\overline{\mathrm{Spin}(9)} = \mathrm{Spin}(9) \ltimes \mathbb{O}^2. \tag{2.83}$$

As usual, we freely identify $\mathbb{O}^2 = \mathbb{R}^{16}$ via the standard basis of $\mathbb{O}$. Also, we shall take advantage of using the following index conventions throughout the rest of this chapter:

$$x = \begin{pmatrix} x_0^0 \\ \vdots \\ x_0^7 \\ x_1^0 \\ \vdots \\ x_1^7 \end{pmatrix} \in \mathbb{R}^{16} \quad \text{and} \quad A = \begin{pmatrix} A_{0,0}^{0,0} & \cdots & A_{0,0}^{0,7} & A_{0,1}^{0,0} & \cdots & A_{0,1}^{0,7} \\ \vdots & & \vdots & \vdots & & \vdots \\ A_{0,0}^{7,0} & \cdots & A_{0,0}^{7,7} & A_{0,1}^{7,0} & \cdots & A_{0,1}^{7,7} \\ A_{1,0}^{0,0} & \cdots & A_{1,0}^{0,7} & A_{1,1}^{0,0} & \cdots & A_{1,1}^{0,7} \\ \vdots & & \vdots & \vdots & & \vdots \\ A_{1,0}^{7,0} & \cdots & A_{1,0}^{7,7} & A_{1,1}^{7,0} & \cdots & A_{1,1}^{7,7} \end{pmatrix} \in \mathfrak{gl}(16, \mathbb{R}).$$

First of all, $\overline{\mathrm{Spin}(9)}$ can be viewed as a subgroup of $\mathrm{GL}(17, \mathbb{R})$ as follows:

$$\overline{\mathrm{Spin}(9)} = \left\{ \begin{pmatrix} g & x \\ 0 & 1 \end{pmatrix} ; g \in \mathrm{Spin}(9), x \in \mathbb{O}^2 \right\}. \tag{2.84}$$

The corresponding Lie algebra is then

$$\overline{\mathfrak{spin}(9)} = \left\{ \begin{pmatrix} A & x \\ 0 & 0 \end{pmatrix} ; A \in \mathfrak{spin}(9), x \in \mathbb{O}^2 \right\} \subset \mathfrak{gl}(17, \mathbb{R}). \tag{2.85}$$

Let us denote the blocks of the Maurer-Cartan form on $\overline{\mathrm{Spin}(9)}$ in the following way:

$$\omega = \begin{pmatrix} \varphi & \theta \\ 0 & 0 \end{pmatrix}, \tag{2.86}$$

i.e. in a point $\overline{g} = \begin{pmatrix} g & x \\ 0 & 1 \end{pmatrix}$, we have

$$\varphi_{\overline{g}} = g^{-1}\mathrm{d}g, \tag{2.87}$$
$$\theta_{\overline{g}} = g^{-1}\mathrm{d}x. \tag{2.88}$$

From now on, let us specialize our discussion to a particular homogeneous space, namely, to the *sphere bundle* $S\mathbb{O}^2 = \mathbb{O}^2 \times S^{15}$ over the octonionic plane. The standard action of $\overline{\mathrm{Spin}(9)}$ on $S\mathbb{O}^2$

$$\begin{pmatrix} g & x \\ 0 & 1 \end{pmatrix} : (y, v) \mapsto (gy + x, gv) \tag{2.89}$$

is transitive according to Proposition 2.16. Let us fix the point $p = (0, E_0) \in S\mathbb{O}^2$, where $E_0 = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \in S^{15}$. Then, according to Theorem 2.30,

$$H = \mathrm{Stab}_p\, \overline{\mathrm{Spin}(9)} = \left\{ \begin{pmatrix} \chi_h & 0 & 0 \\ 0 & h & 0 \\ 0 & 0 & 1 \end{pmatrix} ; h \in \mathrm{Spin}(7) \right\} \cong \mathrm{Spin}(7), \tag{2.90}$$

i.e. $S\mathbb{O}^2 \cong \overline{\mathrm{Spin}(9)}/\mathrm{Spin}(7)$.

Remark 2.51. Notice that it follows from (2.46) that a general element of H is a block diagonal matrix with blocks of size 1, 7, 8, and 1, respectively.

The projection $\pi : \overline{\mathrm{Spin}(9)} \to S\mathbb{O}^2$ is in this case given by

$$\pi \begin{pmatrix} g & x \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} g & x \\ 0 & 1 \end{pmatrix} (0, E_0) = (x, gE_0), \tag{2.91}$$

where the second factor of the image is nothing else but the first column of the matrix $g \in \mathrm{Spin}(9) \subset \mathrm{SO}(16)$.

Proposition 2.52. *The 1-forms*

$$\theta^a_k, \varphi^{a,0}_{k,0}, \quad 0 \le a \le 7, \quad k = 0, 1, \tag{2.92}$$

are horizontal in the sense of (2.80).

Proof. The differential of the projection is

$$\pi_* = (\mathrm{d}x, (\mathrm{d}g)E_0). \tag{2.93}$$

Consider $Y \in \ker \pi_* \subset \mathfrak{X}(\overline{\mathrm{Spin}(9)})$, i.e. $\mathrm{d}x(Y) = 0$ and $\mathrm{d}g(Y)E_0 = 0$. It then follows from (2.87) and (2.88), the former rewritten into $\varphi_{\overline{g}}E_0 = g^{-1}(\mathrm{d}gE_0)$, that $\theta(Y) = 0$ and $\varphi(Y)E_0 = 0$, respectively. $\square$

Let us gather the horizontal forms (2.92) into five $\mathbb{O}$-valued forms as follows: For the standard orthonormal basis $e_0, \ldots, e_7$ of $\mathbb{O}$ we define

$$\alpha = \theta^0_0, \tag{2.94}$$

$$\theta_0 = \sum_{a=1}^{7} e_a \theta^a_0, \tag{2.95}$$

$$\theta_1 = \sum_{a=0}^{7} e_a \theta^a_1 \tag{2.96}$$

$$\varphi_0 = \sum_{a=1}^{7} e_a \varphi^{a,0}_{0,0}, \tag{2.97}$$

$$\varphi_1 = \sum_{a=0}^{7} e_a \varphi^{a,0}_{1,0}. \tag{2.98}$$

Remark 2.53.
(a) Since $\mathrm{Spin}(9) \subset \mathrm{SO}(16)$, one has $\mathfrak{spin}(9) \subset \mathfrak{so}(16)$ and thus $\varphi^{0,0}_{0,0}$ is trivial.
(b) Observe that

$$\theta = \begin{pmatrix} \alpha + \theta_0 \\ \theta_1 \end{pmatrix} \quad \text{and} \quad \varphi E_0 = \begin{pmatrix} \varphi_0 \\ \varphi_1 \end{pmatrix}. \tag{2.99}$$

(c) Notice that we violate, strictly speaking, the notational conventions by not including θ^0_0 in (2.95). There is a good reason for this. Namely, we shall shortly see that the (actually real-valued) 1-form α given by (2.94) stays invariant under the right action of H and descends, thus, to $S\mathbb{O}^2$. According to (2.88), one has

$$\theta^0_0 = (g^{-1}\mathrm{d}x)^0_0 = (g^T \mathrm{d}x)^0_0 = \langle gE_0, \mathrm{d}x \rangle$$

and $\alpha = \theta^0_0$ is therefore nothing else but the pullback, under (2.91), of the *contact form* on $S\mathbb{O}^2$ considered in §1.2.4. In this connection, let us emphasize that *horizontality* as defined therein and as used here in §2.4 are two completely separate notions.
(d) The algebra of $\mathbb{O}$-valued forms will be studied thoroughly in the next chapter. For now, no product of forms (2.94) – (2.98) with each other is necessary.

Lemma 2.54. *Take any $h \in \mathrm{Spin}(7)$ and denote $\tilde{h} = \begin{pmatrix} \chi_{h^{-1}} & 0 & 0 \\ 0 & h^{-1} & 0 \\ 0 & 0 & 1 \end{pmatrix} \in H$. Then*

$$R^*_{\tilde{h}} \alpha = \alpha, \tag{2.100}$$

$$R^*_{\tilde{h}} \theta_0 = \chi_h(\theta_0), \tag{2.101}$$

$$R^*_{\tilde{h}} \theta_1 = h(\theta_1), \tag{2.102}$$

$$R^*_{\tilde{h}} \varphi_0 = \chi_h(\varphi_0), \tag{2.103}$$

$$R^*_{\tilde{h}} \varphi_1 = h(\varphi_1). \tag{2.104}$$

Proof. Let us for convenience also denote $\hat{h} = \begin{pmatrix} \chi_{h^{-1}} & 0 \\ 0 & h^{-1} \end{pmatrix}$. Then, according to (2.75),

$$R^*_{\tilde{h}} \omega = \tilde{h}^{-1} \omega \tilde{h} = \begin{pmatrix} \hat{h}^{-1} & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \varphi & \theta \\ 0 & 0 \end{pmatrix} \begin{pmatrix} \hat{h} & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} \hat{h}^{-1} \varphi \hat{h} & \hat{h}^{-1} \theta \\ 0 & 0 \end{pmatrix}$$

In particular, first, according to (2.99) and (2.46),

$$R_{\tilde{h}}^{*}\begin{pmatrix}\alpha+\theta_0\\ \theta_1\end{pmatrix}=\begin{pmatrix}\chi_h & 0\\ 0 & h\end{pmatrix}\begin{pmatrix}\alpha+\theta_0\\ \theta_1\end{pmatrix}=\begin{pmatrix}\chi_h(\alpha+\theta_0)\\ h(\theta_1)\end{pmatrix}=\begin{pmatrix}\alpha+\chi_h(\theta_0)\\ h(\theta_1)\end{pmatrix},$$

second, according to (2.99) and Remark 2.51,

$$R_{\tilde{h}}^{*}\begin{pmatrix}\varphi_0\\ \varphi_1\end{pmatrix}=R_{\tilde{h}}^{*}\varphi E_0=\hat{h}^{-1}\varphi\hat{h}E_0=\hat{h}^{-1}\varphi E_0=\begin{pmatrix}\chi_h & 0\\ 0 & h\end{pmatrix}\begin{pmatrix}\varphi_0\\ \varphi_1\end{pmatrix}=\begin{pmatrix}\chi_h(\varphi_0)\\ h(\varphi_1)\end{pmatrix}.$$

□

All in all, following the discussion of §2.4.2, one way to describe $\overline{\mathrm{Spin}(9)}$-invariant differential forms on $S\mathbb{O}^2$ is to find such combinations of the 1-forms (2.92) that stay invariant under the transformation (2.100) – (2.104) and descends, thus, from $\overline{\mathrm{Spin}(9)}$ to $S\mathbb{O}^2$. This will be done in Chapter 4, using the contents of §2.3. The crucial advantage of such an approach is that the forms can be then easily differentiated by virtue of the moving frames. In this connection, the Maurer-Cartan equation (2.74) for the Maurer-Cartan form (2.86) on $\overline{\mathrm{Spin}(9)}$ reads

$$\begin{pmatrix}\mathrm{d}\varphi & \mathrm{d}\theta\\ 0 & 0\end{pmatrix}=\mathrm{d}\omega=-\omega\wedge\omega=-\begin{pmatrix}\varphi & \theta\\ 0 & 0\end{pmatrix}\wedge\begin{pmatrix}\varphi & \theta\\ 0 & 0\end{pmatrix}=-\begin{pmatrix}\varphi\wedge\varphi & \varphi\wedge\theta\\ 0 & 0\end{pmatrix}.$$

Entrywise, for $0\le a,b\le 7$ and $k,l=0,1$, one has

$$\mathrm{d}\theta_k^a=-\sum_{j=0}^{1}\sum_{c=0}^{7}\varphi_{k,j}^{a,c}\wedge\theta_j^c, \tag{2.105}$$

$$\mathrm{d}\varphi_{k,l}^{a,b}=-\sum_{j=0}^{1}\sum_{c=0}^{7}\varphi_{k,j}^{a,c}\wedge\varphi_{j,l}^{c,b}. \tag{2.106}$$

However, as explained in §2.4.1, this picture will only be complete when we determine the linear relations among the entries of ω, i.e. the relations describing the inclusion $\overline{\mathfrak{spin}(9)}\subset\mathfrak{gl}(17,\mathbb{R})$. It is clear from (2.85) that these are precisely the defining relations for $\mathfrak{spin}(9)\subset\mathfrak{gl}(16,\mathbb{R})$, leading thus to relations among entries of φ entirely. The rest of this chapter will be devoted to finding these equations.

2.4.4 Algebra of Extended Indices

It will be convenient to extend the set $\{0,1,\dots,7\}$ of indices corresponding to $\mathbb{R}^8$ and to equip it with certain algebraic structures naturally compatible with the algebra $\mathbb{O}$.

Consider the following set of sixteen distinct formal symbols:

$$\mathcal{J}=\{\pm 0,\pm 1,\dots,\pm 7\}, \tag{2.107}$$

i.e. also $0\neq -0$. We define two involutions on $\mathcal{J}$. First, we put

$$-(\pm a)=\mp a,\quad 0\le a\le 7, \tag{2.108}$$

second,

$$\bar{a}=\begin{cases}a, & \text{if } a=\pm 0,\\ -a, & \text{if } a\neq\pm 0.\end{cases} \tag{2.109}$$

We translate this notion to the standard orthonormal basis of $\mathbb{O}$ accordingly:

$$e_{-a} = -e_a, \quad 0 \le a \le 7. \tag{2.110}$$

It is then obvious that

$$e_{\overline{a}} = \overline{e_a} \tag{2.111}$$

holds for any $a \in \mathcal{J}$. Similarly we extend this to matrices $A = (A^{a,b})_{a,b=0}^7$ by

$$A^{-a,b} = A^{a,-b} = -A^{a,b} = -A^{-a,-b}, \quad 0 \le a, b \le 7. \tag{2.112}$$

Finally, we define a product on $\mathcal{J}$: let $a \cdot b = ab$ be the (unique) element of $\mathcal{J}$ such that

$$e_{ab} = e_a e_b. \tag{2.113}$$

Importantly, the identity (2.12) remains valid also within the (extended) indices:

$$\overline{ab} = \overline{b}\,\overline{a}. \tag{2.114}$$

2.4.5 The Lie Subalgebra $\mathfrak{spin}(9) \subset \mathfrak{gl}(16, \mathbb{R})$

Recall from Lemma 2.18 that the Lie algebra $\mathfrak{spin}(9)$ is spanned by the following basis:

$$\{\mathcal{I}_{i,j}\,;\, 0 \le i < j \le 8\}, \tag{2.115}$$

where, for $0 \le i \le 7$,

$$\mathcal{I}_{i,8} = \begin{pmatrix} 0 & -R_{e_i} \\ R_{\overline{e_i}} & 0 \end{pmatrix}, \tag{2.116}$$

and, for $0 \le i < j \le 7$,

$$\mathcal{I}_{i,j} = \begin{pmatrix} R_{e_i} R_{\overline{e_j}} & 0 \\ 0 & R_{\overline{e_i}} R_{e_j} \end{pmatrix}. \tag{2.117}$$

As usual, $e_0, \ldots, e_7$ is the standard orthonormal basis of $\mathbb{O}$.

Lemma 2.55. *If $A \in \mathfrak{spin}(9)$, then for all $0 \le a, b \le 7$ and $k, l = 0, 1$ one has*

$$A^{a,b}_{k,l} = -A^{b,a}_{l,k}, \tag{2.118}$$

$$A^{a,b}_{1,0} = A^{\overline{b}a,0}_{1,0}, \tag{2.119}$$

$$A^{a,b}_{0,0} - A^{\overline{b}a,0}_{0,0} = A^{a,b}_{1,1} - A^{\overline{b}a,0}_{1,1}, \tag{2.120}$$

$$\sum_{c=0}^{7} A^{ac,c}_{1,1} = 4A^{a,0}_{0,0}. \tag{2.121}$$

Proof. First, as $\mathrm{Spin}(9) \subset \mathrm{SO}(16)$, $\mathfrak{spin}(9) \subset \mathfrak{so}(16)$ which is equivalent to (2.118).

Second, if $j < 8$, then $(\mathcal{I}_{i,j})^{a,b}_{1,0} = 0$ holds for any a, b. Thus $A^{a,b}_{1,0} = \sum_{i=0}^{7} \alpha_{i,8} (\mathcal{I}_{i,8})^{a,b}_{1,0}$ and (2.119) follows from

$$(\mathcal{I}_{i,8})^{a,b}_{1,0} = (R_{\overline{e_i}})^{a,b} = \langle e_a, e_b \overline{e_i} \rangle = \langle \overline{e_b} e_a, \overline{e_i} \rangle = (R_{\overline{e_i}})^{\overline{b}a,0} = (\mathcal{I}_{i,8})^{\overline{b}a,0}_{1,0}.$$

Third, $(\mathcal{I}_{i,8})^{a,b}_{0,0} = (\mathcal{I}_{i,8})^{a,b}_{1,1} = 0$ for all a, b, i. Hence $A^{a,b}_{0,0} = \sum_{0 \le i < j \le 7} \alpha_{i,j} (\mathcal{I}_{i,j})^{a,b}_{0,0}$ and similarly for $A^{a,b}_{1,1}$. Then, assuming $j \le 7$, one has

$$(\mathcal{I}_{i,j})^{a,b}_{0,0} - (\mathcal{I}_{i,j})^{\overline{b}a,0}_{0,0} = (R_{e_i} R_{\overline{e_j}})^{a,b} - (R_{e_i} R_{\overline{e_j}})^{\overline{b}a,0}$$

$$
\begin{aligned}
&= \langle e_a, (e_b\overline{e_j})e_i\rangle - \langle \overline{e_b}e_a, \overline{e_j}e_i\rangle \\
&= \langle e_a, (e_b\overline{e_j})e_i\rangle - \langle e_a, e_b(\overline{e_j}e_i)\rangle \\
&= \langle e_a, [e_b, \overline{e_j}, e_i]\rangle \\
&= \langle e_a, [e_b, e_j, \overline{e_i}]\rangle \\
&= \langle e_a, (e_be_j)\overline{e_i}\rangle - \langle e_a, e_b(e_j\overline{e_i})\rangle \\
&= \langle e_a, (e_be_j)\overline{e_i}\rangle - \langle \overline{e_b}e_a, e_j\overline{e_i}\rangle \\
&= (R_{\overline{e_i}}R_{e_j})^{a,b} - (R_{\overline{e_i}}R_{e_j})^{\overline{b}a,0} \\
&= (\mathcal{I}_{i,j})^{a,b}_{1,1} - (\mathcal{I}_{i,j})^{\overline{b}a,0}_{1,1},
\end{aligned}
$$

using (2.18) for the middle equation, and so (2.120) follows.

Finally, for $1 \le j \le 7$ we can write

$$
\begin{aligned}
\sum_{c=0}^{7} (\mathcal{I}_{0,j})^{ac,c}_{1,1} &= \sum_{c=0}^{7} (R_{e_j})^{ac,c} \\
&= \sum_{c=0}^{7} \langle e_ae_c, e_ce_j\rangle \\
&= \sum_{c\in\{0,j\}} \langle e_ae_c, e_je_c\rangle - \sum_{c\notin\{0,j\}} \langle e_ae_c, e_je_c\rangle \\
&= (2-6)\langle e_a, e_j\rangle \\
&= 4\langle e_a, \overline{e_j}\rangle \\
&= 4(R_{\overline{e_j}})^{a,0} \\
&= 4(\mathcal{I}_{0,j})^{a,0}_{0,0}.
\end{aligned}
$$

Similarly, if $1 \le i < j \le 7$, then $0, i, i, |ij|$ are pairwise distinct and

$$
\begin{aligned}
\sum_{c=0}^{7} (\mathcal{I}_{i,j})^{ac,c}_{1,1} &= \sum_{c=0}^{7} (R_{\overline{e_i}}R_{e_j})^{ac,c} \\
&= \sum_{c=0}^{7} \langle e_ae_c, (e_ce_j)\overline{e_i}\rangle \\
&= \sum_{c\in\{0,i,j,|ij|\}} \langle e_ae_c, e_c(e_j\overline{e_i})\rangle - \sum_{c\notin\{0,i,j,|ij|\}} \langle e_ae_c, e_c(e_j\overline{e_i})\rangle \\
&= \sum_{c\in\{0,|ij|\}} \langle e_ae_c, (e_j\overline{e_i})e_c\rangle - \sum_{c\in\{i,j\}} \langle e_ae_c, (e_j\overline{e_i})e_c\rangle + \sum_{c\notin\{0,i,j,|ij|\}} \langle e_ae_c, (e_j\overline{e_i})e_c\rangle \\
&= (2-2+4)\langle e_a, e_j\overline{e_i}\rangle \\
&= 4\langle e_a, \overline{e_j}e_i\rangle \\
&= 4(R_{e_i}R_{\overline{e_j}})^{a,0} \\
&= 4(\mathcal{I}_{i,j})^{a,0}_{0,0}.
\end{aligned}
$$

Altogether, $\sum_{c=0}^{7}(\mathcal{I}_{i,j})^{ac,c}_{1,1} = 4(\mathcal{I}_{i,j})^{a,0}_{0,0}$ holds for all $0 \le i < j \le 7$ and (2.121) follows. □

Corollary 2.56. *Let $A \in \mathfrak{gl}(16, \mathbb{R})$. $A \in \mathfrak{spin}(9)$ if and only if the following relations hold:*

$$
\begin{aligned}
A^{a,b}_{1,0} &= A^{\overline{b}a,0}_{1,0}, && 0 \le a \le 7, \quad 1 \le b \le 7, && (2.122)\\
A^{a,b}_{0,1} &= -A^{\overline{a}b,0}_{1,0}, && 0 \le a, b \le 7, && (2.123)\\
A^{a,a}_{k,k} &= 0, && 0 \le a \le 7, \quad k = 0, 1, && (2.124)
\end{aligned}
$$

$$A^{a,b}_{1,1} = -A^{b,a}_{1,1}, \qquad 1 \le a < b \le 7, \tag{2.125}$$

$$A^{a,0}_{1,1} = 2A^{a,0}_{0,0} - \frac{1}{2}\sum_{c\neq 0,a} A^{ac,c}_{1,1}, \qquad 1 \le a \le 7, \tag{2.126}$$

$$A^{0,a}_{1,1} = -2A^{a,0}_{0,0} + \frac{1}{2}\sum_{c\neq 0,a} A^{ac,c}_{1,1}, \qquad 1 \le a \le 7, \tag{2.127}$$

$$A^{a,b}_{0,0} = -A^{\bar{b}a,0}_{0,0} + A^{a,b}_{1,1} + \frac{1}{2}\sum_{c\neq 0,|\bar{b}a|} A^{(\bar{b}a)c,c}_{1,1}, \quad 1 \le b < a \le 7, \tag{2.128}$$

$$A^{a,b}_{0,0} = A^{\bar{a}b,0}_{0,0} - A^{b,a}_{1,1} - \frac{1}{2}\sum_{c\neq 0,|\bar{a}b|} A^{(\bar{a}b)c,c}_{1,1}, \qquad 1 \le a < b \le 7, \tag{2.129}$$

$$A^{0,a}_{0,0} = -A^{a,0}_{0,0}, \qquad 1 \le a \le 7. \tag{2.130}$$

Proof. For the 'only if' direction, it only remains to show (2.126) and (2.128). Indeed, (2.122) is just a special case of (2.119), and the other relations then follow easily, using skew-symmetry (2.118). Thus, first, if $a \neq 0$,

$$\begin{aligned}
4A^{a,0}_{0,0} &= \sum_{c=0}^{7} A^{ac,c}_{1,1} \\
&= A^{a^2,a}_{1,1} + A^{a,0}_{1,1} + \sum_{c\neq 0,a} A^{ac,c}_{1,1} \\
&= -A^{0,a}_{1,1} + A^{a,0}_{1,1} + \sum_{c\neq 0,a} A^{ac,c}_{1,1} \\
&= 2A^{a,0}_{1,1} + \sum_{c\neq 0,a} A^{ac,c}_{1,1}.
\end{aligned}$$

Second, for $1 \le b < a \le 7$, $\left|\bar{b}a\right| \neq 0$ and so, by (2.120),

$$\begin{aligned}
A^{a,b}_{0,0} &= A^{\bar{b}a,0}_{0,0} + A^{a,b}_{1,1} - A^{\overline{\bar{b}a},0}_{1,1} \\
&= A^{\bar{b}a,0}_{0,0} + A^{a,b}_{1,1} - 2A^{\bar{b}a,0}_{0,0} + \frac{1}{2}\sum_{c\neq 0,\bar{b}a} A^{(\bar{b}a)c,c}_{1,1} \\
&= -A^{\bar{b}a,0}_{0,0} + A^{a,b}_{1,1} + \frac{1}{2}\sum_{c\neq 0,\bar{b}a} A^{(\bar{b}a)c,c}_{1,1}.
\end{aligned}$$

As for the 'if' direction, it is evident that the relations are linearly independent in the given range of indices, in that sense that the corresponding linear functionals on $\mathfrak{gl}(16,\mathbb{R})$ are ($\xi \in \mathfrak{gl}(16,\mathbb{R})^*$ corresponds to the relation $\xi(A) = 0$). Since there are

$$56 + 64 + 16 + 21 + 7 + 7 + 21 + 21 + 7 = 220$$

of them in number, the claim follows by dimension counting. $\square$

Remark 2.57. In a generic matrix $A \in \mathfrak{spin}(9) \subset \mathfrak{gl}(16,\mathbb{R})$, therefore, the 36 entries

$$\begin{aligned}
&A^{a,0}_{0,0}, \quad 1 \le a \le 7, \\
&A^{a,0}_{1,0}, \quad 0 \le a \le 7, \\
&A^{a,b}_{1,1}, \quad 1 \le b < a \le 7,
\end{aligned}$$

are independent while the others are expressed in terms of them via (2.122) – (2.130).

Chapter 3

Octonion-Valued Forms and the Spin(9)-Invariant 8-Form

3.1 Canonical Invariant Forms

We have seen in the previous chapter (cf. Theorem 2.37) that the group Spin(7) can be equivalently characterized as the subgroup of $\mathrm{GL}(\mathbb{O})$ stabilizing the Cayley calibration. Furthermore, it follows easily from the First fundamental theorem 2.39 that

$$\left[\bigwedge^4(\mathbb{O})^*\right]^{\mathrm{Spin}(7)} = \mathrm{span}\{\Phi\}. \tag{3.1}$$

In this sense, the Cayley calibration is the *canonical invariant* 4*-form* of the group Spin(7). It is well known that Spin(7) is not the only one among the groups acting transitively on a sphere (which we listed in Remark 1.35 above) that admits such a canonical invariant (see e.g. [36], Table 1 on p. 311). And in fact most of these groups are then precisely the stabilizers of the respective forms.

First, corresponding to the *unitary group*

$$\mathrm{U}(n) = \{A \in \mathrm{GL}(n,\mathbb{C})\,;\, A^*A = \mathrm{id}\}\,, \tag{3.2}$$

where A^* is the (complex) conjugated transpose of the complex n by n matrix A, is the *Kähler form* ω on $\mathbb{C}^n$. This canonical $\mathrm{U}(n)$-invariant 2-form is usually given by (see e.g. [84], p. 31)

$$\omega = \frac{i}{2}\sum_{j=1}^{n} \mathrm{d}z_j \wedge \overline{\mathrm{d}z_j}, \tag{3.3}$$

where $\mathrm{d}z_1, \ldots, \mathrm{d}z_n$ are the complex coordinate 1-forms on $\mathbb{C}^n$. Notice that in this case $\mathrm{U}(n) \subsetneqq \{g \in \mathrm{GL}(\mathbb{C}^n)\,;\, g^*\omega = \omega\}$.

Remark 3.1. Only for completeness of Remark 1.35, let us recall that the *special unitary group* is defined as $\mathrm{SU}(n) = \{A \in \mathrm{U}(n)\,;\, \det A = 1\}$.

Second, the *(compact) symplectic group* is defined as follows:

$$\mathrm{Sp}(n) = \{A \in \mathrm{GL}(n,\mathbb{H})\,;\, A^*A = \mathrm{id}\}\,, \tag{3.4}$$

where A^* is the (quaternionic) conjugated transpose of A. It acts on $\mathbb{H}^n$ from the left. Consider the groups $\mathrm{Sp}(n) \times \mathrm{Sp}(1)$ and $\mathrm{Sp}(n) \times \mathrm{U}(1)$, where the second factors act on $\mathbb{H}^n$ from the right, by multiplication by a unit quaternion and a unit complex number, respectively. The kernel of their action on $\mathbb{H}^n$ is in both cases $\{(\mathrm{id}, 1), (-\mathrm{id}, -1)\} \cong \mathbb{Z}_2$. In this connection, we define

$$\mathrm{Sp}(n)\mathrm{Sp}(1) = \mathrm{Sp}(n) \times \mathrm{Sp}(1)/\mathbb{Z}_2 \tag{3.5}$$

and similarly $\mathrm{Sp}(n)\mathrm{U}(1)$. Among the symplectic groups, only the groups in the series $\mathrm{Sp}(n)\mathrm{Sp}(1)$ are big enough to admit a canonical invariant form, namely, $\mathrm{Sp}(n)\mathrm{Sp}(1)$ is the stabilizer of the *Kraines* 4*-form* on $\mathbb{H}^n$ defined as follows (see [93] and [118], p. 126):

$$\Omega = \Omega_I \wedge \Omega_I + \Omega_J \wedge \Omega_J + \Omega_K \wedge \Omega_K, \tag{3.6}$$

where for $u = \begin{pmatrix} u_1 \\ \vdots \\ u_n \end{pmatrix} \in \mathbb{H}^n$ and $v = \begin{pmatrix} v_1 \\ \vdots \\ v_n \end{pmatrix} \in \mathbb{H}^n$ we put

$$\Omega_I(u, v) = \sum_i \langle u_i e_1, v_i \rangle, \quad \Omega_J(u, v) = \sum_i \langle u_i e_2, v_i \rangle, \quad \Omega_K(u, v) = \sum_i \langle u_i e_4, v_i \rangle. \tag{3.7}$$

Remark 3.2. Later on we shall see (the expected and well-known fact) that the definition (3.6) of Ω is independent of the particular choice of the basis $\{e_1, e_2, e_4\}$ of $\operatorname{Im}\mathbb{H}$.

Third, similar to $\mathrm{Spin}(7)$ is the description of another exceptional group. Namely, the group

$$\mathrm{G}_2 = \{g \in \mathrm{GL}(\mathbb{O})\,;\, g(xy) = g(x)g(y), \text{ for all } x, y \in \mathbb{O}\} \tag{3.8}$$

of automorphisms of the octonion algebra can be equivalently defined as the stabilizer of the *associative calibration*, the canonical G_2-invariant 3-form on $\operatorname{Im}\mathbb{O}$ given by

$$\phi(x, y, z) = \langle x, yz \rangle. \tag{3.9}$$

This was shown by Bryant in §2 of his seminal paper [44].

Finally, the fact that there exists a non-trivial $\mathrm{Spin}(9)$-invariant 8-form Ψ on $\mathbb{O}^2$ that is moreover unique up to a scaling factor was first observed by Brown and Gray [43]. Berger [23] then showed that, for some constant $c_1 \in \mathbb{R}$,

$$\Psi = c_1 \int_{\mathbb{O}P^1} \pi_\ell^* \, \nu_\ell \, d\ell, \tag{3.10}$$

where ν_ℓ is the volume form on $\ell \in \mathbb{O}P^1$, $\pi_\ell : \mathbb{O}^2 \to \ell$ is the orthogonal projection and $d\ell$ is the Haar measure on (the naturally oriented manifold) $\mathbb{O}P^1 \cong \mathrm{Spin}(9)/\mathrm{Spin}(8)$ (see Theorem 2.25). In spite of elegance of (3.10), it turned out that, algebraically, Ψ is indeed a complicated object and in fact the first algebraic formulas appeared quite recently. Namely, it is due to Castrillón López et al. [45] that, for some $c_2 \in \mathbb{R}$,

$$\Psi = c_2 \sum_{i,j,k,l=0}^{8} \omega_{ij} \wedge \omega_{ik} \wedge \omega_{jl} \wedge \omega_{kl}, \tag{3.11}$$

where $\omega_{ij} = \langle \cdot, \mathcal{I}_{i,j} \cdot \rangle \in \bigwedge^2(\mathbb{O}^2)^*$ and $\mathcal{I}_{i,j} \in \mathrm{End}(\mathbb{O}^2)$ are as in §2.2.2. Another interpretation was given by Parton and Piccini [110] who used the computer algebra system MATHEMATICA to compute, directly from (3.10), all 702 (!) terms of Ψ in the standard basis and proved further that the 8-form is proportional to the fourth coefficient of the characteristic polynomial of the matrix $(\omega_{ij})_{i,j=0}^8$. Very recently, Castrillón López et al. [46] showed that this approach differs from (3.11) just from a combinatorial point of view. For the fact that $\mathrm{Spin}(9)$ is precisely the stabilizer of Ψ, see [45], §3, and [50], §1.

Let us summarize these results into Table 3.1. Each row corresponds to a canonical G-invariant m-form α on V, i.e.

$$\left[\bigwedge^m V^*\right]^G = \operatorname{span}\{\alpha\}. \tag{3.12}$$

Moreover, for $G \neq \mathrm{U}(n)$, one has

$$G = \{g \in \mathrm{GL}(V)\,;\, g^*\alpha = \alpha\}\,. \tag{3.13}$$

G	V	m	canonical invariant	α
$\mathrm{U}(n)$	$\mathbb{C}^n$	2	Kähler form	ω
$\mathrm{Sp}(n)\mathrm{Sp}(1)$	$\mathbb{H}^n$	4	Kraines form	Ω
G_2	$\operatorname{Im}\mathbb{O}$	3	associative calibration	ϕ
$\mathrm{Spin}(7)$	$\mathbb{O}$	4	Cayley calibration	Φ
$\mathrm{Spin}(9)$	$\mathbb{O}^2$	8	?	Ψ

Table 3.1: Canonical invariant forms

Our goal in this chapter is to give *new algebraic formulas* for the canonical invariant forms other than ω. In particular we obtain a simple formula for the $\mathrm{Spin}(9)$-invariant 8-form Ψ that will allow us to recover its expression in the standard basis easily *by hand*.

Central to our approach is the following observation: In (3.3), the *real* form ω is regarded as an element of a bigger (real) algebra $\mathbb{C} \otimes \bigwedge^\bullet(\mathbb{C}^n)^*$ of *complex-valued* forms, equipped with the (wedge) product arising naturally on the tensor product of two (real) algebras and with the conjugation extended from $\mathbb{C}$. A natural question is: The group $\mathrm{U}(n)$ is closely related to the complex numbers, so is it possible to move further along the path $\mathbb{R} - \mathbb{C} - \mathbb{H} - \mathbb{O}$ in order to obtain analogous formulas for the groups corresponding to the other normed division algebras? As we shall see, the answer is indeed 'yes'. However, particular care is required when treating forms with values in the quaternions or octonions due to lack of commutativity and associativity, respectively.

The results of the current chapter were published in [91].

3.2 Octonion-Valued Forms

3.2.1 Alternating Forms

In this section, the algebra of (real) *alternating* forms will be extended by allowing them to take values in the octonions. Basic properties of these objects will be discussed as well as first examples. Assume throughout that V is a d-dimensional (real) vector space.

Definition 3.3. Let $0 \leq k \leq d$. We define

$$\bigwedge\nolimits^k_{\mathbb{O}} V^* = \mathbb{O} \otimes \bigwedge\nolimits^k V^*. \tag{3.14}$$

We call an element of $\bigwedge^k_{\mathbb{O}} V^*$ an *octonion-valued form of degree k on V*. Further, we denote

$$\bigwedge\nolimits^\bullet_{\mathbb{O}} V^* = \mathbb{O} \otimes \bigwedge\nolimits^\bullet V^* = \bigoplus_{k=0}^{d} \bigwedge\nolimits^k_{\mathbb{O}} V^*, \tag{3.15}$$

the graded algebra equipped with the natural product

$$(u \otimes \varphi) \wedge (v \otimes \psi) = (uv) \otimes (\varphi \wedge \psi). \tag{3.16}$$

Notice that the real algebra $\bigwedge_{\mathbb{O}}^{\bullet} V^*$ is neither associative nor alternating. Nonetheless, we still find it natural to denote the product (3.16) with the same wedge symbol, as it extends the standard wedge product on the subalgebra

$$\bigwedge^{\bullet} V^* = \operatorname{span}\{1\} \otimes \bigwedge^{\bullet} V^* \subset \bigwedge_{\mathbb{O}}^{\bullet} V^*.$$

Example 3.4. Let $V = \mathbb{O}^2$. We define the *octonionic coordinate* 1*-forms* $\mathrm{d}x, \mathrm{d}y \in \bigwedge_{\mathbb{O}}^{1}(\mathbb{O}^2)^*$ on $\mathbb{O}^2$ as

$$\mathrm{d}x \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = u_1 \quad \text{and} \quad \mathrm{d}y \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = u_2. \tag{3.17}$$

Let $e_0, \ldots, e_7$ be an orthonormal basis of $\mathbb{O}$ and consider the corresponding canonical basis $\mathrm{d}x^0, \ldots, \mathrm{d}x^7, \mathrm{d}y^0, \ldots, \mathrm{d}y^7$ of $\bigwedge^1(\mathbb{O}^2)^*$, i.e.

$$\mathrm{d}x^i \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = \langle e_i, u_1 \rangle \quad \text{and} \quad \mathrm{d}y^i \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = \langle e_i, u_2 \rangle.$$

Then

$$\mathrm{d}x = \sum_{i=0}^{7} e_i \otimes \mathrm{d}x^i \quad \text{and} \quad \mathrm{d}y = \sum_{i=0}^{7} e_i \otimes \mathrm{d}y^i \tag{3.18}$$

and so it is transparent that $\mathrm{d}x$ and $\mathrm{d}y$ are octonion-valued 1-forms indeed.

From now on, the following conventions will be adhered to, regarding octonion-valued forms. First of all, the tensor-product symbol will be omitted, i.e.

$$u\varphi = u \otimes \varphi \in \bigwedge_{\mathbb{O}}^{\bullet} V^*, \tag{3.19}$$

for the sake of brevity. Further, if $F : \mathbb{O} \to \mathbb{O}$ is an linear function, we define its (linear) extension to $\bigwedge_{\mathbb{O}}^{\bullet} V^*$ by

$$F(u\varphi) = F(u)\varphi. \tag{3.20}$$

Examples of such functions we shall use are the involution, right/left multiplication by an octonion, or the real-part operator.

Proposition 3.5. *For any $\alpha \in \bigwedge_{\mathbb{O}}^{k} V^*$ and $\beta \in \bigwedge_{\mathbb{O}}^{l} V^*$ one has*

$$\overline{\alpha \wedge \beta} = (-1)^{kl}\, \overline{\beta} \wedge \overline{\alpha}. \tag{3.21}$$

Proof. By linearity we may assume $\alpha = u\varphi$ and $\beta = v\psi$ for some $\varphi \in \bigwedge^k V^*$, $\psi \in \bigwedge^l V^*$, and $u, v \in \mathbb{O}$. Then

$$\overline{\alpha \wedge \beta} = \overline{(uv)\varphi \wedge \psi} = \overline{(uv)}\varphi \wedge \psi = (-1)^{kl}(\overline{v}\,\overline{u})\psi \wedge \varphi = (-1)^{kl}\, \overline{\beta} \wedge \overline{\alpha}. \qquad \square$$

To conclude this section, let us make three simple observations that will be useful later on. First, for any $\alpha \in \bigwedge_{\mathbb{O}}^{\bullet} V^*$,

$$\alpha \in \bigwedge^{\bullet} V^* \quad \text{if and only if} \quad \operatorname{Re}(\alpha) = \alpha \quad \text{if and only if} \quad \overline{\alpha} = \alpha. \tag{3.22}$$

Second, for $u\varphi, v\psi \in \bigwedge_{\mathbb{O}}^{\bullet} V^*$ one has

$$\operatorname{Re}\left(u\varphi \wedge \overline{v\psi}\right) = \operatorname{Re}(u\overline{v})\, \varphi \wedge \psi = \langle u, v \rangle\, \varphi \wedge \psi. \tag{3.23}$$

Finally, if $e_0, \ldots, e_7$ is an orthonormal basis of $\mathbb{O}$, then it is a consequence of (3.23) that for any $\alpha, \beta \in \bigwedge_{\mathbb{O}}^{\bullet} V^*$ with $\alpha = \sum_i e_i \alpha^i$ and $\beta = \sum_i e_i \beta^i$ one has

$$\operatorname{Re}(\alpha \wedge \overline{\beta}) = \sum_{i=0}^{7} \alpha^i \wedge \beta^i = \operatorname{Re}(\overline{\alpha} \wedge \beta). \tag{3.24}$$

3.2.2 Differential Forms

It is straightforward to generalize the contents of the previous section to *differential* forms. Let M be a d-dimensional smooth manifold. We define

$$\Omega^k_{\mathbb{O}}(M) = \mathbb{O} \otimes \Omega^k(M) \tag{3.25}$$

and

$$\Omega^\bullet_{\mathbb{O}}(M) = \mathbb{O} \otimes \Omega^\bullet(M) = \bigoplus_{k=0}^{d} \Omega^k_{\mathbb{O}}(M), \tag{3.26}$$

where the latter is turned into a (non-associative) graded algebra by means of

$$(u \otimes \varphi) \wedge (v \otimes \psi) = (uv) \otimes (\varphi \wedge \psi). \tag{3.27}$$

Again, the tensor-product symbol is omitted and linear operators extend naturally from *both* factors of the tensor product. Notice that we did not consider (although we could) any extension from the second factor in the alternating case but it is of particular importance here as it includes the exterior differential or pullback of a smooth map. As for the relations (3.21) – (3.24), they obviously remain valid in the smooth setting so let us, for simplicity, refer to them also in this more general context.

Remark 3.6. To the best of our knowledge, octonion-valued (differential) forms were considered for the very first time in a recent paper by Grigorian [70]. Being more specialized yet more subtle, his perspective slightly differs from ours. Namely, Grigorian considers smooth sections of the bundle $\bigwedge^\bullet(T^*M) \otimes \mathbb{O}M$, where M is a *7-dimensional* smooth manifold and $\mathbb{O}M = (M \times \mathbb{R}) \oplus TM$ is the so-called *octonion bundle* equipped fibrewise with an octonionic multiplication induced by a given G_2-structure (reduction of the frame-bundle structure group to G_2). In this setting, hence, the multiplicative structure on the target space (more precisely its imaginary part) of an octonion-valued form may vary from point to point, unlike in our case. This allows the author to interpret the torsion of the G_2-structure as an $\operatorname{Im}\mathbb{O}$-valued 1-form on M (see [70] for details).

Let us mention that the presence of Grigorian's work was revealed to us after most of the material of this chapter was finished and it did not influence our approach at all.

3.3 Three Toy Examples

It turns out that it is more or less straightforward to rewrite the definitions of the forms Ω, ϕ, and Φ given above (see also Table 3.1) in terms of quaternion- an octonion-valued forms, respectively. For we are, nonetheless, not aware that such formulas have previously appeared in the literature, let us derive them now.

3.3.1 The Kraines 4-Form

Let us begin with the canonical $\mathrm{Sp}(n)\mathrm{Sp}(1)$-invariant 4-form on $\mathbb{H}^n$. Similarly as for $\mathbb{O}$, we define *quaternion-valued forms* on a vector space V:

$$\textstyle\bigwedge^k_{\mathbb{H}} V^* = \mathbb{H} \otimes \bigwedge^k V^* \subset \bigwedge^k_{\mathbb{O}} V^*, \tag{3.28}$$

$0 \leq k \leq d = \dim V$, and

$$\textstyle\bigwedge^\bullet_{\mathbb{H}} V^* = \mathbb{H} \otimes \bigwedge^\bullet V^* = \displaystyle\bigoplus_{k=0}^{d} \textstyle\bigwedge^k_{\mathbb{H}} V^* \subset \bigwedge^\bullet_{\mathbb{O}} V^*. \tag{3.29}$$

The inclusions correspond to $\mathbb{H} \subset \mathbb{O}$ (cf. §2.1.4). Notice that, since $\mathbb{H}$ is associative, associativity of the wedge product (3.16) is also recovered when restricting from $\bigwedge_{\mathbb{O}}^{\bullet} V^*$ to $\bigwedge_{\mathbb{H}}^{\bullet} V^*$. We shall need the following

Lemma 3.7. *Let V be a vector space. For any $\alpha \in \bigwedge_{\mathbb{H}}^{\bullet} V^*$, we have*

$$(R_{e_1} L_{e_1} + R_{e_2} L_{e_2} + R_{e_4} L_{e_4})(\alpha) = -\alpha - 2\overline{\alpha}. \tag{3.30}$$

Proof. Without loss of generality, we may assume that $\alpha = u\varphi$ for some $u \in \mathbb{H}$ and $\varphi \in \bigwedge^{\bullet} V^*$. Then since $\{1, e_1, e_2, e_4\}$ is an orthonormal basis of $\mathbb{H}$, for any $u \in \mathbb{H}$ we can write

$$\begin{aligned} \overline{u} &= \langle 1, \overline{u} \rangle 1 + \langle e_1, \overline{u} \rangle e_1 + \langle e_2, \overline{u} \rangle e_2 + \langle e_4, \overline{u} \rangle e_4 \\ &= \frac{1}{2} \left[(u + \overline{u}) + (e_1 u + \overline{u}\, \overline{e_1}) e_1 + (e_2 u + \overline{u}\, \overline{e_2}) e_2 + (e_4 u + \overline{u}\, \overline{e_4}) e_4 \right] \\ &= \frac{1}{2} \left(u + 4\overline{u} + e_1 u e_1 + e_2 u e_2 + e_4 u e_4 \right), \end{aligned}$$

therefore $e_1 u e_1 + e_2 u e_2 + e_4 u e_4 = -u - 2\overline{u}$ and (3.30) follows. ☐

From now on, we shall consider the vector space $V = \mathbb{H}^n$. First of all, let us define, as in Example 3.4, the *quaternionic coordinate* 1*-forms* $\mathrm{d}w_1, \ldots, \mathrm{d}w_n \in \bigwedge_{\mathbb{H}}^1 (\mathbb{H}^n)^*$ by

$$\mathrm{d}w_i(u) = u_i, \quad 1 \leq i \leq n. \tag{3.31}$$

Recall that u_i is the i-th component of $u \in \mathbb{H}^n$. Then

Theorem 3.8. *The Kraines form on $\mathbb{H}^n$ equals*

$$\Omega = \frac{1}{4} \sum_{i,j=1}^{n} \mathrm{d}w_i \wedge \overline{\mathrm{d}w_j} \wedge \mathrm{d}w_j \wedge \overline{\mathrm{d}w_i}. \tag{3.32}$$

Remark 3.9. If we denote $\Omega_{i,j} = \mathrm{d}w_i \wedge \overline{\mathrm{d}w_j}$, then, by (3.21), $\overline{\Omega_{i,j}} = -\mathrm{d}w_j \wedge \overline{\mathrm{d}w_i}$, and so (3.32) takes the following symmetric form, more resembling (3.3):

$$\Omega = -\frac{1}{4} \sum_{i,j=1}^{n} \Omega_{i,j} \wedge \overline{\Omega_{i,j}}. \tag{3.33}$$

Proof. First, for any $u, v \in \mathbb{H}^n$ and $1 \leq i \leq n$ we have

$$\begin{aligned} 2\langle u_i e_1, v_i \rangle &= u_i e_1 \overline{v_i} - v_i e_1 \overline{u_i} \\ &= \mathrm{d}w_i(u)\, e_1\, \overline{\mathrm{d}w_i(v)} - \mathrm{d}w_i(v)\, e_1\, \overline{\mathrm{d}w_i(u)} \\ &= \left[R_{e_1}(\mathrm{d}w_i) \wedge \overline{\mathrm{d}w_i} \right] (u, v) \\ &= \left[\mathrm{d}w_i \wedge L_{e_1}(\overline{\mathrm{d}w_i}) \right] (u, v), \end{aligned}$$

hence

$$2\Omega_I = \sum_i R_{e_1}(\mathrm{d}w_i) \wedge \overline{\mathrm{d}w_i} = \sum_i \mathrm{d}w_i \wedge L_{e_1}(\overline{\mathrm{d}w_i}),$$

and similarly for Ω_J and Ω_K. The Kraines form (3.6) therefore reads

$$\frac{1}{4} \sum_{i,j} \mathrm{d}w_i \wedge \left[L_{e_1}(\overline{\mathrm{d}w_i}) \wedge R_{e_1}(\mathrm{d}w_j) + L_{e_2}(\overline{\mathrm{d}w_i}) \wedge R_{e_2}(\mathrm{d}w_j) + L_{e_4}(\overline{\mathrm{d}w_i}) \wedge R_{e_4}(\mathrm{d}w_j) \right] \wedge \overline{\mathrm{d}w_j}$$

$$= \frac{1}{4} \sum_{i,j} \mathrm{d}w_i \wedge (R_{e_1} L_{e_1} + R_{e_2} L_{e_2} + R_{e_4} L_{e_4})(\overline{\mathrm{d}w_i} \wedge \mathrm{d}w_j) \wedge \overline{\mathrm{d}w_j}.$$

Further, according to Lemma 3.7 and (3.21),

$$\Omega = \frac{1}{4} \sum_{i,j} \mathrm{d}w_i \wedge (2\overline{\mathrm{d}w_j} \wedge \mathrm{d}w_i - \overline{\mathrm{d}w_i} \wedge \mathrm{d}w_j) \wedge \overline{\mathrm{d}w_j}. \tag{3.34}$$

Now, from (3.24) we have

$$\begin{aligned} \mathrm{d}w_i \wedge \overline{\mathrm{d}w_j} - \mathrm{d}w_j \wedge \overline{\mathrm{d}w_i} &= 2\,\mathrm{Re}(\mathrm{d}w_i \wedge \overline{\mathrm{d}w_j}) \\ &= 2\,\mathrm{Re}(\overline{\mathrm{d}w_i} \wedge \mathrm{d}w_j) \\ &= \overline{\mathrm{d}w_i} \wedge \mathrm{d}w_j - \overline{\mathrm{d}w_j} \wedge \mathrm{d}w_i. \end{aligned}$$

Again by (3.21), this is a real form. Let us denote it by β. Since β is moreover of even degree, it commutes with any element of $\bigwedge_{\mathbb{H}}^{\bullet} V^*$ and hence

$$\begin{aligned} &(2\overline{\mathrm{d}w_j} \wedge \mathrm{d}w_i - \overline{\mathrm{d}w_i} \wedge \mathrm{d}w_j) \wedge \overline{\mathrm{d}w_j} \\ &\quad = 2\overline{\mathrm{d}w_j} \wedge \mathrm{d}w_j \wedge \overline{\mathrm{d}w_i} + 2\overline{\mathrm{d}w_j} \wedge \beta - \overline{\mathrm{d}w_j} \wedge \mathrm{d}w_i \wedge \overline{\mathrm{d}w_j} - \beta \wedge \overline{\mathrm{d}w_j} \\ &\quad = \overline{\mathrm{d}w_j} \wedge \mathrm{d}w_j \wedge \overline{\mathrm{d}w_i} + 2\overline{\mathrm{d}w_j} \wedge \beta - \overline{\mathrm{d}w_j} \wedge \beta - \beta \wedge \overline{\mathrm{d}w_j} \\ &\quad = \overline{\mathrm{d}w_j} \wedge \mathrm{d}w_j \wedge \overline{\mathrm{d}w_i}. \end{aligned}$$

Plugging this identity back into (3.34), the claim follows. □

3.3.2 The Cayley Calibration Revisited

Let $\mathrm{d}x \in \bigwedge_{\mathbb{O}}^1(\mathbb{O})^*$ be the *octonionic coordinate 1-form*, i.e. the *identity*, on $\mathbb{O}$.

Proposition 3.10. *The Cayley calibration on $\mathbb{O}$ equals*

$$\Phi = -\frac{1}{24}(\mathrm{d}x \wedge \overline{\mathrm{d}x}) \wedge (\mathrm{d}x \wedge \overline{\mathrm{d}x}). \tag{3.35}$$

Proof. Since Φ is alternating, using (2.52), one has

$$\begin{aligned} &-4!\,\Phi(x_1, x_2, x_3, x_4) \\ &\quad = 4!\,\Phi(x_1, x_4, x_3, x_2) \\ &\quad = \sum_{\pi \in \mathcal{S}_4} \mathrm{sgn}(\pi)\, \Phi(x_{\pi(1)}, x_{\pi(4)}, x_{\pi(3)}, x_{\pi(2)}) \\ &\quad = \sum_{\pi \in \mathcal{S}_4} \mathrm{sgn}(\pi)\, \langle x_{\pi(1)} \overline{x_{\pi(2)}}, x_{\pi(4)} \overline{x_{\pi(3)}} \rangle \\ &\quad = \frac{1}{2} \sum_{\pi \in \mathcal{S}_4} \mathrm{sgn}(\pi)\, \Big[(x_{\pi(1)} \overline{x_{\pi(2)}})(x_{\pi(3)} \overline{x_{\pi(4)}}) + (x_{\pi(3)} \overline{x_{\pi(4)}})(x_{\pi(1)} \overline{x_{\pi(2)}}) \Big] \\ &\quad = \sum_{\pi \in \mathcal{S}_4} \mathrm{sgn}(\pi)\, (x_{\pi(1)} \overline{x_{\pi(2)}})(x_{\pi(3)} \overline{x_{\pi(4)}}) \\ &\quad = \Big[(\mathrm{d}x \wedge \overline{\mathrm{d}x}) \wedge (\mathrm{d}x \wedge \overline{\mathrm{d}x}) \Big] (x_1, x_2, x_3, x_4). \end{aligned}$$

□

3.3.3 The Associative Calibration

Let us denote the restriction of the *octonionic coordinate* 1-*form* on $\mathbb{O}$ to $\mathrm{Im}\,\mathbb{O}$ by the same symbol, i.e. now we have $\mathrm{d}x \in \bigwedge_{\mathbb{O}}^1(\mathrm{Im}\,\mathbb{O})^*$.

Proposition 3.11. *The associative calibration on* $\operatorname{Im}\mathbb{O}$ *equals*

$$\phi = -\frac{1}{12}\Big[(\mathrm{d}x \wedge \mathrm{d}x) \wedge \mathrm{d}x + \mathrm{d}x \wedge (\mathrm{d}x \wedge \mathrm{d}x)\Big]. \tag{3.36}$$

Proof. Analogous to Proposition 3.10,

$$\begin{aligned}
-3!\,\phi(x_1, x_2, x_3) &= 3!\,\phi(x_1, x_3, x_2) \\
&= \sum_{\pi \in \mathcal{S}_3} \operatorname{sgn}(\pi)\,\phi(x_{\pi(1)}, x_{\pi(3)}, x_{\pi(2)}) \\
&= \sum_{\pi \in \mathcal{S}_3} \operatorname{sgn}(\pi)\,\langle x_{\pi(1)}, x_{\pi(3)} x_{\pi(2)} \rangle \\
&= \frac{1}{2} \sum_{\pi \in \mathcal{S}_3} \operatorname{sgn}(\pi)\,\Big[x_{\pi(1)}(x_{\pi(2)} x_{\pi(3)}) - (x_{\pi(3)} x_{\pi(2)}) x_{\pi(1)}\Big] \\
&= \frac{1}{2} \sum_{\pi \in \mathcal{S}_3} \operatorname{sgn}(\pi)\,\Big[x_{\pi(1)}(x_{\pi(2)} x_{\pi(3)}) + (x_{\pi(1)} x_{\pi(2)}) x_{\pi(3)}\Big] \\
&= \frac{1}{2}\Big[(\mathrm{d}x \wedge \mathrm{d}x) \wedge \mathrm{d}x + \mathrm{d}x \wedge (\mathrm{d}x \wedge \mathrm{d}x)\Big](x_1, x_2, x_3).
\end{aligned}$$

□

3.4 The $\mathrm{Spin}(9)$-Invariant 8-Form

Finally, the notion of octonion-valued forms introduced in §3.2.1 will be applied, this time in its full strength, in order to express the canonical Spin(9)-invariant 8-form Ψ in terms of the octonionic coordinate 1-forms $\mathrm{d}x$ and $\mathrm{d}y$ on $\mathbb{O}^2$. Unlike in the previous cases, the algebraic formulas for Ψ recalled in §3.1 are too complicated to be simply transformed into the desired form. Instead, we shall independently construct a real alternating 8-form on $\mathbb{O}^2$ that is non-trivial and Spin(9)-invariant and must be, therefore, collinear to Ψ. The explicit proportional factor can be then determined by expressing the form in the standard real basis (see Appendix A).

Important Remark 3.12. As we shall, for the sake of space, usually omit commas in subscripts, let us emphasize that *no product of indices* in the sense of §2.4.4 is considered at all in the current chapter.

Let us begin with a technical lemma. Let V be a general vector space. First, assume $\alpha_1, \ldots, \alpha_4 \in \bigwedge_{\mathbb{O}}^{\bullet} V^*$ and define, for the purpose of this section,

$$\mathcal{F}(\alpha_1, \alpha_2, \alpha_3, \alpha_4) = ((\overline{\alpha_1} \wedge \alpha_2) \wedge \overline{\alpha_3}) \wedge \alpha_4. \tag{3.37}$$

Then,

Lemma 3.13. *For any* $\alpha_1, \ldots, \alpha_8 \in \bigwedge_{\mathbb{O}}^{\bullet} V^*$ *and* $u \in \mathbb{O}$ *with* $|u| = 1$ *we have*

$$\begin{aligned}
&\operatorname{Re}\Big[\mathcal{F}(R_u\alpha_1, R_{\overline{u}}\alpha_2, R_{\overline{u}}\alpha_3, R_{\overline{u}}\alpha_4) \wedge \overline{\mathcal{F}(R_u\alpha_5, R_{\overline{u}}\alpha_6, R_{\overline{u}}\alpha_7, R_{\overline{u}}\alpha_8)}\Big] \\
&\quad = \operatorname{Re}\Big[\mathcal{F}(\alpha_1, \alpha_2, \alpha_3, \alpha_4) \wedge \overline{\mathcal{F}(\alpha_5, \alpha_6, \alpha_7, \alpha_8)}\Big],
\end{aligned} \tag{3.38}$$

and

$$\begin{aligned}
&\operatorname{Re}\left[\mathcal{F}(R_u\alpha_1, R_{\overline{u}}\alpha_2, R_{\overline{u}}\alpha_3, R_{\overline{u}}\alpha_4) \wedge \mathcal{F}(R_{\overline{u}}\alpha_5, R_u\alpha_6, R_u\alpha_7, R_u\alpha_8)\right] \\
&\quad = \operatorname{Re}\left[\mathcal{F}(\alpha_1, \alpha_2, \alpha_3, \alpha_4) \wedge \mathcal{F}(\alpha_5, \alpha_6, \alpha_7, \alpha_8)\right].
\end{aligned} \tag{3.39}$$

Proof. As the map $\mathcal{F}$ is multilinear, we may without loss of generality assume $\alpha_i = u_i\varphi_i$ for some $u_i \in \mathbb{O}$ and $\varphi_i \in \bigwedge^\bullet V^*$, $1 \le i \le 8$. Thus, taking the Moufang identities (2.15) and (2.17) and alternativity of $\mathbb{O}$ into account, we can write

$$\begin{aligned}
[[(\overline{u}\,\overline{u_1})(u_2\overline{u})](u\overline{u_3})](u_4\overline{u}) &= [[\overline{u}(\overline{u_1}u_2)\overline{u}](u\overline{u_3})](u_4\overline{u}) \\
&= [\overline{u}[(\overline{u_1}u_2)(\overline{u}(u\overline{u_3}))]](u_4\overline{u}) \\
&= [\overline{u}((\overline{u_1}u_2)\overline{u_3})](u_4\overline{u}) \\
&= \overline{u}[((\overline{u_1}u_2)\overline{u_3})u_4]\overline{u}.
\end{aligned}$$

Then, since $R_{\overline{u}}, L_{\overline{u}} \in \mathrm{O}(\mathbb{O})$ for $|u| = 1$,

$$\langle [[(\overline{u}\,\overline{u_1})(u_2\overline{u})](u\overline{u_3})](u_4\overline{u}), [[(\overline{u}\,\overline{u_5})(u_6\overline{u})](u\overline{u_7})](u_8\overline{u})\rangle = \langle((\overline{u_1}u_2)\overline{u_3})u_4, ((\overline{u_5}u_6)\overline{u_7})u_8\rangle$$

and (3.38) follows from (3.23). Similarly, we have

$$\overline{[[(u\,\overline{u_5})(u_6u)](\overline{u}\,\overline{u_7})](u_8u)} = \overline{u[((\overline{u_5}u_6)\overline{u_7})u_8]u} = \overline{u}\,\overline{[((\overline{u_5}u_6)\overline{u_7})u_8]}\,\overline{u},$$

therefore

$$\langle [[(\overline{u}\,\overline{u_1})(u_2\overline{u})](u\overline{u_3})](u_4\overline{u}), \overline{[[(u\,\overline{u_5})(u_6u)](\overline{u}\,\overline{u_7})](u_8u)}\rangle = \langle((\overline{u_1}u_2)\overline{u_3})u_4, \overline{((\overline{u_5}u_6)\overline{u_7})u_8}\rangle,$$

and (3.39) then follows from (3.23) rewritten in the form

$$\mathrm{Re}\,(w\varphi \wedge v\psi) = \langle w, \overline{v}\rangle\, \varphi \wedge \psi. \qquad \square$$

We shall specialize ourselves to the case $V = \mathbb{O}^2$ throughout the rest of this section. Then, one has the bi-grading $\bigwedge^\bullet_{\mathbb{O}}(\mathbb{O}^2)^* = \bigoplus_{k,l} \bigwedge^{k,l}_{\mathbb{O}}(\mathbb{O}^2)^*$ with respect to $\mathbb{O}^2 = \mathbb{O} \oplus \mathbb{O}$. Let us denote

$$\begin{aligned}
\Psi_{40} &= \mathcal{F}(\mathrm{d}x, \mathrm{d}x, \mathrm{d}x, \mathrm{d}x), \\
\Psi_{31} &= \mathcal{F}(\mathrm{d}y, \mathrm{d}x, \mathrm{d}x, \mathrm{d}x), \\
\Psi_{13} &= \mathcal{F}(\mathrm{d}x, \mathrm{d}y, \mathrm{d}y, \mathrm{d}y), \\
\Psi_{04} &= \mathcal{F}(\mathrm{d}y, \mathrm{d}y, \mathrm{d}y, \mathrm{d}y).
\end{aligned}$$

Clearly, $\Psi_{k,l} \in \bigwedge^{k,l}_{\mathbb{O}}(\mathbb{O}^2)^*$. Notice also that the definition of these 4-forms is independent of the choice of a basis for $\mathbb{O}$, since the same is true for the 1-forms $\mathrm{d}x$ and $\mathrm{d}y$ (remember Example 3.4 above).

Let det be the determinant on $\mathbb{O} \cong \mathbb{R}^8$ such that $\det(e_0, \dots, e_7) = 1$ for the standard basis introduced in §2.1.3, and let us denote

$$\begin{aligned}
\det{}_1 &= (\mathrm{d}x)^* \det, \\
\det{}_2 &= (\mathrm{d}y)^* \det.
\end{aligned}$$

Here the forms $\mathrm{d}x, \mathrm{d}y : \mathbb{O}^2 \to \mathbb{O}$ are regarded as the projections on the first and second factor of $\mathbb{O}^2$, respectively. In the following two lemmas, we present two different ways to construct the determinants from $\mathrm{d}x$ and $\mathrm{d}y$.

Lemma 3.14.

$$\Psi_{40} \wedge \overline{\Psi_{40}} = 8!\,\det{}_1, \tag{3.40}$$

$$\Psi_{04} \wedge \overline{\Psi_{04}} = 8!\,\det{}_2. \tag{3.41}$$

Proof. Let $e_0, \dots, e_7$ be the standard orthonormal basis of $\mathbb{O}$. It follows from (2.10) that

$$R_{\overline{e_i}} R_{e_j} = -R_{\overline{e_j}} R_{e_i}, \quad 0 \le i < j \le 7.$$

By (3.21) and (3.22), we have

$$\Psi_{40} \wedge \overline{\Psi_{40}} = \operatorname{Re} \Psi_{40} \wedge \overline{\Psi_{40}}.$$

Thus, according to (3.23),

$$\begin{aligned}
\Psi_{40} \wedge \overline{\Psi_{40}} &= \sum \langle ((\overline{e_{i_0}} e_{i_1})\overline{e_{i_2}}) e_{i_3}, ((\overline{e_{i_4}} e_{i_5})\overline{e_{i_6}}) e_{i_7} \rangle \, \mathrm{d}x^{i_0} \wedge \cdots \wedge \mathrm{d}x^{i_7} \\
&= \sum \langle R_{e_{i_4}} R_{\overline{e_{i_5}}} R_{e_{i_6}} R_{\overline{e_{i_7}}} R_{e_{i_3}} R_{\overline{e_{i_2}}} R_{e_{i_1}} R_{\overline{e_{i_0}}}(1), 1 \rangle \, \mathrm{d}x^{i_0} \wedge \cdots \wedge \mathrm{d}x^{i_7},
\end{aligned}$$

where the sum extends over all indices $0 \le i_0, \dots, i_7 \le 7$, but clearly only the terms with all indices distinct occur non-trivially. Since both factors in each term of the sum are totally skew-symmetric, we can write

$$\begin{aligned}
\Psi_{40} \wedge \overline{\Psi_{40}} &= 8! \langle R_{e_4} R_{\overline{e_5}} R_{e_6} R_{\overline{e_7}} R_{e_3} R_{\overline{e_2}} R_{e_1} R_{\overline{e_0}}(1), 1 \rangle \, \mathrm{d}x^0 \wedge \cdots \wedge \mathrm{d}x^7 \\
&= 8! \, \mathrm{d}x^0 \wedge \cdots \wedge \mathrm{d}x^7 \\
&= 8! \det{}_1,
\end{aligned}$$

to show (3.40). Notice that it is easily verified by direct computation that

$$R_{e_4} R_{\overline{e_5}} R_{e_6} R_{\overline{e_7}} R_{e_3} R_{\overline{e_2}} R_{e_1} R_{\overline{e_0}}(1) = 1. \tag{3.42}$$

The proof of (3.41) is completely analogous. □

Remark 3.15. Let $e_0, \dots, e_7$ be the standard basis of $\mathbb{O}$. Recall that $\overline{e_i} = \pm e_i$, $e_i e_j = \pm e_j e_i$, and that a product of two basis elements is, at most up to a sign, a member of the basis as well. Therefore we can write

$$e_i(e_j e_k) = \sigma_1 e_i(e_k e_j) = \sigma_2 e_k(e_i e_j) = \sigma_3 (e_i e_j) e_k, \tag{3.43}$$

where the signs $\sigma_1, \sigma_2, \sigma_3 = \pm 1$ depend on i, j, k but are in general independent of each other. The middle equality in (3.43) follows for $i \ne k$ from (2.10) and is trivial if $i = k$. All in all, a particular ordering of any product of the basis elements has effect on the sign of the product at most. In this sense, the aforementioned relation (3.42) implies

$$\prod_{k=0}^{7} e_k = \pm 1. \tag{3.44}$$

Lemma 3.16.

$$\operatorname{Re} \Psi_{40} \wedge \Psi_{40} = -\frac{3}{5} \Psi_{40} \wedge \overline{\Psi_{40}}. \tag{3.45}$$

Proof. We shall work in the standard basis again. For any $0 \le i \le 7$ we denote

$$\Psi_{40}^i = \sum ((\overline{e_{i_0}} e_{i_1})\overline{e_{i_2}}) e_{i_3} \, \mathrm{d}x^{i_0} \wedge \mathrm{d}x^{i_1} \wedge \mathrm{d}x^{i_2} \wedge \mathrm{d}x^{i_3},$$

if the sum runs over all indices with $e_{i_0} e_{i_1} e_{i_2} e_{i_3} = \pm e_i$ (see Remark 3.15). Again we make use of (3.21) and (3.23) to write

$$\begin{aligned}
\Psi_{40}^i \wedge \overline{\Psi_{40}^i} &= \operatorname{Re} \Psi_{40}^i \wedge \overline{\Psi_{40}^i} \\
&= \sum \langle ((\overline{e_{i_0}} e_{i_1})\overline{e_{i_2}}) e_{i_3}, ((\overline{e_{i_4}} e_{i_5})\overline{e_{i_6}}) e_{i_7} \rangle \, \mathrm{d}x^{i_0} \wedge \cdots \wedge \mathrm{d}x^{i_7}
\end{aligned}$$

$$= \sum \langle R_{e_{i_4}} R_{\overline{e_{i_5}}} R_{e_{i_6}} R_{\overline{e_{i_7}}} R_{e_{i_3}} R_{\overline{e_{i_2}}} R_{e_{i_1}} R_{\overline{e_{i_0}}}(1), 1\rangle \, dx^{i_0} \wedge \cdots \wedge dx^{i_7}.$$

Here the sums extend over all indices such that $e_{i_0}e_{i_1}e_{i_2}e_{i_3} = \pm e_i$ (and $e_{i_4}e_{i_5}e_{i_6}e_{i_7} = \pm e_i$ but this is redundant since the inner product would be zero otherwise). As in the proof of Lemma 3.14, due to skew-symmetry we further have

$$\Psi_{40}^i \wedge \overline{\Psi_{40}^i} = n_i (4!)^2 \det{}_1,$$

where n_i denotes the number of combinations of four distinct indices between 0 and 7 whose product is $\pm e_i$.

We claim that $n_0 = 14$, i.e. among all $\binom{8}{4} = 70$ combinations of four distinct indices, 14 products equal ± 1. To see this, assume $e_{i_0}e_{i_1}e_{i_2}e_{i_3} = \pm e_{i_4}e_{i_5}e_{i_6}e_{i_7} = \pm 1$ for $i_0, \ldots, i_7$ all distinct. If one of the indices i_0, i_1, i_2, i_3, say i_0, is zero, then the others are non-zero and $e_{i_1}e_{i_2}e_{i_3} = \pm 1$, hence $e_{i_3} = \pm e_{i_1}e_{i_2}$. There are precisely 7 distinct sets $\{i_1, i_2, i_3\}$ satisfying this, corresponding to the 7 columns of the table in Remark 2.11. Symmetrically, the other 7 combinations occur when $0 \in \{i_4, i_5, i_6, i_7\}$.

Now, $\sum_{i=1}^7 n_i = 70 - 14 = 56$ and therefore, according to (3.24), one finally has

$$\begin{aligned} \operatorname{Re} \Psi_{40} \wedge \Psi_{40} &= \Psi_{40}^0 \wedge \overline{\Psi_{40}^0} - \sum_{i=1}^7 \Psi_{40}^i \wedge \overline{\Psi_{40}^i} \\ &= \left(n_0 - \sum_{i=1}^7 n_i \right) (4!)^2 \det{}_1 \\ &= -\frac{3}{5} \Psi_{40} \wedge \overline{\Psi_{40}}. \end{aligned}$$

□

Let us prove one more auxiliary assertion from the representation theory of Spin(8). For $\sigma = 0, +, -$, we denote by S_σ the space $\mathbb{O}$ equipped with the Spin(8)-module structure ρ_σ, as discussed in §2.2.3 from where the notation is kept.

Lemma 3.17. $\dim \left[\bigwedge^8 (S_+ \oplus S_-)^* \right]^{\mathrm{Spin}(8)} = 5.$

Proof. Regarded as a Spin(8)-module,

$$\bigwedge^8 (S_+ \oplus S_-)^* \cong \bigwedge^8 (S_+ \oplus S_-) = \bigoplus_{k=0}^8 \bigwedge^k S_+ \otimes \bigwedge^{8-k} S_-$$

and thus

$$\dim \left[\bigwedge^8 (S_+ \oplus S_-)^* \right]^{\mathrm{Spin}(8)} = \sum_{k=0}^8 \dim \left[\bigwedge^k S_+ \otimes \bigwedge^{8-k} S_- \right]^{\mathrm{Spin}(8)}.$$

Let us denote this number by d. Since

$$\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \in \mathrm{Spin}(8),$$

the terms of the sum above are trivial for k odd, and because $\dim S_\pm = 8$ and so $\bigwedge^{8-k} S_\pm \cong \bigwedge^k S_\pm$, we in fact have $d = 2d_0 + 2d_2 + d_4$, where

$$d_k = \dim \left[\bigwedge^k S_+ \otimes \bigwedge^k S_- \right]^{\mathrm{Spin}(8)}.$$

Let Γ_μ be an irreducible Spin(8)-module of highest weight μ. In particular, we have $S_0 = \Gamma_{\lambda_1}$, $S_+ = \Gamma_{\lambda_3}$, and $S_- = \Gamma_{\lambda_4}$. Trivially, $\bigwedge^0 S_0 = \Gamma_0$. It is also known that

$\bigwedge^2 S_0 = \Gamma_{\lambda_2}$ is the adjoint representation and that $\bigwedge^4 S_0 = \Gamma_{2\lambda_3} \oplus \Gamma_{2\lambda_4}$ (see [65], §19.2). Applying the triality principle (see §2.2.4), we further obtain $\bigwedge^0 S_+ = \bigwedge^0 S_- = \Gamma_0$, $\bigwedge^2 S_+ = \bigwedge^2 S_- = \Gamma_{\lambda_2}$, and $\bigwedge^4 S_+ = \Gamma_{2\lambda_4} \oplus \Gamma_{2\lambda_1}$, while $\bigwedge^4 S_- = \Gamma_{2\lambda_1} \oplus \Gamma_{2\lambda_3}$. Counting the same factors in the decompositions of exterior powers of S_+ and S_-, we finally conclude $d_0 = d_2 = d_4 = 1$ and thus $d = 5$. □

Now we can finally proceed to the main result of this chapter. Recall that

$$
\begin{aligned}
\Psi_{40} &= ((\overline{\mathrm{d}x} \wedge \mathrm{d}x) \wedge \overline{\mathrm{d}x}) \wedge \mathrm{d}x,\\
\Psi_{31} &= ((\overline{\mathrm{d}y} \wedge \mathrm{d}x) \wedge \overline{\mathrm{d}x}) \wedge \mathrm{d}x,\\
\Psi_{13} &= ((\overline{\mathrm{d}x} \wedge \mathrm{d}y) \wedge \overline{\mathrm{d}y}) \wedge \mathrm{d}y,\\
\Psi_{04} &= ((\overline{\mathrm{d}y} \wedge \mathrm{d}y) \wedge \overline{\mathrm{d}y}) \wedge \mathrm{d}y.
\end{aligned}
$$

Theorem 3.18. *The form*

$$
\Psi_8 = \Psi_{40} \wedge \overline{\Psi_{40}} + 4\,\Psi_{31} \wedge \overline{\Psi_{31}} - 5\left(\Psi_{31} \wedge \Psi_{13} + \overline{\Psi_{13}} \wedge \overline{\Psi_{31}}\right) + 4\,\Psi_{13} \wedge \overline{\Psi_{13}} + \Psi_{04} \wedge \overline{\Psi_{04}} \tag{3.46}
$$

is a non-trivial real multiple of the $\mathrm{Spin}(9)$*-invariant 8-form* Ψ *on* $\mathbb{O}^2$*.*

Remark 3.19. Before we prove this theorem, let us highlight several advantages our approach and the formula it results into have. First, the presented description allows us to verify the invariance and non-triviality with very simple algebraic tools, eliminating thus the role of combinatorics significantly. Second, the formula (3.46) is transparently intrinsic (involves no choice of basis of $\mathbb{O}$) and immediately reveals some non-trivial information about the structure of the form (e.g. its decomposition with respect to the natural bi-grading). Finally, using our formula, we are able to determine explicitly and *without any aid of computer* all the 702 terms of Ψ in the standard basis, explaining thus the pattern Parton and Piccinni [110] observed in their Table 2. These computations are postponed to Appendix A.

Proof. Let us denote

$$
\begin{aligned}
\Psi_{80} &= \Psi_{40} \wedge \overline{\Psi_{40}},\\
\Psi_{62} &= \Psi_{31} \wedge \overline{\Psi_{31}},\\
\Psi_{44} &= -\frac{5}{6}\left(\Psi_{31} \wedge \Psi_{13} + \overline{\Psi_{13}} \wedge \overline{\Psi_{31}}\right) = -\frac{5}{3}\,\mathrm{Re}\,\Psi_{31} \wedge \Psi_{13},\\
\Psi_{26} &= \Psi_{13} \wedge \overline{\Psi_{13}},\\
\Psi_{08} &= \Psi_{04} \wedge \overline{\Psi_{04}}.
\end{aligned}
$$

We have to show that

$$
\Psi_8 = \Psi_{80} + 4\,\Psi_{62} + 6\,\Psi_{44} + 4\,\Psi_{26} + \Psi_{08} \tag{3.47}
$$

is a non-trivial $\mathrm{Spin}(9)$-invariant real 8-form.

First of all, all five summands on the right-hand side of (3.47) clearly belong to $\bigwedge^8_{\mathbb{O}}(\mathbb{O}^2)^*$. In fact, they are all real as seen from (3.21), taking into account that the forms Ψ_{40}, Ψ_{31}, Ψ_{13}, and Ψ_{04} are of even degree. Hence $\Psi_8 \in \bigwedge^8(\mathbb{O}^2)^*$.

Second, we prove that each summand separately is $\mathrm{Spin}(8)$ invariant. By Lemma 2.23, it is sufficient to show invariance under

$$
\begin{pmatrix} R_z & 0 \\ 0 & R_{\overline{z}} \end{pmatrix}, \quad |z| = 1.
$$

Since $R_z, R_{\bar{z}} \in \mathrm{SO}(\mathbb{O})$ for $|z| = 1$, and the determinant on $\mathbb{O}$ is $\mathrm{SO}(\mathbb{O})$ invariant, Ψ_{80} and Ψ_{08} are $\mathrm{Spin}(8)$ invariant according to Lemma 3.14. The invariance of the rest follows at once from Lemma 3.13.

For the final step, let us abbreviate, for $t \in \mathbb{R}$, $c = \cos(t)$, $s = \sin(t)$, and

$$g = \begin{pmatrix} c & s \\ s & -c \end{pmatrix} \in \mathrm{Spin}(9).$$

Further, let $\mathcal{P} : \bigwedge^8(\mathbb{O}^2)^* \to \bigwedge^{8,0}(\mathbb{O}^2)^*$ be the natural projection. Since $g^*\mathrm{d}x = c\mathrm{d}x + s\mathrm{d}y$ and $g^*\mathrm{d}y = s\mathrm{d}x - c\mathrm{d}y$, with help of Lemma 3.16 it is not difficult to see that, for all k, l we consider,

$$\mathcal{P}\left(g^*\Psi_{kl}\right) = c^k s^l\, \Psi_{80}.$$

In particular, this shows that all the five forms Ψ_{kl} are non-trivial. Moreover, because $\Psi_{kl} \in \bigwedge^{k,l}(\mathbb{O}^2)^*$, they are linearly independent and thus, according to Lemma 3.17, span $[\bigwedge^8(\mathbb{O}^2)^*]^{\mathrm{Spin}(8)}$. As $\mathrm{Spin}(8) \subset \mathrm{Spin}(9)$, we have $\Psi \in [\bigwedge^8(\mathbb{O}^2)^*]^{\mathrm{Spin}(8)}$, and therefore there are constants $\kappa = (\kappa_0, \ldots, \kappa_4) \in \mathbb{R}^5$ such that $\Psi_\kappa = \sum_{i=0}^4 \kappa_i \Psi_{8-2i,2i}$ is $\mathrm{Spin}(9)$ invariant. In order to fix κ, we impose the condition of invariance under g. Namely, in particular,

$$\mathcal{P}\left(g^*\Psi_\kappa\right) = \sum_{i=0}^{4} \kappa_i c^{8-2i} s^{2i}\, \Psi_{80}$$

must equal to

$$\mathcal{P}\left(\Psi_\kappa\right) = \kappa_0 \Psi_{80} = \kappa_0 (c^2 + s^2)^4\, \Psi_{80}.$$

It is easily seen that the solution of

$$\sum_{i=0}^{4} \kappa_i c^{8-2i} s^{2i} = \kappa_0 (c^2 + s^2)^4$$

equals uniquely, up to scaling by κ_0, to the binomial coefficients of the fourth-power expansion. In particular, for $\kappa_0 = 1$ we have $\Psi_8 = \Psi_\kappa$ which completes the proof. $\square$

Chapter 4

Spin(9)-Invariant Valuations

4.1 General Framework and Previous Results

In this chapter we introduce a basis of the space of Spin(9)-invariant valuations on the octonionic plane and determine the Bernig-Fu convolution on it. The invariant-form approach in the sense of Theorem 1.29 and the formula (1.38) are the foundations on which our construction will be erected.

Let us recall from §1.3 that $\mathrm{Val}^{\mathrm{Spin}(9)} = \mathrm{Val}(\mathbb{O}^2)^{\mathrm{Spin}(9)}$ equipped with the Bernig-Fu convolution is, a priori, a finite-dimensional commutative associative algebra with unit vol_{16} that is graded with respect to the McMullen decomposition

$$\mathrm{Val}^{\mathrm{Spin}(9)} = \bigoplus_{k=0}^{16} \mathrm{Val}_{16-k}^{\mathrm{Spin}(9)}. \tag{4.1}$$

As usual, we identify $\mathbb{O}^2 = \mathbb{R}^{16}$. Furthermore, the Alesker-Poincaré pairing (1.43) is perfect and convolution with μ_{15} has the hard Lefschetz property (1.47) on $\mathrm{Val}^{\mathrm{Spin}(9)}$.

4.1.1 Examples of Spin(9)-Invariant Valuations

To the best of our knowledge, the problem of Spin(9)-invariant valuations was treated for the first time by Alesker in [13]. Developing the theory of *octoninic plurisubharmonic functions* on $\mathbb{O}^2$, the author was able to introduce a new example of such a valuation, in addition to an array of more or less obvious examples that he also discussed.

First of all, the intrinsic volumes are $\mathrm{SO}(\mathbb{O}^2)$ invariant, hence

$$\mu_k \in \mathrm{Val}_k^{\mathrm{Spin}(9)}, \quad 0 \leq k \leq 16. \tag{4.2}$$

Further, for $K \in \mathcal{K}(\mathbb{O}^2)$, one can consider

$$T_k(K) = \int_{\mathbb{O}P^1} \mu_k(\pi_\ell K)\, \mathrm{d}\ell, \quad 0 \leq k \leq 8, \tag{4.3}$$

where π_ℓ is the orthogonal projection to a octonionic line $\ell \in \mathbb{O}P^1$ (see §2.2.2) and $\mathrm{d}\ell$ is the Spin(9)-invariant Haar measure on $\mathbb{O}P^1$, and

$$U_k(K) = \int_{\overline{\mathbb{O}P^1}} \mu_{k-8}(K \cap \overline{\ell})\, \mathrm{d}\overline{\ell}, \quad 8 \leq k \leq 16, \tag{4.4}$$

where $\overline{\mathbb{O}P^1} = \{x + \ell;\, x \in \mathbb{O}^2, \ell \in \mathbb{O}P^1\}$ is the *affine octonionic projective line* with the Spin(9)-invariant Haar measure $\mathrm{d}\overline{\ell}$. It follows at once from invariance of the measures and Proposition 2.19 that $T_k \in \mathrm{Val}_k^{\mathrm{Spin}(9)}$, $0 \le k \le 8$, and $U_k \in \mathrm{Val}_k^{\mathrm{Spin}(9)}$, $8 \le k \le 16$. We shall comment more on these valuations in §4.5 below.

Finally, for an octonion-valued smooth function $f \in \mathbb{O} \otimes C^\infty(\mathbb{O}^2)$, Alesker defines

$$\frac{\partial f}{\partial \overline{x_i}} = \sum_{a=0}^{7} e_a \frac{\partial f}{\partial x_i^a} \quad \text{and} \quad \frac{\partial f}{\partial x_i} = \sum_{a=0}^{7} \frac{\partial f}{\partial x_i^a} \overline{e_a}, \quad \text{where } \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \in \mathbb{O}^2 \text{ and } x_i = \sum_{a=0}^{7} x_i^a e_a. \tag{4.5}$$

The *octonionic Hessian* $\left(\frac{\partial^2 f}{\partial x_i \partial \overline{x_j}}\right)$ is then, pointwise, a 2x2 octonionic hermitian matrix which has a well-defined real-valued determinant (see also [20]). Alesker extends, in the sense of measures, the notion of det $\left(\frac{\partial^2 f}{\partial x_i \partial \overline{x_j}}\right)$ to much broader class of the so-called octonionic plurisubharmonic functions. In particular, this applies to the *support function* $h_K(X) = \sup_{Y \in K} \langle X, Y \rangle$, $X \in \mathbb{O}^2$, of a convex body $K \in \mathcal{K}(\mathbb{O}^2)$, and

Theorem 4.1 (Alesker [13]). *The functional $\tau_{\mathbb{O}} : \mathcal{K}(\mathbb{O}^2) \to \mathbb{R}$ given by*

$$\tau_{\mathbb{O}}(K) = \int_B \det\left(\frac{\partial^2 h_K}{\partial x_i \partial \overline{x_j}}\right) \mathrm{d}x, \tag{4.6}$$

where $B \subset \mathbb{O}^2$ is the unit ball, is a well-defined element of $\mathrm{Val}_2^{\mathrm{Spin}(9)}$.

Definition 4.2. The valuation τ is called *Alesker-Kazarnovskii octonionic pseudovolume*.

In the article [13], Alesker also raised several questions worth further investigation. First, he pointed out that neither a classification of elements of $\mathrm{Val}^{\mathrm{Spin}(9)}$ nor even the dimension of this space was known to him. Second, he speculated whether the foregoing examples already generate $\mathrm{Val}^{\mathrm{Spin}(9)}$ as an algebra, equipped with either the product or the convolution (see §1.3.2). Finally, he recalled the general invariant-form approach (see §1.2.3) and asked about its relation to the constructions of [13]. For a long time, however, no progress in either of these directions had been reported.

4.1.2 The Dimension

A breakthrough came with recent work of Bernig and Voide who, applying methods developed by the first-named author [28], computed the dimension and in fact all Betti numbers of the algebra $\mathrm{Val}^{\mathrm{Spin}(9)}$. Remarkably, this was possible without explicitly determining the space. Namely, studying certain exact sequences and using enumerative representation-theoretical machinery, the authors showed

Theorem 4.3 (Bernig, Voide [35]). *One has*

k	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16
$\dim \mathrm{Val}_k^{\mathrm{Spin}(9)}$	1	1	2	3	6	10	15	20	27	20	15	10	6	3	2	1	1

In particular,

$$\dim \mathrm{Val}^{\mathrm{Spin}(9)} = 143. \tag{4.7}$$

Moreover, Bernig and Voide [35] defined a valuation $\mu_{\mathrm{sec}} \in \mathrm{Val}_2^+(T_pM)$ canonically assigned to any tangent space of a Riemannian manifold M as follows: $\mathrm{Kl}_{\mu_{\mathrm{sec}}}$ is the sectional curvature at $p \in M$ (for more details we refer to [35] and also to [89], §1.11).

In particular, regarding $\mathbb{O}^2$ as a tangent space of the so-called *octonionic projective plane* $\mathbb{O}P^2$, this construction yields a Spin(9)-invariant valuation satisfying

$$\mu_{\mathrm{sec}} = 4\mu_2 - 3\tau_{\mathbb{O}}. \tag{4.8}$$

Theorem 4.3 certainly answered some of the aforementioned questions, first of all, the one about the dimension. Second, it is verified readily that $\dim_4^{\mathrm{Spin}(9)} = 6$ already requires at least two generators of this degree, proving thus that more generators are certainly needed than Alesker's examples (see §4.4.1 below for a general discussion on the numbers of generators). Even more they are insufficient to constitute a basis.

To sum up, Theorem 4.3 tells us that the algebra $\mathrm{Val}^{\mathrm{Spin}(9)}$ is indeed a complicated object, especially in comparison with its companions discussed in §1.3.4. This fact, however, only amplifies the unanswered questions, and makes it even more desirable to have a deeper understanding of the space as it is very likely that certain structures hidden behind the simplicity of the other known cases may ensue.

4.2 Invariant Forms on the Sphere Bundle $S\mathbb{O}^2$

Recall from §1.2.3 and §1.3.1 that the elements of $\mathrm{Val}^{\mathrm{Spin}(9)}(\mathbb{O}^2)$ are *smooth* valuations and are, therefore, representable by accordingly invariant smooth differential forms on the sphere bundle $S\mathbb{O}^2$ (see also §2.4.3). To this end, the goal of the following section is to determine the space $\Omega^\bullet(S\mathbb{O}^2)^{\overline{\mathrm{Spin}(9)}}$.

Let us outline the strategy based on an observation of Bernig and Voide (see [35], the proof of Proposition 4.2). For the group $\overline{\mathrm{Spin}(9)}$ acts transitively on the sphere bundle $S\mathbb{O}^2$, any invariant differential form on this space is uniquely determined by its value in a single point, say in $p = (0, E_0)$, where $E_0 = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \in S^{15}$. In other words, there is an isomorphism

$$\Omega^\bullet(S\mathbb{O}^2)^{\overline{\mathrm{Spin}(9)}} \cong \left[\bigwedge\nolimits^\bullet (T_p S\mathbb{O}^2)^*\right]^{\mathrm{Stab}_p \overline{\mathrm{Spin}(9)}}. \tag{4.9}$$

According to Theorem 2.30,

$$\mathrm{Stab}_p \overline{\mathrm{Spin}(9)} = \mathrm{Stab}_{E_0} \mathrm{Spin}(9) \cong \mathrm{Spin}(7). \tag{4.10}$$

This group acts diagonally on the tangent space $T_p S\mathbb{O}^2 = T_0\mathbb{O}^2 \oplus T_{E_0} S^{15}$ that further decomposes into irreducible Spin(7)-modules as follows: By Corollaries 2.31 and 2.32,

$$T_p S\mathbb{O}^2 = \mathbb{R} \oplus \mathrm{Im}\,\mathbb{O} \oplus \mathbb{O} \oplus \mathrm{Im}\,\mathbb{O} \oplus \mathbb{O}. \tag{4.11}$$

(4.9) thus becomes

$$\Omega^\bullet(S\mathbb{O}^2)^{\overline{\mathrm{Spin}(9)}} \cong \left[\bigwedge\nolimits^\bullet (\mathbb{R} \oplus \mathrm{Im}\,\mathbb{O} \oplus \mathbb{O} \oplus \mathrm{Im}\,\mathbb{O} \oplus \mathbb{O})^*\right]^{\mathrm{Spin}(7)}. \tag{4.12}$$

Further, the image of the contact form α under this isomorphism is given by, see (1.25),

$$\alpha_p(Z_p) = \langle E_0, \mathrm{d}\pi(Z_p)\rangle, \quad Z_p \in T_p S\mathbb{O}^2, \tag{4.13}$$

where $\pi : S\mathbb{O}^2 \to \mathbb{O}^2$ is the canonical projection. (4.13) is nothing else but the projection to the first factor of (4.11) and hence (4.12) finally induces

$$\Omega_h^\bullet(S\mathbb{O}^2)^{\overline{\mathrm{Spin}(9)}} \cong \left[\bigwedge\nolimits^\bullet (\mathrm{Im}\,\mathbb{O} \oplus \mathbb{O} \oplus \mathrm{Im}\,\mathbb{O} \oplus \mathbb{O})^*\right]^{\mathrm{Spin}(7)}. \tag{4.14}$$

4.2.1 Spin(7)-Invariant Alternating Forms

The algebra $[\bigwedge^\bullet(\mathrm{Im}\,\mathbb{O}\oplus\mathbb{O}\oplus\mathrm{Im}\,\mathbb{O}\oplus\mathbb{O})^*]^{\mathrm{Spin}(7)}$ will be now described in terms of a generating set. Our main ingredient is the FFT 2.47, and an important role is also played by the algebra of octonion-valued forms, as developed in §3.2.1. In a manner similar to the various invariants constructed in §3.3 and §3.4, the basic building blocks are the octonionic coordinate 1-forms

$$\theta_0,\theta_1,\varphi_0,\varphi_1 \in \textstyle\bigwedge^1_{\mathbb{O}}(\mathrm{Im}\,\mathbb{O}\oplus\mathbb{O}\oplus\mathrm{Im}\,\mathbb{O}\oplus\mathbb{O})^*$$

defined naturally by:

$$\theta_0 : \begin{pmatrix} u\\ x\\ v\\ y\end{pmatrix} \mapsto u, \quad \theta_1 : \begin{pmatrix} u\\ x\\ v\\ y\end{pmatrix} \mapsto x, \quad \varphi_0 : \begin{pmatrix} u\\ x\\ v\\ y\end{pmatrix} \mapsto v, \quad \text{and} \quad \varphi_1 : \begin{pmatrix} u\\ x\\ v\\ y\end{pmatrix} \mapsto y. \tag{4.15}$$

As usual, we denote, for the standard basis $e_0,\dots,e_7$ of $\mathbb{O}$,

$$\theta_0 = \sum_{a=1}^{7} e_a\theta_0^a, \quad \theta_1 = \sum_{a=0}^{7} e_a\theta_1^a, \quad \varphi_0 = \sum_{a=1}^{7} e_a\varphi_0^a, \quad \text{and} \quad \varphi_1 = \sum_{a=0}^{7} e_a\varphi_1^a. \tag{4.16}$$

Remark 4.4. Observe that there is notational conflict between (4.15) and (2.95) – (2.98). In the next section, however, the relation between these, strictly speaking, different objects will be clarified and, in fact, taken advantage of. At the same time, it will become apparent that the task of this section is completely equivalent to the problem of constructing right-Spin(7)-invariant forms on $\overline{\mathrm{Spin}(9)}$, as discussed in §(2.4.3).

Let us establish some more notation. First, one has the natural tetra-grading:

$$\left[\textstyle\bigwedge^\bullet(\mathrm{Im}\,\mathbb{O}\oplus\mathbb{O}\oplus\mathrm{Im}\,\mathbb{O}\oplus\mathbb{O})^*\right]^{\mathrm{Spin}(7)} = \bigoplus_{k,l,m,n} \textstyle\bigwedge^{k,l,m,n}, \tag{4.17}$$

where we abbreviate

$$\textstyle\bigwedge^{k,l,m,n} = \left[\bigwedge^{k,l,m,n}(\mathrm{Im}\,\mathbb{O}\oplus\mathbb{O}\oplus\mathrm{Im}\,\mathbb{O}\oplus\mathbb{O})^*\right]^{\mathrm{Spin}(7)}. \tag{4.18}$$

Second, by $\mathcal{L}$ and $\mathcal{R}$ we denote the *from-the-left* and *from-the-right* ordered products of octonion-valued forms $\alpha_1,\dots,\alpha_n \in \bigwedge^1_{\mathbb{O}}(\mathrm{Im}\,\mathbb{O}\oplus\mathbb{O}\oplus\mathrm{Im}\,\mathbb{O}\oplus\mathbb{O})^*$, i.e.

$$\mathcal{L}(\alpha_1,\dots,\alpha_n) = ((\cdots((\alpha_1\wedge\alpha_2)\wedge\alpha_3)\wedge\cdots)\wedge\alpha_n), \tag{4.19}$$

$$\mathcal{R}(\alpha_1,\dots,\alpha_n) = (\alpha_1\wedge(\cdots\wedge(\alpha_{n-2}\wedge(\alpha_{n-1}\wedge\alpha_n))\cdots)). \tag{4.20}$$

Finally, for a non-negative integer m, we write

$$\mathcal{L}(\alpha_1,\dots,\alpha_{k-1},\alpha_k[m],\alpha_{k+1},\dots,\alpha_n) = \mathcal{L}(\alpha_1,\dots,\alpha_{k-1},\overbrace{\alpha_k,\dots,\alpha_k}^{m\text{-times}},\alpha_{k+1},\dots,\alpha_n), \tag{4.21}$$

and similarly for $\mathcal{R}$.

Theorem 4.5. *The algebra* $[\bigwedge^\bullet(\operatorname{Im}\mathbb{O}\oplus\mathbb{O}\oplus\operatorname{Im}\mathbb{O}\oplus\mathbb{O})^*]^{\operatorname{Spin}(7)}$ *is generated by the following 96 elements:*

(a) *1 element*

$$[1,0,1,0]_p = -\operatorname{Re}(\theta_0\wedge\varphi_0)\,,$$

(b) *8 elements*

$$[k,0,7-k,0]_p = \sum_{\pi\in\mathcal{S}_7}\operatorname{sgn}(\pi)\,\theta_0^{\pi(1)}\wedge\cdots\wedge\theta_0^{\pi(k)}\wedge\varphi_0^{\pi(k+1)}\wedge\cdots\wedge\varphi_0^{\pi(7)},\quad 0\le k\le 7,$$

(c) *36 elements*

$$[k_1,1,k_2,1]_p = \operatorname{Re}\left[\mathcal{R}\left(\overline{\theta_1},\theta_0[k_1],\varphi_0[k_2],\varphi_1\right)\right],\quad 0\le k_1+k_2\le 7,$$

(d) *36 elements*

$$\begin{aligned}
[k_1,2,k_2,0]_p &= \operatorname{Re}\left[\mathcal{R}\left(\overline{\theta_1},\theta_0[k_1],\varphi_0[k_2],\theta_1\right)\right],\\
[k_1,0,k_2,2]_p &= \operatorname{Re}\left[\mathcal{R}(\overline{\varphi_1},\theta_0[k_1],\varphi_0[k_2],\varphi_1)\right],\quad k_1+k_2\in\{1,2,5,6\},
\end{aligned}$$

(e) *15 elements*

$$\begin{aligned}
[k_1,4,k_2,0]_p &= \operatorname{Re}\left[\mathcal{L}\left(\theta_0[k_1],\varphi_0[k_2],\theta_1,\overline{\theta_1},\theta_1,\overline{\theta_1}\right)\right],\\
[k_1,3,k_2,1]_p &= \operatorname{Re}\left[\mathcal{L}\left(\theta_0[k_1],\varphi_0[k_2],\theta_1,\overline{\theta_1},\theta_1,\overline{\varphi_1}\right)\right],\\
[k_1,2,k_2,2]_p &= \operatorname{Re}\left[\mathcal{L}\left(\theta_0[k_1],\varphi_0[k_2],\theta_1,\overline{\theta_1},\varphi_1,\overline{\varphi_1}\right)\right],\\
[k_1,1,k_2,3]_p &= \operatorname{Re}\left[\mathcal{L}(\theta_0[k_1],\varphi_0[k_2],\theta_1,\overline{\varphi_1},\varphi_1,\overline{\varphi_1})\right],\\
[k_1,0,k_2,4]_p &= \operatorname{Re}\left[\mathcal{L}(\theta_0[k_1],\varphi_0[k_2],\varphi_1,\overline{\varphi_1},\varphi_1,\overline{\varphi_1})\right],\quad 0\le k_1+k_2\le 1.
\end{aligned}$$

Remark 4.6.

(a) Observe that each generator is uniquely characterized by its tetra-degree, namely,

$$[k,l,m,n]_p\in\bigwedge\nolimits^{k,l,m,n}. \tag{4.22}$$

This fact advocates our choice of notation.

(b) Related to Remark 2.48 is the following note: The dimensions of the subspaces

$$\bigwedge\nolimits^{K,M} = \bigoplus_{\substack{k+l=K\\ m+n=M}}\bigwedge\nolimits^{k,l,m,n}$$

were computed by Bernig and Voide (see [35], Proposition 4.2). Comparison with their result asserts that there exist (in fact numerous) relations among the algebra generators we listed in the preceding theorem. For instance, simple combinatorics shows that there are 12 monomials in the generators in bi-degree $(6,1)$ whereas $\dim\bigwedge^{6,1}=10$. In fact, MAPLE computation in coordinates shows

$$2\,[0,2,1,0]_p\wedge[2,2,0,0]_p-[1,2,0,0]_p\wedge[1,2,1,0]_p+3\,[1,0,1,0]_p\wedge[1,4,0,0]_p=0, \tag{4.23}$$

and

$$2\,[1,1,0,1]_p\wedge[0,4,0,0]_p-[1,2,0,0]_p\wedge[0,3,0,1]_p+3\,[0,1,0,1]_p\wedge[1,4,0,0]_p=0. \tag{4.24}$$

It would be certainly interesting to prove (4.23) and (4.24) as well as the relations that appear in other bi-degrees without computer assistance. Our attempts in this direction, based on manipulations of the $\mathbb{O}$-valued 1-forms which the generators are built of, did not, unfortunately, meet with success. Similarly, no structural result about the relations à la the SFT is known to us in this case.

Proof. First of all, conventions similar to those of §2.3.3 will be adhered to within the proof: u, v will always refer to the vector module $\operatorname{Im}\mathbb{O}$, while x, y to the spin module $\mathbb{O}$ of $\mathrm{Spin}(7)$. Consider an arbitrary $\phi \in \bigwedge^{k,l,m,n}$. Then

$$\begin{aligned} p &= p(u_1, \dots, u_k, v_1, \dots, v_m, x_1, \dots, x_l, y_1, \dots, y_n) \\ &= \phi\left(\begin{pmatrix} u_1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \dots, \begin{pmatrix} u_k \\ 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ x_1 \\ 0 \\ 0 \end{pmatrix}, \dots, \begin{pmatrix} 0 \\ x_l \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ v_1 \\ 0 \end{pmatrix}, \dots, \begin{pmatrix} 0 \\ 0 \\ v_m \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 0 \\ y_1 \end{pmatrix}, \dots, \begin{pmatrix} 0 \\ 0 \\ 0 \\ y_n \end{pmatrix}\right) \end{aligned}$$

is a multilinear $\mathrm{Spin}(7)$-invariant polynomial, i.e. $p \in P_{k+m,l+n}$. According to Theorem 2.47, p must be a polynomial in the generating elements listed therein, hence a linear combination of the following monomials:

$$\prod_{j=1}^{J} p_j\big(u_{\kappa(k_{j-1}+1)}, \dots, u_{\kappa(k_j)}, v_{\mu(m_{j-1}+1)}, \dots, v_{\mu(m_j)}, x_{\lambda(l_{j-1}+1)}, \dots, x_{\lambda(l_j)}, y_{\nu(n_{j-1}+1)}, \dots, y_{\nu(n_j)}\big), \tag{4.25}$$

for some permutations $\kappa \in \mathcal{S}_k$, $\lambda \in \mathcal{S}_l$, $\mu \in \mathcal{S}_m$, $\nu \in \mathcal{S}_n$, some integers

$$\begin{aligned} 0 = k_0 \le k_1 \le \cdots \le k_J = k, \\ 0 = l_0 \le l_1 \le \cdots \le l_J = l, \\ 0 = m_0 \le m_1 \le \cdots \le m_J = m, \\ 0 = n_0 \le n_1 \le \cdots \le n_J = n, \end{aligned}$$

and some generators $p_j \in P_{\Delta k_j + \Delta m_j, \Delta l_j + \Delta n_j}$ where

$$\begin{aligned} \Delta k_j &= k_j - k_{j-1}, \\ \Delta l_j &= l_j - l_{j-1,} \\ \Delta m_j &= m_j - m_{j-1}, \\ \Delta n_j &= n_j - n_{j-1}. \end{aligned}$$

Without loss of generality, we may assume that ϕ is such that p actually equals to (4.25). If we denote

$$V = \operatorname{Im}\mathbb{O} \oplus \mathbb{O} \oplus \operatorname{Im}\mathbb{O} \oplus \mathbb{O} \quad \text{and} \quad N = k + l + m + n,$$

ϕ can also be regarded as an element of

$$(V^*)^{\otimes N} = \left(V^{\otimes N}\right)^*.$$

In this respect, consider another element of this space, namely, let

$$\tilde{\phi} = \sum_{\mathcal{I}} \alpha_{\mathcal{I}}\, \theta_0^{i_1^{(u)}} \otimes \cdots \otimes \theta_0^{i_k^{(u)}} \otimes \theta_1^{i_1^{(x)}} \otimes \cdots \otimes \theta_1^{i_l^{(x)}} \otimes \varphi_0^{i_1^{(v)}} \otimes \cdots \otimes \varphi_0^{i_m^{(v)}} \otimes \varphi_1^{i_1^{(y)}} \otimes \cdots \otimes \varphi_1^{i_n^{(y)}},$$

where the summation is taken over all multiindices

$$\begin{aligned} \mathcal{I} &= \left(i_1^{(u)}, \dots, i_k^{(u)}, i_1^{(x)}, \dots, i_l^{(x)}, i_1^{(v)}, \dots, i_m^{(v)}, i_1^{(y)}, \dots, i_n^{(y)}\right) \\ &\in \{1, \dots, 7\}^k \times \{0, \dots, 7\}^l \times \{1, \dots, 7\}^m \times \{0, \dots, 7\}^n, \end{aligned}$$

and

$$\alpha_{\mathcal{I}} = \prod_{j=1}^{J} p_j(e_{i^{(u)}_{\kappa(k_{j-1}+1)}}, \ldots, e_{i^{(u)}_{\kappa(k_j)}}, e_{i^{(v)}_{\mu(m_{j-1}+1)}}, \ldots, e_{i^{(v)}_{\mu(m_j)}}, e_{i^{(x)}_{\lambda(l_{j-1}+1)}}, \ldots, e_{i^{(x)}_{\lambda(l_j)}}, e_{i^{(y)}_{\nu(n_{j-1}+1)}}, \ldots, e_{i^{(y)}_{\nu(n_j)}}),$$

Then for any

$$\begin{aligned} Z = &\begin{pmatrix} u_1 \\ 0 \\ 0 \\ 0 \end{pmatrix} \otimes \cdots \otimes \begin{pmatrix} u_k \\ 0 \\ 0 \\ 0 \end{pmatrix} \otimes \begin{pmatrix} 0 \\ x_1 \\ 0 \\ 0 \end{pmatrix} \otimes \cdots \otimes \begin{pmatrix} 0 \\ x_l \\ 0 \\ 0 \end{pmatrix} \\ &\otimes \begin{pmatrix} 0 \\ 0 \\ v_1 \\ 0 \end{pmatrix} \otimes \cdots \otimes \begin{pmatrix} 0 \\ 0 \\ v_m \\ 0 \end{pmatrix} \otimes \begin{pmatrix} 0 \\ 0 \\ 0 \\ y_1 \end{pmatrix} \otimes \cdots \otimes \begin{pmatrix} 0 \\ 0 \\ 0 \\ y_n \end{pmatrix} \end{aligned} \tag{4.26}$$

we obviously have

$$\phi(Z) = \tilde{\phi}(Z). \tag{4.27}$$

Let us now apply the alternation operator Alt : $(V^*)^{\otimes N} \to \bigwedge^N V^*$, defined by

$$(\operatorname{Alt} \varphi)(W_1 \otimes \cdots \otimes W_N) = \frac{1}{N!} \sum_{\pi \in \mathcal{S}_N} \operatorname{sgn}(\pi)\, \varphi(W_{\pi(1)} \otimes \cdots \otimes W_{\pi(N)})$$

(see [131], pp. 202–205, and the proofs of Theorem 2.1.2 in [109] and of Lemma 3.3 in [24]). On one hand, we have

$$\operatorname{Alt} \phi = \phi, \tag{4.28}$$

on the other,

$$\operatorname{Alt} \tilde{\phi} = c \sum_{\mathcal{I}} \widetilde{\alpha}_{\mathcal{I}}\, \theta_0^{i_1^{(u)}} \wedge \cdots \wedge \theta_0^{i_k^{(u)}} \wedge \theta_1^{i_1^{(x)}} \wedge \cdots \wedge \theta_1^{i_l^{(x)}} \wedge \varphi_0^{i_1^{(v)}} \wedge \cdots \wedge \varphi_0^{i_m^{(v)}} \wedge \varphi_1^{i_1^{(y)}} \wedge \cdots \wedge \varphi_1^{i_n^{(y)}},$$

where $c = \frac{\operatorname{sgn}(\kappa)\operatorname{sgn}(\lambda)\operatorname{sgn}(\mu)\operatorname{sgn}(\nu)}{N!}$ is a normalizing constant and

$$\widetilde{\alpha}_{\mathcal{I}} = \prod_{j=1}^{J} p_j(e_{i^{(u)}_{k_{j-1}+1}}, \ldots, e_{i^{(u)}_{k_j}}, e_{i^{(v)}_{m_{j-1}+1}}, \ldots, e_{i^{(v)}_{m_j}}, e_{i^{(x)}_{l_{j-1}+1}}, \ldots, e_{i^{(x)}_{l_j}}, e_{i^{(y)}_{n_{j-1}+1}}, \ldots, e_{i^{(y)}_{n_j}}).$$

(4.27) and (4.28) together imply

$$\phi(Z) = (\operatorname{Alt} \phi)(Z) = (\operatorname{Alt} \tilde{\phi})(Z).$$

However, since any multilinear alternating form that is homogeneous of tetra-degree (k, l, m, n) is already determined by its values on tensors of type (4.26), we in fact have

$$\phi = \operatorname{Alt} \tilde{\phi}.$$

Altogether, ϕ is a polynomial (with respect to wedge product) in the following forms:

$$\begin{aligned} &\sum_{\mathcal{I}'} q(e_{i^{(u)}_1}, \ldots, e_{i^{(u)}_{k'}}, e_{i^{(v)}_1}, \ldots, e_{i^{(v)}_{m'}}, e_{i^{(x)}_1}, \ldots, e_{i^{(x)}_{l'}}, e_{i^{(y)}_1}, \ldots, e_{i^{(y)}_{n'}}) \\ &\quad \times \theta_0^{i_1^{(u)}} \wedge \cdots \wedge \theta_0^{i_{k'}^{(u)}} \wedge \theta_1^{i_1^{(x)}} \wedge \cdots \wedge \theta_1^{i_{l'}^{(x)}} \wedge \varphi_0^{i_1^{(v)}} \wedge \cdots \wedge \varphi_0^{i_{m'}^{(v)}} \wedge \varphi_1^{i_1^{(y)}} \wedge \cdots \wedge \varphi_1^{i_{n'}^{(y)}}, \end{aligned} \tag{4.29}$$

where k', l', m', n' are non-negative integers,

$$\begin{aligned}\mathcal{I}' &= \left(i_1^{(u)}, \dots, i_{k'}^{(u)}, i_1^{(x)}, \dots, i_{l'}^{(x)}, i_1^{(v)}, \dots, i_{m'}^{(v)}, i_1^{(y)}, \dots, i_{n'}^{(y)}\right) \\ &\in \{1, \dots, 7\}^{k'} \times \{0, \dots, 7\}^{l'} \times \{1, \dots, 7\}^{m'} \times \{0, \dots, 7\}^{n'},\end{aligned}$$

and $q \in P_{k'+m', l'+n'}$ is a generating polynomial from Theorem 2.47.

In the rest of the proof we will separately investigate all possible combinations of a generator q and integers k', l', m', n' that may occur.

(a) Consider $q = \langle u_1, u_2 \rangle \in P_{2,0}$. Then $k' + m' = 2$ and $l' = n' = 0$, so we distinguish three cases here. First, if $k' = 2$ and $m' = 0$,

$$\sum_{i_1, i_2 = 1}^{7} \langle e_{i_1}, e_{i_2} \rangle \theta_0^{i_1} \wedge \theta_0^{i_2} = \sum_{i_1 = 1}^{7} \theta_0^{i_1} \wedge \theta_0^{i_1} = 0.$$

Second, for $k' = m' = 1$ we have

$$\begin{aligned}\sum_{i_1, i_2 = 1}^{7} \langle e_{i_1}, e_{i_2} \rangle \theta_0^{i_1} \wedge \varphi_0^{i_2} &= \sum_{i_1, i_2 = 1}^{7} \mathrm{Re}(\overline{e_{i_1}} e_{i_2}) \theta_0^{i_1} \wedge \varphi_0^{i_2} \\ &= - \sum_{i_1, i_2 = 1}^{7} \mathrm{Re}(e_{i_1} e_{i_2}) \theta_0^{i_1} \wedge \varphi_0^{i_2} \\ &= - \mathrm{Re}\left(\theta_0 \wedge \varphi_0\right).\end{aligned}$$

Third, if $k' = 0$ and $m' = 2$, then again

$$\sum_{i_1, i_2 = 1}^{7} \langle e_{i_1}, e_{i_2} \rangle \varphi_0^{i_1} \wedge \varphi_0^{i_2} = 0.$$

(b) Let $q = \det(u_1, \dots, u_7) \in P_{7,0}$. Then $l' = n' = 0$, $m' = 7 - k'$ and for $0 \le k' \le 7$ one has

$$\begin{aligned}&\sum_{i_1, \dots, i_7 = 1}^{7} \det(e_{i_1}, \dots, e_{i_7})\, \theta_0^{i_1} \wedge \dots \wedge \theta_0^{i_{k'}} \wedge \varphi_0^{i_{k'+1}} \wedge \dots \wedge \varphi_0^{i_7} \\ &\quad = \sum_{\pi \in \mathcal{S}_7} \det(e_{\pi(1)}, \dots, e_{\pi(7)})\, \theta_0^{\pi(1)} \wedge \dots \wedge \theta_0^{\pi(k')} \wedge \varphi_0^{\pi(k'+1)} \wedge \dots \wedge \varphi_0^{\pi(7)}, \\ &\quad = \sum_{\pi \in \mathcal{S}_7} \mathrm{sgn}(\pi)\, \theta_0^{\pi(1)} \wedge \dots \wedge \theta_0^{\pi(k')} \wedge \varphi_0^{\pi(k'+1)} \wedge \dots \wedge \varphi_0^{\pi(7)}.\end{aligned}$$

(c) For $0 \le r \le 7$, let $q = \langle L_{u_1} \cdots L_{u_r}(x_1), x_2 \rangle \in P_{r,2}$. In this case $n' = 2 - l'$, $0 \le l' \le 2$, and $m' = r - k'$, $0 \le k' \le r$. First, if $l' = 2$, we have

$$\begin{aligned}&\sum_{j_1, j_2 = 0}^{7} \sum_{i_1, \dots, i_r = 1}^{7} \langle L_{e_{i_1}} \cdots L_{e_{i_r}}(e_{j_1}), e_{j_2} \rangle \theta_0^{i_1} \wedge \dots \wedge \theta_0^{i_{k'}} \wedge \theta_1^{j_1} \wedge \theta_1^{j_2} \wedge \varphi_0^{i_{k'+1}} \wedge \dots \wedge \varphi_0^{i_r} \\ &= \pm \sum_{j_1, j_2 = 0}^{7} \sum_{i_1, \dots, i_r = 1}^{7} \mathrm{Re}\left[L_{\overline{e_{j_2}}} L_{e_{i_1}} \cdots L_{e_{i_r}}(e_{j_1})\right] \theta_1^{j_2} \wedge \theta_0^{i_1} \wedge \dots \wedge \theta_0^{i_{k'}} \wedge \varphi_0^{i_{k'+1}} \wedge \dots \wedge \varphi_0^{i_r} \wedge \theta_1^{j_1} \\ &= \pm \mathrm{Re}\left[\mathcal{R}\left(\overline{\theta_1}, \theta_0[k'], \varphi_0[r - k'], \theta_1\right)\right],\end{aligned}$$

$0 \le k' \le r$, where the sign reflects the number of transpositions needed for commuting the coordinate 1-forms to the appropriate order. Exactly in the same way we obtain

$$\pm \mathrm{Re}\left[\mathcal{R}\left(\overline{\theta_1}, \theta_0[k'], \varphi_0[r - k'], \varphi_1\right)\right]$$

when $l' = 1$, and, for $l' = 0$,

$$\pm \operatorname{Re}\left[\mathcal{R}\left(\overline{\varphi_1}, \theta_0[k'], \varphi_0[r-k'], \varphi_1\right)\right].$$

In fact, we may exclude the cases $l' = 2$ and $l' = 0$ if $r \in \{0, 3, 4, 7\}$. Let us explain the case $l' = 2$, the other being completely similar. First, if $r = 0$, by (3.21) we have

$$\operatorname{Re}\left[\mathcal{R}\left(\overline{\theta_1}, \theta_0[0], \varphi_0[0], \theta_1\right)\right] = \operatorname{Re}\left(\overline{\theta_1} \wedge \theta_1\right) = \frac{1}{2}\left(\overline{\theta_1} \wedge \theta_1 - \overline{\theta_1} \wedge \theta_1\right) = 0.$$

Second, using $L_u L_v + L_v L_u = -2\langle u, v\rangle$, $u, v \in \operatorname{Im}\mathbb{O}$, one has

$$\begin{aligned}
\operatorname{Re}\left[L_{\overline{e_{j_2}}} L_{e_{i_1}} \cdots L_{e_{i_r}}(e_{j_1})\right] &= \langle L_{e_{i_1}} \cdots L_{e_{i_r}}(e_{j_1}), e_{j_2}\rangle \\
&= \langle e_{j_1}, L_{\overline{e_{i_r}}} \cdots L_{\overline{e_{i_1}}}(e_{j_2})\rangle \\
&= (-1)^r \langle e_{j_1}, L_{e_{i_r}} \cdots L_{e_{i_1}}(e_{j_2})\rangle \\
&= (-1)^{r+(r-1)+\cdots+1} \langle e_{j_1}, L_{e_{i_1}} \cdots L_{e_{i_r}}(e_{j_2})\rangle + \tilde{q}(e_{j_1}, e_{j_2}, e_{i_1}, \ldots, e_{i_r}) \\
&= (-1)^{\frac{r}{2}(r+1)} \operatorname{Re}\left[L_{\overline{e_{j_1}}} L_{e_{i_1}} \cdots L_{e_{i_r}}(e_{j_2})\right] + \tilde{q}(e_{j_1}, e_{j_2}, e_{i_1}, \ldots, e_{i_r}),
\end{aligned}$$

where $\tilde{q}$ is a linear combination of polynomials $\tilde{q}_1\tilde{q}_2$ with $\tilde{q}_1 \in P_{2,0}$ and $\tilde{q}_2 \in P_{r-2,2}$. So,

$$\begin{aligned}
&2\operatorname{Re}\left[\mathcal{R}\left(\overline{\theta_1}, \theta_0[k'], \varphi_0[r-k'], \theta_1\right)\right] \\
&= \sum \left\{\operatorname{Re}\left[L_{\overline{e_{j_2}}} L_{e_{i_1}} \cdots L_{e_{i_r}}(e_{j_1})\right] + (-1)^{\frac{r}{2}(r+1)} \operatorname{Re}\left[L_{\overline{e_{j_1}}} L_{e_{i_1}} \cdots L_{e_{i_r}}(e_{j_2})\right] + \tilde{q}\right\} \\
&\qquad \times \theta_1^{j_2} \wedge \theta_0^{i_1} \wedge \cdots \wedge \theta_0^{i_{k'}} \wedge \varphi_0^{i_{k'+1}} \wedge \cdots \wedge \varphi_0^{i_r} \wedge \theta_1^{j_1} \\
&= \sum \left\{\left(1 - (-1)^{\frac{r}{2}(r+1)}\right) \operatorname{Re}\left[L_{\overline{e_{j_2}}} L_{e_{i_1}} \cdots L_{e_{i_r}}(e_{j_1}) + \tilde{q}\right]\right\} \\
&\qquad \times \theta_1^{j_2} \wedge \theta_0^{i_1} \wedge \cdots \wedge \theta_0^{i_{k'}} \wedge \varphi_0^{i_{k'+1}} \wedge \cdots \wedge \varphi_0^{i_r} \wedge \theta_1^{j_1},
\end{aligned}$$

and therefore for $r \in \{3, 4, 7\}$, $[k', 2, r-k', 0]_p$ is either zero (if $k' \in \{0, r\}$) or a multiple of $[1, 0, 1, 0]_p \wedge [k'-1, 2, r-1-k', 0]_p$.

Remark 4.7. It follows easily from the proof of Theorem 2.47, see in particular (2.66), that in the statement of the theorem, the polynomials

$$\langle 1, ((x_{k_1}\overline{x_{k_2}})x_{k_3})\overline{x_{k_4}}\rangle \quad \text{and} \quad \langle 1, (((u_j x_{k_1})\overline{x_{k_2}})x_{k_3})\overline{x_{k_4}}\rangle$$

may equivalently replace $\Phi(x_{k_1}, x_{k_2}, x_{k_3}, x_{k_4})$ and $\Phi(u_j x_{k_1}, x_{k_2}, x_{k_3}, x_{k_4})$, respectively.

(d) Assume $q = \langle 1, ((x_{k_1}\overline{x_{k_2}})x_{k_3})\overline{x_{k_4}}\rangle \in P_{0,4}$, $k' = m' = 0$, $n' = 4 - l'$. For $l' = 4$,

$$\sum_{j_1,\ldots,j_4=0}^{7} \langle 1, ((e_{j_1}\overline{e_{j_2}})e_{j_3})\overline{e_{j_4}}\rangle \theta_1^{j_1} \wedge \theta_1^{j_2} \wedge \theta_1^{j_3} \wedge \theta_1^{j_4} = \operatorname{Re}\left[\mathcal{L}\left(\theta_1, \overline{\theta_1}, \theta_1, \overline{\theta_1}\right)\right].$$

Similarly, it is straightforward that one arrives at

$$\begin{aligned}
&\operatorname{Re}\left[\mathcal{L}\left(\theta_1, \overline{\theta_1}, \theta_1, \overline{\varphi_1}\right)\right], \\
&\operatorname{Re}\left[\mathcal{L}\left(\theta_1, \overline{\theta_1}, \varphi_1, \overline{\varphi_1}\right)\right], \\
&\operatorname{Re}\left[\mathcal{L}(\theta_1, \overline{\varphi_1}, \varphi_1, \overline{\varphi_1})\right], \\
&\operatorname{Re}\left[\mathcal{L}(\varphi_1, \overline{\varphi_1}, \varphi_1, \overline{\varphi_1})\right],
\end{aligned}$$

depending on whether l' is 3, 2, 1, or 0, respectively.

(e) Completely analogous to (d) is the last case $q = \langle 1, (((u_j x_{k_1})\overline{x_{k_2}})x_{k_3})\overline{x_{k_4}} \rangle \in P_{1,4}$. Just for instance, when $k' = 0$, $l' = 2$, $m' = 1$, and $n' = 2$, one has

$$\sum_{j_1,\dots,j_4=0}^{7} \sum_{i=1}^{7} \langle 1, (((e_i e_{j_1})\overline{e_{j_2}})e_{j_3})\overline{e_{j_4}} \rangle \theta_1^{j_1} \wedge \theta_1^{j_2} \wedge \varphi_0^{i} \wedge \varphi_1^{j_3} \wedge \varphi_1^{j_4} = \mathrm{Re}\left[\mathcal{L}\left(\varphi_0, \theta_1, \overline{\theta_1}, \varphi_1, \overline{\varphi_1}\right)\right].$$

Since it is clear from the construction that all the considered forms are Spin(7) invariant and at the same time we have exhausted all the possibilities provided by Theorem 2.47, the proof is completed. □

Important Remark 4.8. It is clear from the proof that if we rescale the generators by

$$[k, l, m, n]_p \mapsto \frac{1}{k!\, l!\, m!\, n!}[k, l, m, n]_p, \tag{4.30}$$

their coefficients in the standard basis $\theta_0^1, \dots, \theta_0^7, \theta_1^0, \dots, \theta_1^7, \varphi_0^1, \dots, \varphi_0^7, \varphi_1^0, \dots, \varphi_1^7$ remain integers. Let us, therefore, *redefine hereby the generating forms* according to (4.30).

Naturally, the rescaled generators are more convenient to work with in implementation and practical computation. Of the same spirit is the following observation. First of all, let us denote

$$\kappa(i_1, \dots, i_k) = \sum_{a=1}^{k}(i_a - a) = \sum_{a=1}^{k} i_a - \frac{1}{2}k(k+1) \tag{4.31}$$

and, for completeness, $\kappa() = 0$. Then

Proposition 4.9. *Let $k \in \mathbb{N}$ and $m \in \mathbb{N}_0$. Let further $i_1, \dots, i_k, j_1, \dots, j_m \in \mathbb{N}$ be such that $i_1 < i_2 < \cdots < i_k$, $j_1 < j_2 < \cdots < j_m$, and $\{i_1, \dots, i_k, j_1, \dots, j_m\} = \{1, \dots, k+m\}$. Then the parity of $\kappa(i_1, \dots, i_k)$ is the same as that of the permutation $(i_1, \dots, i_k, j_1, \dots, j_m)$, i.e.*

$$(-1)^{\kappa(i_1,\dots,i_k)} = \mathrm{sgn}(i_1, \dots, i_k, j_1, \dots, j_m). \tag{4.32}$$

Proof. We use induction on m. First, if $m = 0$, then $i_a = a$ for $1 \le a \le k$, consequently $\kappa(i_1, \dots, i_k) = 0$ and (4.32) follows. Second, assume (4.32) holds for $m \in \mathbb{N}_0$. Then

$$\mathrm{sgn}(i_1, \dots, i_k, j_1, \dots, j_{m+1}) = (-1)^{k-l}\,\mathrm{sgn}(i_1, \dots, i_l, j_1, i_{l+1}, \dots, i_k, j_2, \dots, j_{m+1}),$$

where $i_l < j_1 < i_{l+1}$, i.e. $j_1 = l + 1$. By induction hypothesis this equals $(-1)^n$, where

$$\begin{aligned} n = k - l + \kappa(i_1, \dots, i_l, j_1, i_{l+1}, \dots, i_k) &= k - l + \sum_{a=1}^{k} i_a + j_1 - \frac{1}{2}(k+1)(k+2) \\ &= \sum_{a=1}^{k} i_a + (k+1) - \frac{1}{2}(k+1)(k+2) \\ &= \sum_{a=1}^{k} i_a - \frac{1}{2}k(k+1) \\ &= \kappa(i_1, \dots, i_k), \end{aligned}$$

which completes the proof. □

Now, the (rescaled) generator $[k,0,7-k,0]_p$, $0 \leq k \leq 7$, equals

$$\begin{aligned}&\frac{1}{k!(7-k)!}\sum_{\pi\in\mathcal{S}_7}\operatorname{sgn}(\pi)\theta_0^{\pi(1)}\wedge\cdots\wedge\theta_0^{\pi(k)}\wedge\varphi_0^{\pi(k+1)}\wedge\cdots\wedge\varphi_0^{\pi(7)}\\&\qquad=\sum\operatorname{sgn}(i_1,\ldots,i_k,i_{k+1},\ldots,i_7)\theta_0^{i_1}\wedge\cdots\wedge\theta_0^{i_k}\wedge\varphi_0^{i_{k+1}}\wedge\cdots\wedge\varphi_0^{i_7},\end{aligned}$$

where the sum on the right-hand side extends over all integers $1 \leq i_1 < \cdots < i_k \leq 7$, and by $i_{k+1} < \cdots < i_7$ we denote the integers satisfying $\{i_1,\ldots,i_7\} = \{1,\ldots,7\}$. Hence, according to Proposition 4.9,

$$[k,0,7-k,0]_p = \sum_{1\leq i_1<\cdots<i_k\leq 7}(-1)^{\kappa(i_1,\ldots,i_k)}\theta_0^{i_1}\wedge\cdots\wedge\theta_0^{i_k}\wedge\varphi_0^{i_{k+1}}\wedge\cdots\wedge\varphi_0^{i_7}. \tag{4.33}$$

4.2.2 $\overline{\mathrm{Spin}(9)}$-Invariant Differential Forms

The generators from Theorem 4.5 can be also regarded as elements of

$$\left[\bigwedge{}^\bullet(T_pS\mathbb{O}^2)^*\right]^{\mathrm{Spin}(7)}. \tag{4.34}$$

More precisely, let us identify the 96 forms with their images under $(\mathrm{id} - \alpha_p)^*$, where α_p is again viewed as the projection to the first factor of

$$T_pS\mathbb{O}^2 = \mathbb{R}\oplus\operatorname{Im}\mathbb{O}\oplus\mathbb{O}\oplus\operatorname{Im}\mathbb{O}\oplus\mathbb{O}. \tag{4.35}$$

Then, the algebra (4.34) is generated by them and by the 1-form α_p. Let us extend our notation also to the latter generator by putting (cf. Remark 4.6(a))

$$[1,0,0,0]_p = \alpha_p. \tag{4.36}$$

Remark 4.10. It may be tempting to regard α_p as the (missing) real part of the octonion-valued 1-form θ_0. However, this would mean to replace θ_0 by its imaginary part in the statement of Theorem 4.5 which would somewhat decrease the readability of the result.

Finally, we apply the isomorphism (4.9):

Theorem 4.11. *Let $[k,l,n,m] \in \Omega^\bullet(S\mathbb{O}^2)^{\overline{\mathrm{Spin}(9)}}$ be the (unique) form whose value in the point p is $[k,l,m,n]_p$. Then the algebra $\Omega^\bullet(S\mathbb{O}^2)^{\overline{\mathrm{Spin}(9)}}$ is generated by the following 97 elements:*

$$\begin{array}{ll}[1,0,0,0], & \\ [1,0,1,0], & \\ [k,0,7-k,0], & 0\leq k\leq 7,\\ [k_1,1,k_2,1], & 0\leq k_1+k_2\leq 7,\\ [k_1,2,k_2,0], & k_1+k_2\in\{1,2,5,6\},\\ [k_1,0,k_2,2], & k_1+k_2\in\{1,2,5,6\},\\ [k_1,4,k_2,0], & 0\leq k_1+k_2\leq 1,\\ [k_1,3,k_2,1], & 0\leq k_1+k_2\leq 1,\\ [k_1,2,k_2,2], & 0\leq k_1+k_2\leq 1,\\ [k_1,1,k_2,3], & 0\leq k_1+k_2\leq 1,\\ [k_1,0,k_2,4], & 0\leq k_1+k_2\leq 1.\end{array} \tag{4.37}$$

4.3 Exterior Derivatives

Strictly speaking, we have now done enough to describe the vector space $\mathrm{Val}^{\mathrm{Spin}(9)}$. Indeed, according to Theorem 1.29, each invariant valuation, apart from the Lebesque measure vol_{16}, is represented by a 15-form that is a polynomial in the generators listed in Theorem 4.11. However, as discussed in §1.2.4, there are more forms than valuations, the precise proportion being expressed by virtue of the second-order differential Rumin operator (see Theorem 1.33). Thus, in order to find a basis and to determine the algebra structure on $\mathrm{Val}^{\mathrm{Spin}(9)}$, more work needs to be done. With this regard, the purpose of the section that follows is to develop a method for differentiating the invariant forms.

4.3.1 The Octonionic Structure Equations

We shall make use of Cartan's moving frames, as pre-prepared for this purpose in §2.4 above. Let $\pi : \overline{\mathrm{Spin}(9)} \to S\mathbb{O}^2$ be the natural projection (2.91). First, it was showed in Remark 2.53 (c) that

$$\pi^*[1,0,0,0] = \alpha, \tag{4.38}$$

where the right-hand side is given by (2.94). Let us now extend this relation to the rest of the generators (4.37). Namely, for a generator $[k,l,m,n] \in \Omega^\bullet(S\mathbb{O}^2)^{\overline{\mathrm{Spin}(9)}}$, let

$$]k,l,m,n[\, \in \Omega^\bullet(\overline{\mathrm{Spin}(9)}) \subset \Omega^\bullet_{\mathbb{O}}(\overline{\mathrm{Spin}(9)}) \tag{4.39}$$

be given formally by the same expression as $[k,l,m,n]$ in the point $p = (0, E_0)$, i.e. as in Theorem 4.5, *but* with 1-forms (4.15) replaced by (2.95) – (2.98). Theorem 4.5 together with Lemma 2.54 implies that $]k,l,m,n[$ is right $\mathrm{Spin}(7)$ invariant. Since the other two assumptions of Lemma 2.50 are satisfied trivially, (4.39) descends to a $\overline{\mathrm{Spin}(9)}$-form on $S\mathbb{O}^2$. In fact,

Proposition 4.12. *For any* $[k,l,m,n]$ *from* (4.37), *one has*

$$\pi^*[k,l,m,n] = \,]k,l,m,n[. \tag{4.40}$$

Proof. Since both sides of (4.40) are $\overline{\mathrm{Spin}(9)}$-invariant (the left-hand one by Lemma 2.50) and $\overline{\mathrm{Spin}(9)}$ acts transitively on itself, it is enough to verify (4.40) in a point, say in the identity, which follows immediately from (2.87), (2.88), (2.99), and (2.93). □

This at ones generalizes to

Corollary 4.13. *Consider* $\beta \in \Omega^\bullet(S\mathbb{O}^2)^{\overline{\mathrm{Spin}(9)}}$ *and let* $\beta_p = f(\alpha_p, \theta_0, \theta_1, \varphi_0, \varphi_1)$ *be its value in the point p. Then*

$$\pi^*\beta = f(\alpha, \theta_0, \theta_1, \varphi_0, \varphi_1) \tag{4.41}$$

where, on the right-hand-side, the arguments are taken in the sense of (2.94) – (2.98).

Remark 4.14. $f(\alpha, \theta_0, \theta_1, \varphi_0, \varphi_1)$ is meant to be a polynomial in

$$\alpha, \theta_0^1, \ldots, \theta_0^7, \theta_1^0, \ldots, \theta_1^1, \varphi_0^1, \ldots, \varphi_0^7, \varphi_1^0, \ldots, \varphi_1^1.$$

Notice that (4.41) can be differentiated by means of the moving frames. In this connection, let us now adjust the Maurer-Cartan equations on $\overline{\mathrm{Spin}(9)}$

$$\mathrm{d}\theta_k^a = -\sum_{j=0}^{1}\sum_{c=0}^{7} \varphi_{k,j}^{a,c} \wedge \theta_j^c, \tag{2.105}$$

$$\mathrm{d}\varphi_{k,l}^{a,b} = -\sum_{j=0}^{1}\sum_{c=0}^{7}\varphi_{k,j}^{a,c}\wedge\varphi_{j,l}^{c,b} \tag{2.106}$$

into a compact form that can be regarded as an *octonionic* version of the *structure equations* and that will be particularly convenient for our purpose.

First, it is necessary to employ the description of the Lie algebra $\mathfrak{spin}(9)$ derived in §2.4.5. In particular, according to Remark 2.57, the Maurer-Cartan form on $\overline{\mathrm{Spin}(9)}$ is the collection of the following 52 (independent) 1-forms:

$$\begin{aligned}
&\alpha = \theta_0^0, && \\
&\theta_0^a, && 1 \le a \le 7, \\
&\theta_1^a, && 0 \le a \le 7, \\
&\varphi_0^a = \varphi_{0,0}^{a,0}, && 1 \le a \le 7, \\
&\varphi_1^a = \varphi_{1,0}^{a,0}, && 0 \le a \le 7, \\
&\varphi_{1,1}^{a,b}, && 1 \le b < a \le 7,
\end{aligned} \tag{4.42}$$

whereas the other entries of φ are expressed in terms of them via (2.122) – (2.130).

Second is the following observation: Differentiating (4.41) yields

$$\pi^*\mathrm{d}\beta = \mathrm{d}f(\alpha,\theta_0,\theta_1,\varphi_0,\varphi_1). \tag{4.43}$$

Since $\mathrm{d}\beta \in \Omega^\bullet(S\mathbb{O}^2)^{\overline{\mathrm{Spin}(9)}}$, we may apply Corollary 4.13 to conclude that (4.43) is again expressed entirely in terms of $\alpha, \theta_0^a, \theta_1^a, \varphi_0^a, \varphi_1^a$ and contains, a priori, no $\varphi_{1,1}^{a,b}$. It is, hence, sufficient to consider the Maurer-Cartan equations (subject to the relations (2.122) – (2.130)) modulo the following congruence relation:

$$\equiv \quad \mathrm{mod}\ \mathbb{O}\otimes\mathrm{span}\{\varphi_{1,1}^{a,b}\,;\,1\le b<a\le 7\}. \tag{4.44}$$

Lemma 4.15. *On $\Omega_{\mathbb{O}}^\bullet(\overline{\mathrm{Spin}(9)})$, one has*

$$\begin{aligned}
\mathrm{d}(\alpha+\theta_0) &\equiv -\varphi_0\wedge(\alpha+\theta_0) - \theta_1\wedge\overline{\varphi_1}, &(4.45)\\
\mathrm{d}\theta_1 &\equiv (\alpha+\theta_0)\wedge\varphi_1 - \varphi_0\wedge\theta_1 + \theta_1\wedge\varphi_0, &(4.46)\\
\mathrm{d}\varphi_0 &\equiv -\varphi_0\wedge\varphi_0 - \varphi_1\wedge\overline{\varphi_1}, &(4.47)\\
\mathrm{d}\varphi_1 &\equiv \varphi_1\wedge\varphi_0. &(4.48)
\end{aligned}$$

Remark 4.16. Considering separately the real and the imaginary part of (4.45) yields formulas for $\mathrm{d}\alpha$ and $\mathrm{d}\theta_0$, respectively. More generally, inner product with e_i gives us the exterior derivative of the i-th component of either of the four octonion-valued forms.

Proof. First of all, according to Corollary 2.56 (see also Remark 2.57) we have

$$\begin{aligned}
&\varphi_{1,0}^{a,b} \equiv \varphi_{1,0}^{\bar{b}a,0}, && 0\le a\le 7,\quad 1\le b\le 7,\\
&\varphi_{0,1}^{a,b} \equiv -\varphi_{1,0}^{\bar{a}b,0}, && 0\le a,b\le 7,\\
&\varphi_{k,k}^{a,a} \equiv 0, && 0\le a\le 7,\quad k=0,1,\\
&\varphi_{1,1}^{a,b} \equiv 0, && 1\le a<b\le 7,\\
&\varphi_{1,1}^{a,0} \equiv 2\varphi_{0,0}^{a,0}, && 1\le a\le 7,\\
&\varphi_{1,1}^{0,a} \equiv -2\varphi_{0,0}^{a,0}, && 1\le a\le 7,\\
&\varphi_{0,0}^{a,b} \equiv -\varphi_{0,0}^{\bar{b}a,0}, && 1\le b<a\le 7,
\end{aligned}$$

$$\varphi^{a,b}_{0,0} \equiv \varphi^{\overline{a}b,0}_{0,0}, \qquad 1 \le a < b \le 7,$$
$$\varphi^{0,a}_{0,0} \equiv -\varphi^{a,0}_{0,0}, \qquad 1 \le a \le 7.$$

Observe that, since for $a, b > 0$ such that $a \neq b$ one has $\overline{b}a = -a\overline{b}$, $\overline{a}b = a\overline{b}$ and $\overline{a} = -a$, the last three relations can be written together as

$$\varphi^{a,b}_{0,0} \equiv \varphi^{a\overline{b},0}_{0,0}, \quad 0 \le a, b \le 7.$$

Now we can proceed to computing the differentials modulo the congruence (4.44). All summations are taken from 0 to 7 throughout the rest of the proof. First,

$$\begin{aligned}
\mathrm{d}(\alpha + \theta_0) &= \sum_a e_a \mathrm{d}\theta^a_0 \\
&= -\sum_{a,c} e_a \varphi^{a,c}_{0,0} \wedge \theta^c_0 - \sum_{a,c} e_a \varphi^{a,c}_{0,1} \wedge \theta^c_1 \\
&\equiv -\sum_{a,c} e_a \varphi^{a\overline{c},0}_{0,0} \wedge \theta^c_0 + \sum_{a,c} e_a \varphi^{\overline{a}c,0}_{1,0} \wedge \theta^c_1 \\
&= -\sum_{b,c} e_b e_c \varphi^{b,0}_{0,0} \wedge \theta^c_0 + \sum_{b,c} e_c \overline{e_b} \varphi^{b,0}_{1,0} \wedge \theta^c_1 \\
&= -\varphi_0 \wedge (\alpha + \theta_0) - \sum_{b,c} e_c \overline{e_b} \theta^c_1 \wedge \varphi^{b,0}_{1,0} \\
&= -\varphi_0 \wedge (\alpha + \theta_0) - \theta_1 \wedge \overline{\varphi_1}.
\end{aligned}$$

Let us comment on the substitution we made use of in the fourth step. It is easily seen that $\{|a\overline{c}| \,;\, 0 \le a \le 7\} = \{0, \ldots, 7\}$ holds for any c. Thus, instead of summing over a, we may sum over $b = a\overline{c}$ in $\sum_a e_a \varphi^{a\overline{c},0}_{0,0}$, where $e_a = e_{bc} = e_b e_c$ then holds. Further, if $|b| = -b$, then $e_{|b|} = -e_b$ but also $\varphi^{|b|,0}_{0,0} = -\varphi^{b,0}_{0,0}$, so we may in fact sum over $|b|$, which is what we do. This trick will be used repeatedly in the sequel. Second,

$$\begin{aligned}
\mathrm{d}\theta_1 &= \sum_a e_a \mathrm{d}\theta^a_1 \\
&= -\sum_{a,c} e_a \varphi^{a,c}_{1,0} \wedge \theta^c_0 - \sum_{a,c} e_a \varphi^{a,c}_{1,1} \wedge \theta^c_1 \\
&\equiv -\sum_{a,c} e_a \varphi^{\overline{c}a,0}_{1,0} \wedge \theta^c_0 - \sum_{a} e_a \varphi^{a,0}_{1,1} \wedge \theta^0_1 - \sum_{c} e_0 \varphi^{0,c}_{1,1} \wedge \theta^c_1 \\
&\equiv -\sum_{b,c} e_c e_b \varphi^{b,0}_{1,0} \wedge \theta^c_0 - 2\sum_{a} e_a \varphi^{a,0}_{0,0} \wedge \theta^0_1 + 2\sum_{c} \varphi^{c,0}_{0,0} \wedge \theta^c_1 \\
&= \sum_{b,c} e_c e_b \theta^c_0 \wedge \varphi^{b,0}_{1,0} - 2\varphi_0 \wedge \operatorname{Re}\theta_1 + 2\operatorname{Re}(\varphi_0 \wedge \overline{\theta_1}) \\
&= (\alpha + \theta_0) \wedge \varphi_1 - \varphi_0 \wedge (\theta_1 + \overline{\theta_1}) + \varphi_0 \wedge \overline{\theta_1} - \theta_1 \wedge \overline{\varphi_0} \\
&= (\alpha + \theta_0) \wedge \varphi_1 - \varphi_0 \wedge \theta_1 + \theta_1 \wedge \varphi_0,
\end{aligned}$$

where (3.21) and (3.24) were used as well as $\overline{\varphi_0} = -\varphi_0$. Third,

$$\begin{aligned}
\mathrm{d}\varphi_0 &= \sum_a e_a \mathrm{d}\varphi^{a,0}_{0,0} \\
&= -\sum_{a,c} e_a \varphi^{a,c}_{0,0} \wedge \varphi^{c,0}_{0,0} - \sum_{a,c} e_a \varphi^{a,c}_{0,1} \wedge \varphi^{c,0}_{1,0} \\
&\equiv -\sum_{a,c} e_a \varphi^{a\overline{c},0}_{0,0} \wedge \varphi^{c,0}_{0,0} + \sum_{a,c} e_a \varphi^{\overline{a}c,0}_{1,0} \wedge \varphi^{c,0}_{1,0} \\
&= -\sum_{b,c} e_b e_c \varphi^{b,0}_{0,0} \wedge \varphi^{c,0}_{0,0} + \sum_{b,c} e_c \overline{e_b} \varphi^{b,0}_{1,0} \wedge \varphi^{c,0}_{1,0}
\end{aligned}$$

$$= -\varphi_0 \wedge \varphi_0 - \sum_{b,c} e_c \overline{e_b} \varphi_{1,0}^{c,0} \wedge \varphi_{1,0}^{b,0}$$
$$= -\varphi_0 \wedge \varphi_0 - \varphi_1 \wedge \overline{\varphi_1}.$$

Finally, according to (3.21), (3.24), and $\overline{\varphi_0} = -\varphi_0$ again,

$$\begin{aligned}
\mathrm{d}\varphi_1 &= \sum_a e_a \mathrm{d}\varphi_{1,0}^{a,0} \\
&= -\sum_{a,c} e_a \varphi_{1,0}^{a,c} \wedge \varphi_{0,0}^{c,0} - \sum_{a,c} e_a \varphi_{1,1}^{a,c} \wedge \varphi_{1,0}^{c,0} \\
&\equiv -\sum_{a,c} e_a \varphi_{1,0}^{\bar{c}a,0} \wedge \varphi_{0,0}^{c,0} - \sum_a e_a \varphi_{1,1}^{a,0} \wedge \varphi_{1,0}^{c,0} - \sum_c e_0 \varphi_{1,1}^{0,c} \wedge \varphi_{1,0}^{c,0} \\
&\equiv -\sum_{b,c} e_c e_b \varphi_{1,0}^{b,0} \wedge \varphi_{0,0}^{c,0} - 2\sum_a e_a \varphi_{0,0}^{a,0} \wedge \varphi_{1,0}^{0,0} + 2\sum_c \varphi_{0,0}^{c,0} \wedge \varphi_{1,0}^{c,0} \\
&= \sum_{b,c} e_c e_b \varphi_{0,0}^{c,0} \wedge \varphi_{1,0}^{b,0} - 2\varphi_0 \operatorname{Re}\varphi_1 + 2\operatorname{Re}(\varphi_0 \wedge \overline{\varphi_1}) \\
&= \varphi_0 \wedge \varphi_1 - \varphi_0 \wedge (\varphi_1 + \overline{\varphi_1}) + \varphi_0 \wedge \overline{\varphi_1} - \varphi_1 \wedge \overline{\varphi_0} \\
&= \varphi_1 \wedge \varphi_0.
\end{aligned}$$ □

4.3.2 Exterior Differentials of the Generating Forms

Let us conclude this section by an explicit recipe for differentiating the generators (4.37) of the algebra $\Omega^\bullet(S\mathbb{O}^2)^{\overline{\mathrm{Spin}(9)}}$. The three main ingredients are: Lemma 2.50, the Octonionic structure equations (Lemma 4.15), and the anti-derivation property of d.

First of all, consider a subalgebra $\mathcal{S}_h \subset \Omega^\bullet(\overline{\mathrm{Spin}(9)})$ generated by

$$\alpha, \theta_0^1, \ldots, \theta_0^7, \theta_1^0, \ldots, \theta_1^1, \varphi_0^1, \ldots, \varphi_0^7, \varphi_1^0, \ldots, \varphi_1^1, \tag{4.49}$$

and let us define a linear anti-derivative operator d_h on $\mathcal{S}_h$ as follows (see Lemma 4.15):

$$\mathrm{d}_h(\alpha + \theta_0) = -\varphi_0 \wedge (\alpha + \theta_0) - \theta_1 \wedge \overline{\varphi_1}, \tag{4.50}$$
$$\mathrm{d}_h\theta_1 = (\alpha + \theta_0) \wedge \varphi_1 - \varphi_0 \wedge \theta_1 + \theta_1 \wedge \varphi_0, \tag{4.51}$$
$$\mathrm{d}_h\varphi_0 = -\varphi_0 \wedge \varphi_0 - \varphi_1 \wedge \overline{\varphi_1}, \tag{4.52}$$
$$\mathrm{d}_h\varphi_1 = \varphi_1 \wedge \varphi_0. \tag{4.53}$$

Considering coordinates of these octonion-valued forms in the standard basis of $\mathbb{O}$ then gives values of d_h on the generators (4.49). Recall from §4.3.1 that $\pi^* \left[\Omega^\bullet(S\mathbb{O}^2)\right] \subset \mathcal{S}_h$ and that, on the former space, $\mathrm{d} = \mathrm{d}_h$.

Theorem 4.17. *Consider a generator $[k, l, m, n] \in \Omega^\bullet(S\mathbb{O}^2)^{\overline{\mathrm{Spin}(9)}}$. First, there is f such that*

$$\mathrm{d}_h\Big(]k, l, m, n[\Big) = f(\alpha, \theta_0, \theta_1, \varphi_0, \varphi_1). \tag{4.54}$$

Second, one has

$$(\mathrm{d}[k, l, m, n])_p = f(\alpha_p, \theta_0, \theta_1, \varphi_0, \varphi_1) \tag{4.55}$$

where, on the right-hand-side of (4.55)*, the arguments are taken in the sense of* (4.15).

Remark 4.18.
(a) Remember that $]k,l,m,n[= \pi^*[k,l,m,n]$ is given by formally the same expression as $[k,l,m,n]_p$. Therefore, for any $\beta \in \Omega^\bullet(S\mathbb{O}^2)^{\overline{\mathrm{Spin}(9)}}$, Theorem 4.17 allows us to deduce $(\mathrm{d}\beta)_p$ *only* from β_p.
(b) According to Theorem 4.5, $(\mathrm{d}\beta)_p$ is a polynomial in the generators $[a,b,c,d]_p$. It is just the matter of notation that $\mathrm{d}\beta$ is then given by precisely the same polynomial, but in the corresponding generators $[a,b,c,d]$.

Proof. First, since $]k,l,m,n[\in \pi^*\Omega^\bullet(S\mathbb{O}^2)$, we have

$$\mathrm{d}_h\Big(]k,l,m,n[\Big) = \mathrm{d}\Big(]k,l,m,n[\Big) = \pi^*\mathrm{d}[k,l,m,n] = \mathrm{d}\pi^*[k,l,m,n].$$

As have already seen in §4.3.1, because $\mathrm{d}[k,l,m,n] \in \Omega^\bullet(S\mathbb{O}^2)^{\overline{\mathrm{Spin}(9)}}$, Corollary 4.13 indeed implies existence of a polynomial f such that (4.54) holds.

Second, according to Lemma 2.50, $f(\alpha,\theta_0,\theta_1,\varphi_0,\varphi_1)$ is right $\mathrm{Spin}(7)$ invariant. Let $\beta_p = f(\alpha_p,\theta_0,\theta_1,\varphi_0,\varphi_1) \in \bigwedge^\bullet(T_pS\mathbb{O}^2)$, i.e. the arguments taken in the sense of (4.15). Then Theorem 4.5 and the transformation rules in Lemma 2.54 imply that β_p is $\mathrm{Spin}(7)$ invariant, i.e. it is the value in p of a certain $\beta \in \Omega^\bullet(S\mathbb{O}^2)^{\overline{\mathrm{Spin}(7)}}$. According to Corollary 4.13 and the construction of this form, one has

$$\pi^*\beta = f(\alpha,\theta_0,\theta_1,\varphi_0,\varphi_1) = \mathrm{d}_h\Big(]k,l,m,n[\Big) = \pi^*\mathrm{d}[k,l,m,n].$$

Finally, the uniqueness part of Lemma 2.50 yields $\beta = \mathrm{d}[k,l,m,n]$. In particular, in the point p, one has (4.55). $\square$

Example 4.19. Sometimes it is necessary to split the derivation rules (4.50) – (4.53) into the real coordinates, for instance in order to differentiate the 'determinantal' generators $[k,0,7-k,0]$, c.f. (4.33). In other cases, however, it may be more convenient to work with the octonion-valued forms. To illustrate this, consider the following computation in $\Omega^\bullet_{\mathbb{O}}(\overline{\mathrm{Spin}(9)})$:

$$\begin{aligned}
\mathrm{d}_h\,]1,0,0,0[&= \mathrm{d}_h\alpha \\
&= \mathrm{Re}\left[-\varphi_0\wedge(\alpha+\theta_0) - \theta_1\wedge\overline{\varphi_1}\right] \\
&= -\mathrm{Re}(\varphi_0)\wedge\alpha - \mathrm{Re}(\varphi_0\wedge\theta_0) - \mathrm{Re}(\theta_1\wedge\overline{\varphi_1}) \\
&= -\frac{1}{2}(\varphi_0\wedge\theta_0 - \theta_0\wedge\varphi_0) - \mathrm{Re}(\overline{\theta_1}\wedge\varphi_1) \\
&= \mathrm{Re}(\theta_0\wedge\varphi_0) - \mathrm{Re}(\overline{\theta_1}\wedge\varphi_1) \\
&= -\,]1,0,1,0[-]0,1,0,1[,
\end{aligned}$$

where we used (3.21), (3.24), and $\overline{\alpha}=\alpha$, $\overline{\theta_0}=-\theta_0$, $\overline{\varphi_0}=-\varphi_0$. We conclude that

$$(\mathrm{d}\alpha)_p = -[1,0,1,0]_p - [0,1,0,1]_p \in \bigwedge\nolimits^\bullet(T_pS\mathbb{O}^2)^{\mathrm{Spin}(7)},$$

and, finally, on $\Omega^\bullet(S\mathbb{O}^2)^{\overline{\mathrm{Spin}(9)}}$ we have

$$\mathrm{d}\alpha = -[1,0,1,0] - [0,1,0,1]. \tag{4.56}$$

4.4 Integral Geometry on the Octonionic Plane

The contents of the previous sections and chapters will be finally synthesized into a description of the algebra of Spin(9)-invariant valuations and, consequently, into the Principal kinematic formula on the octonionic plane. This section forms the nucleus of our thesis.

4.4.1 Minimal Generating Set

In order to formulate precisely a particular aspect of our main result, we shall need the following statement of algebraic nature which is probably well known to algebraists and follows, most likely, from certain more general considerations. Let us, nonetheless, give a direct proof.

Consider a commutative associative unital graded (real) algebra $\mathcal{A} = \bigoplus_{k=0}^{n} \mathcal{A}_k$ with $\mathcal{A}_0 \cong \mathbb{R}$ and $1 < \dim \mathcal{A} < \infty$. $\mathcal{A}$ is finitely generated, e.g. by a basis. Hence the set

$$M = \{l \in \mathbb{N}\,;\text{ there are } g_1, \ldots, g_l \in \mathcal{A} \text{ that generate } \mathcal{A}\} \tag{4.57}$$

has a minimum. Denote $m = \min M$. A generating set $\{g_1, \ldots, g_m\} \subset \mathcal{A}$ of cardinality m is then said to be *minimal*.

Lemma 4.20. *$\mathcal{A}$ admits a minimal generating set consisting entirely of homogeneous elements.*

Proof. Let us construct a homogeneous generating set as follows: First, choose a basis $h_1, \ldots, h_{m_1}$ of $\mathcal{A}_1$. Second, pick a basis of $\mathcal{A}_1^2 \subset \mathcal{A}_2$ and complete it to a basis of $\mathcal{A}_2$ by adding $h_{m_1+1}, \ldots, h_{m_2}$. Similarly, in the k-th step, pick a basis of

$$\sum \mathcal{A}_1^{i_1} \cdots \mathcal{A}_{k-1}^{i_{k-1}} \subset \mathcal{A}_k,$$

where the sum of vector spaces runs over all tuples $(i_1, \ldots, i_{k-1})$ of non-negative integers such that $\sum_{l=1}^{k-1} l \cdot i_l = k$ and complete it to a basis of $\mathcal{A}_k$ by adding $h_{m_{k-1}+1}, \ldots, h_{m_k}$. Continue until $k = n$. Obviously, $\{h_1, \ldots, h_{m_n}\}$ is a homogeneous generating set.

Next, consider a minimal generating set $\{g_1, \ldots, g_m\}$. We shall show that necessarily $m_n = n$. Clearly, we may assume $\{g_1, \ldots, g_m\} \subset \bigoplus_{k=1}^{n} \mathcal{A}_k$ without loss of any generality. Then there are constants $a_{i,j} \in \mathbb{R}$ and polynomials $p_i \in \mathbb{R}[\lambda_1, \ldots, \lambda_m]$ with $p_i(0) = \frac{\partial p_i}{\partial \lambda_j}(0) = 0$, for all $1 \le i \le m_n$ and $1 \le j \le m$, such that

$$h_i = \sum_{j=1}^{m} a_{i,j} g_j + p_i(g_1, \ldots, g_m), \quad 1 \le i \le m_n.$$

Conversely, there are $\alpha_j \in \mathbb{R}$ and $q_j \in \mathbb{R}[\lambda, \ldots, \lambda_{m_n}]$ with $q_j(0) = \frac{\partial q_j}{\partial \lambda_1}(0) = 0$, for all $1 \le j \le m$, such that

$$g_j = \alpha_j h_1 + q_i(h_1, \ldots, h_{m_n}), \quad 1 \le j \le m. \tag{4.58}$$

Together, we have

$$h_i = \left(\sum_{j=1}^{m} a_{i,j} \alpha_j \right) h_1 + r_i(h_1, \ldots, h_{m_n}), \quad 1 \le i \le m_n, \tag{4.59}$$

for some $r_i \in \mathbb{R}[\lambda_1, \ldots, \lambda_{m_n}]$ with $r_j(0) = \frac{\partial r_j}{\partial \lambda_1}(0) = 0$, $1 \le j \le m$. It is clear from the construction that any h_i cannot be expressed as a polynomial in $h_1, \ldots, h_{m_n}$ with trivial constant- and h_i-linear term (otherwise it would not be a basis vector for $\mathcal{A}_{k_i}$, $h_i \in \mathcal{A}_{k_i}$). In particular, this applies to h_1. Thus, collecting in (4.59) the coefficients standing in front of h_1, we conclude that the following linear system must have a solution in $\mathbb{R}^m$:

$$\begin{pmatrix} a_{1,1} & \cdots & a_{1,m} \\ \vdots & & \vdots \\ a_{m_n,1} & \cdots & a_{m_n,m} \end{pmatrix} \begin{pmatrix} \alpha_1 \\ \vdots \\ \alpha_m \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}, \tag{4.60}$$

in particular,

$$\operatorname{rank} \begin{pmatrix} a_{2,1} & \cdots & a_{2,m} \\ \vdots & & \vdots \\ a_{m_n,1} & \cdots & a_{m_n,m} \end{pmatrix} < m. \tag{4.61}$$

Since $\{g_1, \dots, g_m\}$ is minimal, we have $m_n \geq m$. Suppose $m_n > m$. Then (4.61) implies existence of $l \in \{2, \dots, m_n\}$ such that the l-th row of the matrix in (4.60) is a linear combination of the others. However, we can repeat the same construction for any other member of our homogeneous generating set, in particular for h_l. Namely, we have

$$g_j = \tilde{\alpha}_j h_l + \tilde{q}_i(h_1, \dots, h_{m_n}), \quad 1 \leq j \leq m,$$

for some $\tilde{\alpha}_j \in \mathbb{R}$ and $\tilde{q}_j \in \mathbb{R}[\lambda, \dots, \lambda_{m_n}]$ with $\tilde{q}_j(0) = \frac{\partial \tilde{q}_j}{\partial \lambda_l}(0) = 0$, for all $1 \leq j \leq m$, instead of (4.58) and consequently there must be a solution of

$$\begin{pmatrix} a_{1,1} & \cdots & a_{1,m} \\ \vdots & & \vdots \\ a_{m_n,1} & \cdots & a_{m_n,m} \end{pmatrix} \begin{pmatrix} \tilde{\alpha}_1 \\ \vdots \\ \tilde{\alpha}_m \end{pmatrix} = \begin{pmatrix} \vdots \\ 0 \\ 1 \\ 0 \\ \vdots \end{pmatrix}, \tag{4.62}$$

where, on the right-hand side, the 1 is in the l-th row. But this is impossible as the l-th row of the matrix on the left is a linear combination of the others. Hence we conclude that $m_n = m$ in fact. $\square$

Remark 4.21. Any minimal homogeneous generating set $\{h_1, \dots, h_m\}$ obviously arises through the construction described in the first part of the previous proof. Observe also that the quantities

$$\operatorname{gen}_k \mathcal{A} = \#\left(\{h_1, \dots, h_m\} \cap \mathcal{A}_k\right), \quad 1 \leq k \leq n. \tag{4.63}$$

are well-defined invariants of $\mathcal{A}$.

4.4.2 The Algebra $\mathrm{Val}(\mathbb{O}^2)^{\mathrm{Spin}(9)}$

Theorem 4.22. $\left(\mathrm{Val}(\mathbb{O}^2)^{\mathrm{Spin}(9)}, *\right)$ *is generated, as an algebra, by the following 12 elements:*

$$\begin{aligned}
t &= \frac{1}{14}\left[\!\!\left[[0,4,0,0] \wedge [0,4,0,0] \wedge [7,0,0,0]\right]\!\!\right] \in \mathrm{Val}_{16-1}, \\
s &= \frac{1}{14}\left[\!\!\left[[0,4,0,0] \wedge [0,4,0,0] \wedge [6,0,1,0]\right]\!\!\right] \in \mathrm{Val}_{16-2}, \\
v &= \frac{1}{14}\left[\!\!\left[[0,4,0,0] \wedge [0,4,0,0] \wedge [5,0,2,0]\right]\!\!\right] \in \mathrm{Val}_{16-3}, \\
u_1 &= \frac{1}{14}\left[\!\!\left[[0,4,0,0] \wedge [0,4,0,0] \wedge [4,0,3,0]\right]\!\!\right] \in \mathrm{Val}_{16-4}, \\
u_2 &= \left[\!\!\left[[0,4,0,0] \wedge [0,1,2,1] \wedge [7,0,0,0]\right]\!\!\right] \in \mathrm{Val}_{16-4},
\end{aligned}$$

$$
\begin{aligned}
w_1 &= \frac{1}{14}\Big[\Big[[0,4,0,0]\wedge[0,4,0,0]\wedge[3,0,4,0]\Big]\Big] \in \mathrm{Val}_{16-5},\\
w_2 &= \Big[\Big[[0,4,0,0]\wedge[0,0,2,2]\wedge[7,0,0,0]\Big]\Big] \in \mathrm{Val}_{16-5},\\
w_3 &= \Big[\Big[[0,4,0,0]\wedge[0,0,0,4]\wedge[7,0,0,0]\Big]\Big] \in \mathrm{Val}_{16-5},\\
x_1 &= \frac{1}{14}\Big[\Big[[0,4,0,0]\wedge[0,4,0,0]\wedge[2,0,5,0]\Big]\Big] \in \mathrm{Val}_{16-6},\\
x_2 &= \Big[\Big[[0,4,0,0]\wedge[1,1,1,1]\wedge[4,0,3,0]\Big]\Big] \in \mathrm{Val}_{16-6},\\
y &= \frac{1}{14}\Big[\Big[[0,4,0,0]\wedge[0,4,0,0]\wedge[1,0,6,0]\Big]\Big] \in \mathrm{Val}_{16-7},\\
z &= \frac{1}{14}\Big[\Big[[0,4,0,0]\wedge[0,4,0,0]\wedge[0,0,7,0]\Big]\Big] \in \mathrm{Val}_{16-8},
\end{aligned}
$$

subject to 93 independent homogeneous relations specified in Appendix B, such that

$$
\begin{aligned}
\mathrm{Val}_{16-0}^{\mathrm{Spin}(9)} &= \mathrm{span}\left\{\mathrm{vol}_{16}\right\},\\
\mathrm{Val}_{16-1}^{\mathrm{Spin}(9)} &= \mathrm{span}\left\{t\right\},\\
\mathrm{Val}_{16-2}^{\mathrm{Spin}(9)} &= \mathrm{span}\left\{t^2, s\right\},\\
\mathrm{Val}_{16-3}^{\mathrm{Spin}(9)} &= \mathrm{span}\left\{t^3, ts, v\right\},\\
\mathrm{Val}_{16-4}^{\mathrm{Spin}(9)} &= \mathrm{span}\left\{t^4, t^2s, tv, s^2, u_1, u_2\right\},\\
\mathrm{Val}_{16-5}^{\mathrm{Spin}(9)} &= \mathrm{span}\left\{t^5, t^3s, t^2v, ts^2, tu_1, tu_2, sv, w_1, w_2, w_3\right\},\\
\mathrm{Val}_{16-6}^{\mathrm{Spin}(9)} &= \mathrm{span}\left\{t^6, t^4s, t^3v, t^2s^2, t^2u_1, t^2u_2, tsv, tw_1, tw_2, tw_3, s^3, su_1, su_2, x_1, x_2\right\},\\
\mathrm{Val}_{16-7}^{\mathrm{Spin}(9)} &= \mathrm{span}\Big\{t^7, t^5s, t^4v, t^3s^2, t^3u_1, t^3u_2, t^2sv, t^2w_1, t^2w_2, t^2w_3,\\
&\qquad ts^3, tsu_1, tsu_2, tx_1, tx_2, s^2v, sw_1, sw_2, sw_3, y\Big\},\\
\mathrm{Val}_{16-8}^{\mathrm{Spin}(9)} &= \mathrm{span}\Big\{t^8, t^6s, t^5v, t^4s^2, t^4u_1, t^4u_2, t^3sv, t^3w_1, t^3w_2, t^3w_3, t^2s^3, t^2su_1, t^2su_2,\\
&\qquad t^2x_1, t^2x_2, ts^2v, tsw_1, tsw_2, tsw_3, ty, s^4, s^2u_1, s^2u_2, sx_1, vw_2, vw_3, z\Big\},\\
\mathrm{Val}_{16-9}^{\mathrm{Spin}(9)} &= \mathrm{span}\Big\{t^9, t^7s, t^6v, t^5s^2, t^5u_1, t^5u_2, t^4sv, t^4w_1, t^4w_2, t^4w_3,\\
&\qquad t^3s^3, t^3su_1, t^3su_2, t^3x_1, t^3x_2, t^2s^2v, t^2sw_1, t^2sw_2, t^2sw_3, t^2y\Big\},\\
\mathrm{Val}_{16-10}^{\mathrm{Spin}(9)} &= \mathrm{span}\Big\{t^{10}, t^8s, t^7v, t^6s^2, t^6u_1, t^6u_2, t^5sv, t^5w_1,\\
&\qquad t^5w_2, t^5w_3, t^4s^3, t^4su_1, t^4su_2, t^4x_1, t^4x_2\Big\},\\
\mathrm{Val}_{16-11}^{\mathrm{Spin}(9)} &= \mathrm{span}\left\{t^{11}, t^9s, t^8v, t^7s^2, t^7u_1, t^7u_2, t^6sv, t^6w_1, t^6w_2, t^6w_3\right\},\\
\mathrm{Val}_{16-12}^{\mathrm{Spin}(9)} &= \mathrm{span}\left\{t^{12}, t^{10}s, t^9v, t^8s^2, t^8u_1, t^8u_2\right\},\\
\mathrm{Val}_{16-13}^{\mathrm{Spin}(9)} &= \mathrm{span}\left\{t^{13}, t^{11}s, t^{10}v\right\},\\
\mathrm{Val}_{16-14}^{\mathrm{Spin}(9)} &= \mathrm{span}\left\{t^{14}, t^{12}s\right\},\\
\mathrm{Val}_{16-15}^{\mathrm{Spin}(9)} &= \mathrm{span}\left\{t^{15}\right\},\\
\mathrm{Val}_{16-16}^{\mathrm{Spin}(9)} &= \mathrm{span}\left\{t^{16}\right\},
\end{aligned}
$$

where each of the sets is also linearly independent, i.e. a basis of the respective subspace.

Proof. We shall construct a basis and determine the convolution at the same time, using an inductive algorithm analogous to the first part of the proof of Lemma 4.20. Let us keep the notation (4.18) and to extend it also to the bi-degree ($\bigwedge^{l,m}$) and the degree ($\bigwedge^n$) throughout the proof.

First of all, recall from §1.2.3 and §1.3.1 that for any $\mu \in \mathrm{Val}_{16-k}^{\mathrm{Spin}(9)}$, $1 \leq k \leq 16$, there exists $\omega \in \Omega^{16-k,k-1}(S\mathbb{O}^2)^{\overline{\mathrm{Spin}(9)}}$ with $\mu = [[\omega]]$. According to Theorem 4.11, ω is a polynomial in the 97 generators listed therein.

Let us describe how to compute the Rumin differential $\mathrm{D}\omega$ defined in §1.2.4. Recall that (see also Proposition 1.37)

$$\mathrm{D}\omega = \mathrm{d}(\omega + \alpha \wedge \xi) = \mathrm{d}\omega + \mathrm{d}\alpha \wedge \xi - \alpha \wedge \mathrm{d}\xi, \tag{4.64}$$

where $\xi \in \Omega_h^{14}(S\mathbb{O}^2)^{\overline{\mathrm{Spin}(9)}}$ is the unique element such that (4.64) is vertical, i.e. with

$$\mathrm{d}\omega + \mathrm{d}\alpha \wedge \xi \equiv 0 \mod \alpha. \tag{4.65}$$

Because all the forms here are invariant, we may do all the computations in the point $p = (0, E_0) \in S\mathbb{O}^2$, $E_0 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$. The first step, thus, is to describe the space

$$\bigwedge^{14} \cong \Omega_h^{14}(S\mathbb{O}^2)^{\overline{\mathrm{Spin}(9)}}.$$

To this end, we consider all possible combinations of the generators given in Theorem 4.5 that give degree 14. Using MAPLE, we express them in the standard dual basis given by (4.15) and pick a basis among them. Second, we compute $(\mathrm{d}\omega)_p$ according to Theorem 4.17. This is done again in coordinates and with MAPLE. Observe from the differentiating rules (4.50) – (4.53) that $\mathrm{d}\omega$ has bi-degree $(16-k, k)$. Third, we need to solve the linear problem (4.65) for ξ_p (also (4.56)). To this end, we simply let the computer to find the right linear combination of the basis vectors constructed earlier. The following two simplifications are available: First, it is now clear that $\xi_p \in \bigwedge^{15-k,k-1}$. Second, it is convenient to consider

$$\bigwedge^{15-k,k-1} = \bigoplus_m \left[\bigoplus_j \bigwedge^{j,15-k-j,m+j,k-1-m-j} \right] \tag{4.66}$$

since the multiplication by $(\mathrm{d}\alpha)_p \in \bigwedge^{1,1}$ then acts diagonally with respect to the outer grading.

Everything is prepared now to start the algorithm. Theorems 1.33, 1.41, and 4.3 will be used in the sequel. We start with $k = 1$ and take

$$\omega = [0,4,0,0] \wedge [0,4,0,0] \wedge [7,0,0,0] \in \Omega^{15,0}(S\mathbb{O}^2)^{\overline{\mathrm{Spin}(9)}}.$$

Using the considerations from the first part of the proof, we let the computer to find $(\mathrm{D}\omega)_p \neq 0$. Consequently, $[[\omega]] \neq 0$, and since $\dim \mathrm{Val}_{16-1}^{\mathrm{Spin}(9)} = 1$, $[[\omega]]$ in fact spans $\mathrm{Val}_{16-1}^{\mathrm{Spin}(9)}$. Expressing ω in coordinates reveals that these are all integers divisible by 14 - we rescale ω accordingly and get the first generator t. In the second step, we first use the formulas (1.38) and (1.40) for the Bernig-Fu convolution and the Kernel theorem to compute $t^2 = t * t \neq 0$. As $\dim \mathrm{Val}_{16-2}^{\mathrm{Spin}(9)} = 2$, we need one new generator here to complete a basis. We choose s as above. As for the induction, assume we have found a basis for every $\mathrm{Val}_{16-j}^{\mathrm{Spin}(9)}$, $j = 1, 2, \ldots, k-1$. Consider all possible $\mu * \nu \in \mathrm{Val}_{16-k}$, where μ, ν are some basis vectors from our bases and choose the biggest linearly independent set among them. Those that do not fit into this set are expressible in those that do. We express them and record the corresponding relations in Appendix B. If the linearly independent set is not a basis of $\mathrm{Val}_{16-k}^{\mathrm{Spin}(9)}$ (i.e. of cardinality $\dim \mathrm{Val}_{16-k}^{\mathrm{Spin}(9)}$, see Theorem 4.3), we complete it to one by adding new generator(s). We continue until $k = 16$. $\square$

Let us briefly comment on certain aspects of the computational part of the proof. In general, we face two problems of linear algebra: We need to perform linear operations, in particular the wedge product, on the space $\bigwedge^\bullet(\mathrm{Im}\,\mathbb{O} \oplus \mathbb{O} \oplus \mathrm{Im}\,\mathbb{O} \oplus \mathbb{O})$, and we need to solve systems of linear equations, i.e. to invert matrices. As for the latter, this is unproblematic: Using in particular the decomposition (4.66), the problem is reduced to inverting rational matrices of size at most 110 which is done instantly by MAPLE.

The first task, however, is more subtle. Observe that the dimension of the space is

$$2^{30} = 1\,073\,741\,824.$$

Therefore, to compute a wedge product, for instance, the computer may need to conduct a huge number of basic operations. All our attempts to use conventional MAPLE procedures and packages for work with exterior algebra failed to be able to multiply generic forms in real time. Instead, the following idea turned out to be extremely useful and, in fact, made it even possible to bring the necessary computations to an end: First, the standard basis of $\bigwedge^\bullet(\mathrm{Im}\,\mathbb{O} \oplus \mathbb{O} \oplus \mathrm{Im}\,\mathbb{O} \oplus \mathbb{O})$ is in a bijective correspondence with the set

$$\{(K, L, M, N)\,;\, 0 \leq K, M \leq 2^7 - 1 = 127 \text{ and } 0 \leq L, N \leq 2^8 - 1 = 255\}.$$

Using the usual notation (4.15), it is convenient to redenote $\theta_1^8 = \theta_1^0$ and $\varphi_1^8 = \varphi_1^0$. Then, we identify the basis vector

$$\theta_0^{i_1^{(u)}} \wedge \cdots \wedge \theta_0^{i_k^{(u)}} \wedge \theta_1^{i_1^{(x)}} \wedge \cdots \wedge \theta_1^{i_l^{(x)}} \wedge \varphi_0^{i_1^{(v)}} \wedge \cdots \wedge \varphi_0^{i_m^{(v)}} \wedge \varphi_1^{i_1^{(y)}} \wedge \cdots \wedge \varphi_1^{i_n^{(y)}},$$

where

$$\begin{aligned} 1 &\leq i_1^{(u)} < \cdots < i_k^{(u)} \leq 7, \\ 1 &\leq i_1^{(x)} < \cdots < i_l^{(x)} \leq 8, \\ 1 &\leq i_1^{(v)} < \cdots < i_m^{(v)} \leq 7, \\ 1 &\leq i_1^{(y)} < \cdots < i_n^{(y)} \leq 8, \end{aligned}$$

with (K, L, M, N), where

$$K = \sum_{j=1}^{k} 2^{i_j^{(u)}-1}, \quad L = \sum_{j=1}^{l} 2^{i_j^{(x)}-1}, \quad M = \sum_{j=1}^{m} 2^{i_j^{(v)}-1}, \quad \text{and} \quad N = \sum_{j=1}^{n} 2^{i_j^{(y)}-1}.$$

In this representation, the wedge product is fairly simple: Namely, it is easily seen that it equals either

$$(K, L, M, N) \wedge (K', L', M', N') = \pm(K + K', L + L', M + M', N + N') \tag{4.67}$$

or it is trivial. The pairs of four-tuples for which this is the case can be easily computed (or seen) as well as the signs in (4.67). Moreover, this can be done just *once*, stored, and then read within each elementary operation. The 'difficult' part of the operation $\wedge$ is therefore transformed into the *addition of integers*. Similarly, for the Hodge $*$, one has

$$*(K, L, M, N) = \pm(127 - K, 255 - L, 127 - M, 255 - N). \tag{4.68}$$

It is clear from the proof of Theorem 4.22 that the (homogeneous) generating set we construct is minimal in the sense of §4.4.1. Consequently (see Remark 4.21),

Corollary 4.23. *The algebra* $\mathrm{Val}^{\mathrm{Spin}(9)}$ *satisfies*

k	1	2	3	4	5	6	7	8
$\mathrm{gen}_k \mathrm{Val}^{\mathrm{Spin}(9)}$	1	1	1	2	3	2	1	1

.

Remark 4.24. Notice that the previous statement is with respect to the codegree if $\mathrm{Val}(\mathbb{O}^2)^{\mathrm{Spin}(9)}$ is equipped with the Bernig-Fu convolution. Equivalently, one may consider its image under Alesker-Fourier transform, i.e. being endowed with the Alesker product. In that case Corollary 4.23 holds for the degree of homogeneity.

To compare the statement of Corollary 4.23 with its counterparts in the other known cases listed in §1.3.4, observe that, in particular,

k	1
$\mathrm{gen}_k \mathrm{Val}^{\mathrm{SO}(n)}$	1

,

k	1	2
$\mathrm{gen}_k \mathrm{Val}^{\mathrm{U}(n)}$	1	1

,

k	1	2	3	4
$\mathrm{gen}_k \mathrm{Val}^{\mathrm{Sp}(2)\mathrm{Sp}(1)}$	1	1	1	1

.

Similarly, one gets analogous tables for $\mathrm{SU}(n)$, G_2, and $\mathrm{Spin}(7)$. However, the numbers gen_k in these three reaming cases *disobey* a magical phenomenon that can be observed in the four tables above: Apart from gen_1 (which corresponds to the first intrinsic volume and may be perhaps viewed somewhat special) the remaining non-zero numbers of generators display the following 'hard-Lefschetz-type' behaviour:

$$\mathrm{gen}_2 \leq \mathrm{gen}_3 \leq \cdots \leq \mathrm{gen}_{\frac{d}{2}+1}, \quad \text{and} \quad \mathrm{gen}_k = \mathrm{gen}_{d+2-k}, \quad 2 \leq k \leq d, \tag{4.69}$$

where d is the dimension of the underlying normed division algebra. (4.69) is rather trivial in the three latter cases but the things are more interesting on the octonionic plane. In particular, the existence of the second generator in degree 6 is not required by dimensional reasons but rather it reflects the (unexpected) existence of a non-trivial relation of this degree (see the first paragraph of Appendix B). Although we believe that there might be some underlying principle, perhaps related to the structure of the corresponding normed division algebras, we currently have no understanding of this phenomenon at all.

4.4.3 The Principal Kinematic Formula

We shall explicitly determine the Principal kinematic formula on the octonionic plane, using the results of the previous section and the FTAIG (Theorem 1.52) in its special version (1.52). The main step is to compute the matrix of the Alesker-Poincaré pairing. Recall that this could be deduced from knowledge of the Bernin-Fu-convolution product on $\mathrm{Val}^{\mathrm{Spin}(9)}$ via (1.43).

Let us denote the Betti numbers of the algebra (see Theorem 4.3) by

$$d_k = \dim \mathrm{Val}_{16-k}^{\mathrm{Spin}(9)}, \quad 0 \leq k \leq 16, \tag{4.70}$$

and let

$$\Psi_k^{(1)}, \ldots, \Psi_k^{(d_k)} \tag{4.71}$$

be the basis of $\mathrm{Val}_{16-k}^{\mathrm{Spin}(9)}$ given in Theorem 4.22. Thus, the basis of $\mathrm{Val}^{\mathrm{Spin}(9)}$ is

$$\Psi_0^{(1)}, \Psi_1^{(1)}, \Psi_2^{(1)}, \Psi_2^{(2)}, \ldots, \Psi_k^{(1)}, \ldots, \Psi_k^{(d_k)}, \ldots, \Psi_{16}^{(1)}. \tag{4.72}$$

Observe also it was chosen such that

$$\Psi_8^{(i)} = t^{8-k} * \Psi_k^{(i)} \quad \text{and} \quad \Psi_{16-k}^{(i)} = t^{8-k} * \Psi_8^{(i)}, \quad 0 \leq k \leq 8, \quad 1 \leq i \leq d_k, \tag{4.73}$$

in particular,

$$\Psi_{16-k}^{(i)} = t^{16-2k} * \Psi_k^{(i)}, \quad 0 \leq k \leq 8, \quad 1 \leq i \leq d_k. \tag{4.74}$$

It is easily seen from (1.43) that the matrix of the Alesker-Poincaré pairing with respect to (4.72) has the following (symmetric) block anti-diagonal form

$$M = \begin{pmatrix} & & & & M_0 \\ & & & \iddots & \\ & & M_k & & \\ & \iddots & & & \\ M_{16} & & & & \end{pmatrix}, \tag{4.75}$$

where a single block M_k is of size d_k and given by

$$(M_k)_{i,j} = \left(\Psi_k^{(i)} * \Psi_{16-k}^{(j)}\right)_0. \tag{4.76}$$

(4.74) yields immediately

$$M_k = M_{16-k}, \tag{4.77}$$

In particular, M_k is symmetric. In fact, the middle block contains all the others: from (4.73) one conclude that for any $0 \leq k \leq 8$ and $1 \leq i \leq d_k$,

$$(M_k)_{i,j} = (M_8)_{i,j}, \quad 0 \leq k \leq 8, \quad 1 \leq i \leq d_k \tag{4.78}$$

(cf. Bernig's discussion of Corollary 3.6 in [27]).

With MAPLE, M_8 is easily computed in terms of invariant forms, using the formula (1.38). Equally simple is to invert it as well as its submatrices and, thus, to obtain

$$M^{-1} = \begin{pmatrix} & & & & M_0^{-1} \\ & & & \iddots & \\ & & M_k^{-1} & & \\ & \iddots & & & \\ M_{16}^{-1} & & & & \end{pmatrix}. \tag{4.79}$$

Consequently, keeping the notation of this paragraph, it follows from (1.52) that

Theorem 4.25. *The principal kinematic formula in $\mathbb{O}^2$ reads*

$$\int_{\overline{\mathrm{Spin}(9)}} \chi(K \cap \bar{g}L) = \sum_{k=0}^{16} \sum_{i,j=1}^{d_k} (M_k^{-1})_{i,j} \Psi_k^{(i)}(K)\, \Psi_{16-k}^{(j)}(L), \quad K, L \in \mathcal{K}(\mathbb{O}^2). \tag{4.80}$$

Remark 4.26. The explicit expression of the (biggest) middle part $k = 8$ of (4.80) is, for illustration, given in Appendix C.

4.5 Kubota-Type $\mathrm{Spin}(9)$-Invariant Valuations

In the final part of this chapter we shall study the Kubota-type valuations

$$T_k(K) = \int_{\mathbb{O}P^1} \mu_k(\pi_\ell K)\,\mathrm{d}\ell, \quad 0 \le k \le 8, \tag{4.81}$$

introduced by Alesker [13] that were recalled in §4.1.1. As for the terminology we use, cf. the classical Kubota formulas (1.11). First, we prove their non-triviality and second, we express them in the basis of $\mathrm{Val}^{\mathrm{Spin}(9)}$ that was given in §4.4.2.

4.5.1 Invariant Measures on $\mathbb{O}P^1$ and $\overline{\mathbb{O}P^1}$

For we shall need to explicitly compute the integrals (4.81) in the sequel, let us recall that the $\mathrm{Spin}(9)$-invariant measure on $\mathbb{O}P^1$ is well known. Namely, in the stereographic coordinates $\mathbb{O}P^1\backslash\{\ell_\infty\} \to \mathbb{O} : \ell_a \mapsto a$, one has (see [71])

$$\mathrm{d}\ell_a = \frac{c\,\mathrm{d}a}{\left(1+|a|^2\right)^8}, \tag{4.82}$$

where $\mathrm{d}a$ is the Lebesgue measure on $\mathbb{O} = \mathbb{R}^8$, and $c \in \mathbb{R}$ a normalizing constant. Let us calibrate c such that $\mathrm{d}\ell_a$ is a probability measure. To this end,

Proposition 4.27.

$$\frac{1}{c} = \int_{\mathbb{O}} \frac{\mathrm{d}a}{\left(1+|a|^2\right)^8} = \frac{\pi^4}{840}. \tag{4.83}$$

Proof. Using spherical coordinates in $\mathbb{O}$, where the surface area of the unit sphere S^7 is $8\omega_8 = 8 \cdot \frac{\pi^4}{4!} = \frac{\pi^4}{3}$, and an easy substitution $1 + r^2 = y(r)$, we can write

$$\begin{aligned}
\int_{\mathbb{O}} \frac{\mathrm{d}a}{\left(1+|a|^2\right)^8} &= \frac{\pi^4}{3} \int_0^\infty \frac{r^7}{(1+r^2)^8}\,\mathrm{d}r \\
&= \frac{\pi^4}{6} \int_1^\infty \frac{(y-1)^3}{y^8}\,\mathrm{d}y \\
&= \frac{\pi^4}{6} \int_1^\infty \left(\frac{1}{y^5} - \frac{3}{y^6} + \frac{3}{y^7} - \frac{1}{y^8}\right)\mathrm{d}y \\
&= \frac{\pi^4}{6} \left[-\frac{1}{4y^4} + \frac{3}{5y^5} - \frac{3}{6y^6} + \frac{1}{7y^7}\right]_1^\infty \\
&= \frac{\pi^4}{6} \left(\frac{1}{4} - \frac{3}{5} + \frac{1}{2} - \frac{1}{7}\right) \\
&= \frac{\pi^4}{840}.
\end{aligned}$$

□

The invariant measure $\mathrm{d}\bar{\ell}$ on $\overline{\mathbb{O}P^1}$ is constructed accordingly: To any measurable function f on $\overline{\mathbb{O}P^1}$ we first assign a function $\tilde{f}$ on $\mathbb{O}P^1 \times \mathbb{O}^2$ by $\tilde{f}(\ell, x) = f(\ell + x)$. Then we require

$$\int_{\overline{\mathbb{O}P^1}} f(\bar{\ell})\,\mathrm{d}\bar{\ell} = \int_{\mathbb{O}P^1} \left(\int_{\ell^\perp} \tilde{f}(\ell, x)\,\mathrm{d}x\right)\mathrm{d}\ell$$

(cf. [88], §7.1).

4.5.2 Non-Triviality

Let us show that the valuations T_k, $0 \leq k \leq 8$, are non-trivial in that sense that they are not multiples of the intrinsic volumes of the respective degree. Notice that we do not yet use our description of $\mathrm{Val}^{\mathrm{Spin}(9)}$ (given in §4.4) at all.

We shall need the following well-known description of the operator Λ (which holds in fact much more general). It explains why Λ is usually called the *derivation operator*. See also (1.4).

Lemma 4.28 (Bernig, Fu [30], Corollary 1.8). *For any $\phi \in \mathrm{Val}^{\mathrm{Spin}(9)}$ and $K \in \mathcal{K}$ one has*

$$(\Lambda\phi)(K) = (\mu_{15} * \phi)(K) = \frac{1}{2} \frac{\mathrm{d}}{\mathrm{d}\lambda}\bigg|_{\lambda=0} \phi(K_\lambda). \tag{4.84}$$

Let us also recall the standard integral-geometric notation (see [88]): We put

$$[0] = 1 \quad \text{and} \quad [k] = \frac{k\omega_k}{2\omega_{k-1}}, \quad k \in \mathbb{N}. \tag{4.85}$$

Then for $[k]! = [k][k-1]\cdots[1]$ we define

$$\begin{bmatrix} n \\ k \end{bmatrix} = \frac{[n]!}{[k]![n-k]!}, \quad 0 \leq k \leq n. \tag{4.86}$$

Proposition 4.29. *For $1 \leq k \leq 8$,*

$$\Lambda T_k = [9-k]\, T_{k-1}. \tag{4.87}$$

Proof. Take $K \in \mathcal{K}$ and $\lambda > 0$. According to the Kubota formula (1.11) and the Steiner formula (1.10), we have

$$\begin{aligned}
T_k(K_\lambda) &= \int_{\mathbb{O}P^1} \mu_k(\pi_\ell K_\lambda)\,\mathrm{d}\ell \\
&= \begin{bmatrix} 8 \\ k \end{bmatrix} \int_{\mathbb{O}P^1} \left(\int_{\mathrm{Gr}_k(\ell)} \mathrm{vol}_k(\pi_E \pi_\ell K_\lambda)\,\mathrm{d}E \right) \mathrm{d}\ell \\
&= \begin{bmatrix} 8 \\ k \end{bmatrix} \int_{\mathbb{O}P^1} \left(\int_{\mathrm{Gr}_k(\ell)} \mathrm{vol}_k((\pi_E \pi_\ell K)_\lambda)\,\mathrm{d}E \right) \mathrm{d}\ell \\
&= \begin{bmatrix} 8 \\ k \end{bmatrix} \int_{\mathbb{O}P^1} \left(\sum_{j=0}^{k} \int_{\mathrm{Gr}_k(\ell)} \omega_j \mu_{k-j}(\pi_E \pi_\ell K) \lambda^j\,\mathrm{d}E \right) \mathrm{d}\ell.
\end{aligned}$$

Differentiating in $\lambda = 0$ and Lemma 4.28 then yield

$$\begin{aligned}
(\Lambda T_k)(K) &= \begin{bmatrix} 8 \\ k \end{bmatrix} \frac{\omega_1}{2} \int_{\mathbb{O}P^1} \left(\int_{\mathrm{Gr}_k(\ell)} \mu_{k-1}(\pi_E \pi_\ell K)\,\mathrm{d}E \right) \mathrm{d}\ell \\
&= \begin{bmatrix} 8 \\ k \end{bmatrix} \begin{bmatrix} k \\ k-1 \end{bmatrix} \int_{\mathbb{O}P^1} \left(\int_{\mathrm{Gr}_k(\ell)} \left(\int_{\mathrm{Gr}_{k-1}(E)} \mathrm{vol}_{k-1}(\pi_F \pi_E \pi_\ell K)\,\mathrm{d}F \right) \mathrm{d}E \right) \mathrm{d}\ell \\
&= \begin{bmatrix} 8 \\ k \end{bmatrix} \begin{bmatrix} k \\ k-1 \end{bmatrix} \int_{\mathbb{O}P^1} \left(\int_{\mathrm{Gr}_{k-1}(\ell)} \mathrm{vol}_{k-1}(\pi_F \pi_\ell K)\,\mathrm{d}F \right) \mathrm{d}\ell \\
&= \begin{bmatrix} 8 \\ k \end{bmatrix} \begin{bmatrix} k \\ k-1 \end{bmatrix} \begin{bmatrix} 8 \\ k-1 \end{bmatrix}^{-1} \int_{\mathbb{O}P^1} \mu_{k-1}(\pi_\ell K)\,\mathrm{d}\ell \\
&= \begin{bmatrix} 8 \\ k \end{bmatrix} \begin{bmatrix} k \\ k-1 \end{bmatrix} \begin{bmatrix} 8 \\ k-1 \end{bmatrix}^{-1} T_{k-1}(K),
\end{aligned}$$

where we used the Kubota formula again, back and forth, as well as the probability normalization of the invariant measures on Grassmannians. (4.87) now follows from

$$\begin{bmatrix}8\\k\end{bmatrix}\begin{bmatrix}k\\k-1\end{bmatrix}\begin{bmatrix}8\\k-1\end{bmatrix}^{-1} = \frac{[8]![k]![k-1]![8-k+1]!}{[k]![8-k]![k-1]![1]![8]!} = \frac{[9-k]!}{[1]![8-k]!} = [9-k].$$

□

By induction, it follows at once

Corollary 4.30. *For* $0 \le k \le 8$*,*

$$\Lambda^k T_8 = [k]!\, T_{8-k}. \tag{4.88}$$

Lemma 4.31. *The valuations* $\mu_2, T_2 \in \mathrm{Val}^{\mathrm{Spin}(9)}$ *are linearly independent.*

Proof. According to Klain's Embedding theorem 1.21 and Example 1.22, it is enough to show that Kl_{T_2} is not constant. To this end, consider the following two convex bodies in $\mathbb{O}^2$:

$$K_1 = \mathrm{conv}\left\{\begin{pmatrix}0\\0\end{pmatrix}, \begin{pmatrix}e_0\\0\end{pmatrix}, \begin{pmatrix}e_1\\0\end{pmatrix}\right\} \quad \text{and} \quad K_2 = \mathrm{conv}\left\{\begin{pmatrix}0\\0\end{pmatrix}, \begin{pmatrix}e_0\\0\end{pmatrix}, \begin{pmatrix}0\\e_0\end{pmatrix}\right\}.$$

Since $\mu_2(K_i) = \mathrm{vol}_2(K_i) = \frac{1}{2}$ clearly holds for $i = 1, 2$, in order to prove the statement, it is sufficient to show $T_2(K_1) \neq T_2(K_2)$.

Let $a \in \mathbb{O}$. Then the set

$$\frac{1}{\sqrt{1+|a|^2}}\begin{pmatrix}e_0\\e_0 a\end{pmatrix}, \dots, \frac{1}{\sqrt{1+|a|^2}}\begin{pmatrix}e_7\\e_7 a\end{pmatrix}$$

is an orthonormal basis of the octonionic line ℓ_a. Consequently, the orthogonal projection $\pi_a : \mathbb{O}^2 \to \ell_a$ is given by

$$\pi_a\begin{pmatrix}x\\y\end{pmatrix} = \frac{1}{1+|a|^2}\sum_{j=0}^{7}\left(\langle x, e_j\rangle + \langle y, e_j a\rangle\right)\begin{pmatrix}e_j\\e_j a\end{pmatrix}.$$

In particular, one has, first,

$$\pi_a\begin{pmatrix}e_0\\0\end{pmatrix} = \frac{1}{1+|a|^2}\begin{pmatrix}e_0\\e_0 a\end{pmatrix}, \tag{4.89}$$

second,

$$\pi_a\begin{pmatrix}e_1\\0\end{pmatrix} = \frac{1}{1+|a|^2}\begin{pmatrix}e_1\\e_1 a\end{pmatrix}, \tag{4.90}$$

and finally,

$$\pi_a\begin{pmatrix}0\\e_0\end{pmatrix} = \frac{1}{1+|a|^2}\sum_{j=0}^{7}\langle e_0, e_j a\rangle\begin{pmatrix}e_j\\e_j a\end{pmatrix} = \frac{\mathrm{Re}(a)}{1+|a|^2}\begin{pmatrix}e_0\\e_0 a\end{pmatrix} - \frac{1}{1+|a|^2}\sum_{j=1}^{7}\langle e_j, a\rangle\begin{pmatrix}e_j\\e_j a\end{pmatrix}. \tag{4.91}$$

Now we can explicitly compute $T_2(K_1)$ and $T_2(K_2)$ using (4.82).

First, since the projections (4.89) and (4.90) are obviously perpendicular, one has

$$2\mu_2(\pi_a K_1) = \left\| \frac{1}{1+|a|^2} \begin{pmatrix} e_0 \\ e_0 a \end{pmatrix} \right\| \cdot \left\| \frac{1}{1+|a|^2} \begin{pmatrix} e_1 \\ e_1 a \end{pmatrix} \right\| = \frac{1}{1+|a|^2}.$$

Hence, integration completely analogous to the proof of Proposition 4.27 above yields

$$\begin{aligned} 2T_2(K_1) &= \frac{840}{\pi^4} \int_{\mathbb{O}} \frac{\mathrm{d}a}{\left(1+|a|^2\right)^9} \\ &= 280 \int_0^\infty \frac{r^7}{(1+r^2)^9} \,\mathrm{d}r \\ &= 140 \int_1^\infty \frac{(y-1)^3}{y^9} \,\mathrm{d}y \\ &= 140 \int_1^\infty \left(\frac{1}{y^6} - \frac{3}{y^7} + \frac{3}{y^8} - \frac{1}{y^9} \right) \mathrm{d}y \\ &= 140 \left[-\frac{1}{5y^5} + \frac{3}{6y^6} - \frac{3}{7y^7} + \frac{1}{8y^8} \right]_1^\infty \\ &= 140 \left(\frac{1}{5} - \frac{1}{2} + \frac{3}{7} - \frac{1}{8} \right) \\ &= \frac{1}{2}. \end{aligned}$$

Second, the two parts of the projection on the right-hand side of (4.91) are obviously parallel and perpendicular, respectively, to (4.89), one has

$$\begin{aligned} 2\mu_2(\pi_a K_2) &= \left\| \frac{1}{1+|a|^2} \begin{pmatrix} e_0 \\ e_0 a \end{pmatrix} \right\| \cdot \left\| \frac{1}{1+|a|^2} \sum_{j=1}^{7} \langle e_j, a \rangle \begin{pmatrix} e_j \\ e_j a \end{pmatrix} \right\| \\ &= \frac{1}{1+|a|^2} \sqrt{\sum_{j=1}^{7} \langle e_j, a \rangle^2} \\ &= \frac{|\mathrm{Im}(a)|}{1+|a|^2}. \end{aligned}$$

Now, using spherical coordinates in $\mathrm{Im}\,\mathbb{O}$, where the surface area of the unit sphere S^6 is $7\omega_7 = 7 \cdot \frac{2^4 \pi^3}{7!!} = \frac{16\pi^3}{15}$, substitution $1 + x^2 + r^2 = y(r)$, and the known integral

$$\int_0^\infty \frac{\mathrm{d}x}{(1+x^2)^5} = \frac{35\pi}{256}$$

(see e.g. [67], p. 327, §3.249), we have

$$\begin{aligned} 2T_2(K_1) &= \frac{840}{\pi^4} \int_{\mathbb{O}} \frac{|\mathrm{Im}(a)|}{\left(1+|a|^2\right)^9} \,\mathrm{d}a \\ &= \frac{896}{\pi} \int_{\mathbb{R}} \left(\int_0^\infty \frac{r^7}{(1+r^2+x^2)^9} \,\mathrm{d}r \right) \mathrm{d}x \\ &= \frac{448}{\pi} \int_{\mathbb{R}} \left(\int_{1+x^2}^\infty \frac{(y-x^2-1)^3}{y^9} \,\mathrm{d}y \right) \mathrm{d}x \\ &= \frac{448}{\pi} \int_{\mathbb{R}} \left(\int_{1+x^2}^\infty \left(\frac{1}{y^6} - \frac{3(1+x^2)}{y^7} + \frac{3(1+x^2)^2}{y^8} - \frac{(1+x^2)^3}{y^9} \right) \mathrm{d}y \right) \mathrm{d}x \end{aligned}$$

$$
\begin{aligned}
&= \frac{448}{\pi}\int_{\mathbb{R}}\left[-\frac{1}{5y^5}+\frac{3(1+x^2)}{6y^6}-\frac{3(1+x^2)^2}{7y^7}+\frac{(1+x^2)^3}{8y^8}\right]_{1+x^2}^{\infty}\mathrm{d}x\\
&= \frac{8}{5\pi}\int_{\mathbb{R}}\frac{\mathrm{d}x}{(1+x^2)^5}\\
&= \frac{16}{5\pi}\int_0^{\infty}\frac{\mathrm{d}x}{(1+x^2)^5}\\
&= \frac{7}{16}.
\end{aligned}
$$

Therefore, $T_2(K_1) \neq T_2(K_2)$ as desired. $\square$

Remark 4.32. Consider $E_1, E_2 \in \mathrm{Gr}_2(\mathbb{O}^2)$ as follows:

$$
E_1 = \operatorname{span}\left\{\begin{pmatrix} e_0 \\ 0\end{pmatrix}, \begin{pmatrix} e_1 \\ 0\end{pmatrix}\right\} \quad \text{and} \quad E_2 = \operatorname{span}\left\{\begin{pmatrix} e_0 \\ 0\end{pmatrix}, \begin{pmatrix} 0 \\ e_0\end{pmatrix}\right\}.
$$

It is easily observed from the previous proof that

$$
\mathrm{Kl}_{T_2}(E_0) = \frac{1}{2} \quad \text{and} \quad \mathrm{Kl}_{T_2}(E_1) = \frac{7}{16}. \tag{4.92}
$$

Using the same considerations as on p. 853 of [35], namely that $\mathrm{Kl}_{\mu_2} \equiv 1$, $\mathrm{Kl}_{\tau_{\mathbb{O}}}(E_1) = 0$, $\mathrm{Kl}_{\tau_{\mathbb{O}}}(E_2) = 1$, and $\dim \mathrm{Val}_2^{\mathrm{Spin}(9)} = 2$, from (4.92) we conclude

$$
T_2 = \frac{1}{2}\mu_2 - \frac{1}{16}\tau_{\mathbb{O}}. \tag{4.93}
$$

Corollary 4.33. *Let $2 \le k \le 8$. The valuations $\mu_k, T_k \in \mathrm{Val}^{\mathrm{Spin}(9)}$ are linearly independent.*

Proof. Assume that $T_k = \lambda\mu_k$ for some $\lambda \neq 0$. Then, according to Proposition 4.29 and Theorem 1.53, there are $c_0 \neq 0$ and $c_1 \neq 0$ such that

$$
c_0 T_2 = \Lambda^{k-2} T_k = \lambda \Lambda^{k-2}\mu_k = c_1\mu_2
$$

which is in contradiction with Lemma 4.31. $\square$

4.5.3 Expressions in the Monomial Basis

Finally, let us now express the Kubota-type valuations in terms of the basis of $\mathrm{Val}^{\mathrm{Spin}(9)}$ introduced in Theorem 4.22. We shall first deduce the expression of T_8 from the Principal kinematic formula (4.80) and extend it consequently to T_k, $0 \le k \le 7$, using the relation (4.88).

First of all, with the normalization we chose,

Proposition 4.34. *One has $T_8 = U_8$.*

Proof. For any $K \in \mathcal{K}$,

$$
U_8(K) = \int_{\overline{\mathbb{O}P^1}} \chi(K \cap \bar{\ell})\,\mathrm{d}\bar{\ell} = \int_{\mathbb{O}P^1}\left(\int_{\ell^\perp}\chi(K\cap(x+\ell))\,\mathrm{d}x\right)\mathrm{d}\ell.
$$

For a given $x \in \ell^\perp$, $\chi(K \cap (x+\ell)) = 1$ if and only if there exist $k \in K$ and $l \in \ell$ with $x = k - l$ which clearly occurs if and only if $x \in \pi_{\ell^\perp}K$. Otherwise $\chi(K\cap(x+\ell)) = 0$. Therefore,

$$
\int_{\ell^\perp}\chi(K\cap(x+\ell))\,\mathrm{d}x = \mathrm{vol}_8(\pi_{\ell^\perp}K).
$$

To complete the proof, observe that $\ell_a^\perp = \ell_{-\frac{a}{|a|^2}}$ for $0 \neq a \in \mathbb{O}$, and $\ell_0^\perp = \ell_\infty$. $\square$

In what follows, the notation of §4.4.3 will be kept. Consider the following eight-dimensional convex body:

$$D=\left\{\begin{pmatrix}0\\x\end{pmatrix}\in\mathbb{O}^2\,;\,x\in\mathbb{O},|x|\le 1\right\}\in\mathcal{K}(\mathbb{O}^2). \tag{4.94}$$

First of all, observe that span $D=\ell_\infty$. Consequently, one has

Lemma 4.35. *For any $K\in\mathcal{K}(\mathbb{O}^2)$,*

$$U_8=\sum_{i=1}^{d_8}\left[\sum_{j=1}^{d_8}(M_8^{-1})_{i,j}\Psi_8^{(j)}(D)\right]\Psi_8^{(i)}. \tag{4.95}$$

Proof. We apply the principal kinematic formula (4.80) to $L=\lambda D$, $\lambda>0$:

$$\begin{aligned}\int_{\overline{\mathrm{Spin}(9)}}\chi(K\cap\bar{g}\lambda D)&=\sum_{k=0}^{16}\sum_{i,j=1}^{d_k}(M_k^{-1})_{i,j}\Psi_k^{(i)}(K)\,\Psi_{16-k}^{(j)}(\lambda D)\\&=\lambda^8\sum_{i,j=1}^{d_8}(M_8^{-1})_{i,j}\Psi_8^{(i)}(K)\,\Psi_8^{(j)}(D)+O(\lambda^7)\end{aligned}$$

holds for any $K\in\mathcal{K}(\mathbb{O}^2)$. Dividing by λ^8 and sending $\lambda\to\infty$ then yields (4.95). □

As we shall see, the convex body D was chosen such that it is that easy to evaluate the basis elements of $\mathrm{Val}_8^{\mathrm{Spin}(9)}$ given in Theorem 4.22 on it. In fact, the normal cycle is $\mathrm{nc}(D)=N_1\cup N_2\in S\mathbb{O}^2$, where

$$N_1=\left\{\left(\begin{pmatrix}0\\x\end{pmatrix},\begin{pmatrix}v\\\lambda x\end{pmatrix}\right)\,;\,\lambda\in\mathbb{R},x,v\in\mathbb{O},\lambda>0,|x|=1,\lambda^2+|v|^2=1\right\} \tag{4.96}$$

and

$$N_2=\left\{\left(\begin{pmatrix}0\\x\end{pmatrix},\begin{pmatrix}v\\0\end{pmatrix}\right)\,;\,x,v\in\mathbb{O},|x|\le|v|=1\right\}. \tag{4.97}$$

Take any $\omega\in\Omega^{8,7}(S\mathbb{O}^2)^{\overline{\mathrm{Spin}(9)}}$. Then clearly

$$\int_{\mathrm{nc}(D)}\omega=\int_{N_2}\omega. \tag{4.98}$$

Let us keep the notation from the proof of Theorem 4.22, in particular the abbreviations $\bigwedge^{k,l,m,n}$ and $\bigwedge^{k,l}$. Remember that we only know explicitly $\omega_p\in\bigwedge^{8,7}$ (strictly speaking, modulo α, which is insignificant). First, observe that $p\in N_2$. Second, for some $c\in\mathbb{R}$, one has

$$\omega_p=c\,\theta_1^0\wedge\cdots\wedge\theta_1^7\wedge\varphi_0^1\wedge\cdots\wedge\varphi_1^7+\tilde{\omega}_p,\quad\text{where }\tilde{\omega}_p\in\bigoplus_{l+m<15}\bigwedge^{8-l,l,m,7-m}.$$

Clearly, $\tilde{\omega}_p\big|_{T_pN_2}\equiv 0$ and hence $\omega\big|_{T_pN_2}=c\,\theta_1^0\wedge\cdots\wedge\theta_1^7\wedge\varphi_0^1\wedge\cdots\wedge\varphi_1^7$. Third, for any other point $q\in N_2$, there is $g\in\overline{\mathrm{Spin}(8)}$ such that $p=gq$. Since ω is invariant,

$$\omega_q=g^*\omega_p=c\,g^*(\theta_1^0\wedge\cdots\wedge\theta_1^7\wedge\varphi_0^1\wedge\cdots\wedge\varphi_1^7)+g^*\tilde{\omega}_p.$$

Finally, recall from §2.2.3 that Spin(8) acts diagonally on $\mathbb{O}^2 = \mathbb{O} \oplus \mathbb{O}$ and thus g^* preserves the tetra-degree. In particular, $g^*\tilde{\omega}_p \in \bigoplus_{l+m<15} \bigwedge^{8-l,l,m,7-m}$ is again trivial on T_qN_2. Altogether,

$$[[\omega]](D) = \int_{\mathrm{nc}(D)} \omega = c\,\mathrm{vol}_{8+7}(N_2) = c \cdot 8(\omega_8)^2. \tag{4.99}$$

To make our consideration explicit, let us first denote

$$\widetilde{T}_k = \frac{1}{\omega_8^2[8-k]!} T_k, \quad 0 \leq k \leq 8.$$

Then we have

$$\begin{aligned}
\widetilde{T}_8 = {} & \frac{97}{14192640}t^8 + \frac{727}{17740800}t^6 s + \frac{43}{184800}t^5 v - \frac{53}{7096320}t^4 s^2 + \frac{4703}{4435200}t^4 u_1 \\
& - \frac{1}{46200}t^3 sv + \frac{3229}{831600}t^3 w_1 + \frac{29}{53222400}t^2 s^3 - \frac{47}{950400}t^2 su_1 + \frac{593}{55440}t^2 x_1 \\
& + \frac{1}{1663200}ts^2 v - \frac{13}{166320}tsw_1 + \frac{13}{660}ty - \frac{1}{212889600}s^4 + \frac{1}{2661120}s^2 u_1 \\
& - \frac{1}{15840}sx_1 + \frac{1}{55}z, \\
\widetilde{T}_7 = {} & \frac{1961}{2531917440}t^9 + \frac{373}{82052880}t^7 s + \frac{2063}{82052880}t^6 v - \frac{101}{123079320}t^5 s^2 + \frac{1133}{10256610}t^5 u_1 \\
& - \frac{55}{24615864}t^4 sv + \frac{28193}{73847592}t^4 w_1 + \frac{1}{18461898}t^3 s^3 - \frac{43}{9230949}t^3 su_1 + \frac{25}{26299}t^3 x_1 \\
& + \frac{1}{18461898}t^2 s^2 v, - \frac{1}{157794}t^2 sw_1 + \frac{1}{714}t^2 y, \\
\widetilde{T}_6 = {} & \frac{67}{116121600}t^{10} + \frac{41}{12902400}t^8 s + \frac{13}{806400}t^7 v - \frac{1}{1843200}t^6 s^2 + \frac{101}{1612800}t^6 u_1 \\
& - \frac{1}{806400}t^5 sv + \frac{1}{5600}t^5 w_1, + \frac{1}{38707200}t^4 s^3 - \frac{1}{537600}t^4 su_1 + \frac{1}{3360}t^4 x_1, \\
\widetilde{T}_5 = {} & \frac{313}{884822400}t^{11} + \frac{383}{221205600}t^9 s + \frac{61}{8192800}t^8 v - \frac{1}{3686760}t^7 s^2 \\
& + \frac{1}{43890}t^7 u_1 - \frac{1}{2633400}t^6 sv + \frac{1}{25080}t^6 w_1, \\
\widetilde{T}_4 = {} & \frac{79}{447068160}t^{12} + \frac{1}{1354752}t^{10} s + \frac{1}{423360}t^9 v - \frac{1}{13547520}t^8 s^2 + \frac{1}{241920}t^8 u_1, \\
\widetilde{T}_3 = {} & \frac{107}{1470268800}t^{13} + \frac{1}{4039200}t^{11} s + \frac{1}{2570400}t^{10}, \\
\widetilde{T}_2 = {} & \frac{1}{55351296}t^{14} + \frac{1}{21288960}t^{12} s, \\
\widetilde{T}_1 = {} & -\frac{1}{259459200}t^{15}.
\end{aligned}$$

Remark 4.36. Observe that, in particular, each of these valuations lie in the subalgebra generated by the seven 'first-order' generators $t, s, v, u_1, w_1, x_1, y, z$.

Chapter 5

Hodge-Riemann Bilinear Relations

5.1 Kähler Manifolds

There is a fascinating phenomenon in the background of the recent developments in algebraic integral geometry. Namely, and as also the used terminology suggests, many of the remarkable algebraic structures on valuations we recalled in §1.3.2 above have counterparts in the theory of cohomology on compact Kähler manifolds. Motivated by the results of the previous chapter, we hope this analogy may be pushed even further.

Let us begin with a brief review of certain aspects of the theory of Kähler manifolds that turns out to be relevant to valuations. Our references are [53] and [84]. Unlike in the preceding chapters, we shall assume all differential forms to be complex valued.

First of all, recall that each complex manifold M carries naturally an almost complex structure J induced by the multiplication by i in a local holomorphic chart $z_k = x_k + iy_k$, $1 \leq k \leq n = \dim_{\mathbb{C}} M$, around $u \in M$ as follows:

$$J : \partial_{x_k} \mapsto \partial_{y_k} \quad \text{and} \quad J : \partial_{y_k} \mapsto -\partial_{x_k}. \tag{5.1}$$

Then the (complexified) tangent space at u decomposes canonically

$$\mathbb{C} \otimes T_u M = T_u^+ \oplus T_u^- \tag{5.2}$$

into eigenspaces of J, corresponding to the eigenvalues $\pm i$. The algebra of differential forms on M is graded accordingly:

$$\Omega^\bullet(M) = \bigoplus_{p,q=0}^{n} \Omega^{p,q}(M) \quad \text{and} \quad \overline{\Omega^{p,q}(M)} = \Omega^{q,p}(M). \tag{5.3}$$

Definition 5.1. A *Kähler manifold* is a complex manifold M equipped with a Riemannian metric g compatible with the almost complex structure J such that the induced form

$$\omega(X,Y) = g(JX,Y), \quad X,Y \in \mathfrak{X}(M), \tag{5.4}$$

is closed. ω is then called the *Kähler form* on M.

Observe that ω is a real form of degree $(1,1)$ and M is canonically oriented by ω^n. Throughout the rest of this section we shall assume that M is a *compact* Kähler manifold with the Kähler form ω.

The k-th *de Rham cohomology* of M is given by

$$H^k = H^k(M) = \frac{\ker\left(\mathrm{d} : \Omega^k(M) \to \Omega^{k+1}(M)\right)}{\mathrm{im}\left(\mathrm{d} : \Omega^{k-1}(M) \to \Omega^k(M)\right)}, \quad 0 \leq k \leq 2n. \tag{5.5}$$

Since M is compact, one has $\dim H^k(M) < \infty$. Further, like any other compact oriented Riemannian manifold, M satisfies

Theorem 5.2 (Poincaré Duality). *The pairing* $\mathrm{pd} : H^k \times H^{2n-k} \to \mathbb{C}$ *given by*

$$\mathrm{pd}([\alpha], [\beta]) = \int_M \alpha \wedge \beta \tag{5.6}$$

is perfect, i.e. non-degenerate.

Taking now the complex structure of M into consideration, denote $\overline{\partial} = \mathcal{P}^{p,q+1} \circ \mathrm{d}$, where $\mathcal{P}^{p,q} : \Omega^\bullet(M) \to \Omega^{p,q}(M)$ is the natural projection. It is easy to verify $\overline{\partial}^2 = 0$ as well as the Leibnitz rule for $\overline{\partial}$. The (p, q)*-Dolbeault cohomology* of M is then defined as

$$H^{p,q} = H^{p,q}(M) = \frac{\ker\left(\overline{\partial} : \Omega^{p,q}(M) \to \Omega^{p,q+1}(M)\right)}{\mathrm{im}\left(\overline{\partial} : \Omega^{p,q-1}(M) \to \Omega^{p,q}(M)\right)}, \quad 0 \leq p, q \leq n. \tag{5.7}$$

It turns out that (5.5) and (5.7) are compatible with each other. This is the famous

Theorem 5.3 (Hodge decomposition)**.**

$$H^k = \bigoplus_{p+q=k} H^{p,q} \quad \textit{and} \quad \overline{H^{p,q}} = H^{q,p}. \tag{5.8}$$

It follows at once from the above properties of $\overline{\partial}$ that

$$H^\bullet = H^\bullet(M) = \bigoplus_{p,q=0}^{n} H^{p,q} \tag{5.9}$$

is a graded ring with respect to $[\alpha] \wedge [\beta] = [\alpha \wedge \beta]$.

Crucial for the theory of Kähler manifolds is the so-called *Lefschetz map* given by

$$L : H^{p,q} \to H^{p+1,q+1} : [\alpha] \mapsto [\alpha] \wedge [\omega]. \tag{5.10}$$

For $0 \leq p + q = k \leq n$, let us define the subspaces of *primitive elements* as

$$P^{q,p} = H^{p,q} \cap \ker(L^{n-k+1}) \quad \text{and} \quad P^k = \bigoplus_{p+q=k} P^{q,p}. \tag{5.11}$$

The importance of the Lefschetz map is at once fully revealed by the following

Theorem 5.4 (Hard Lefschetz Theorem)**.** *For* $0 \leq k \leq n$*, the map*

$$L^{n-k} : H^k \to H^{2n-k} \tag{5.12}$$

is an isomorphism, and

$$H^k = \bigoplus_{j \geq 0} L^j P^{k-2j}. \tag{5.13}$$

In the *Lefschetz decomposition* (5.13) we put $P^m = 0$ if $m < 0$. Observe also that L respects the bi-grading (5.8). In particular, for $0 \leq k \leq \frac{n}{2}$, one $L^{n-2k} : H^{k,k} \xrightarrow{\sim} H^{n-k,n-k}$.

Finally, the Lefschetz map and the notion of primitiveness are remarkably related to the Poincaré pairing (5.6). Namely, consider the induced pairing $Q : H^k \times H^k \to \mathbb{C}$:

$$Q(\cdot, \cdot) = \mathrm{pd}\left(L^{n-k}(\cdot), \cdot\right) = \mathrm{pd}\left(\cdot, L^{n-k}(\cdot)\right). \tag{5.14}$$

Explicitly,

$$Q([\alpha], [\beta]) = \int_M \alpha \wedge \beta \wedge \omega^{n-k}. \tag{5.15}$$

Theorem 5.5 (Hodge-Riemann Bilinear Relations). *For any non-zero primitive cohomology class* $[\xi] \in P^{p,q}$, $0 \leq p+q = k \leq n$, *one has*

$$i^{p-q}(-1)^{\frac{1}{2}(p+q)(p+q-1)}Q\Big([\xi], \overline{[\xi]}\Big) > 0. \tag{5.16}$$

Remark 5.6. Observe that for $p+q = 2l$ even,

$$i^{p-q}(-1)^{\frac{1}{2}(p+q)(p+q-1)} = i^{2p-2l}(-1)^{l(2l-1)} = (-1)^{p-l+l} = (-1)^p. \tag{5.17}$$

5.2 Algebraic Structures on Smooth Valuations

Let us turn back to smooth valuations on convex bodies, this time in a more general setting than in Chapter 1. Namely, we shall assume that they take values in the *complex numbers* and Val^∞ thus becomes a complex vector space. This is sometimes useful (see Bernig's description [24] of $\mathrm{Val}^{\mathrm{SU}(n)}$), sometimes necessary (for example to consider the Alesker-Fourier transform of odd valuations - see e.g. [32], Theorem 1), and in any case standard. However, when one deals with G-invariant valuations entirely, there clearly exists a basis of Val^G of real-valued elements which can be represented, via (1.22), by real-valued differential forms. This is a very convenient assumption to work with, especially when one wants, for instance, to temporarily twist the algebra of forms with the octonions, like we did on our way to $\mathrm{Val}^{\mathrm{Spin}(9)}$.

As anticipated in §1.3.2, the whole array of algebraic structures listed therein is not only available on G-invariant valuations but in much greater generality of Val^∞. Let us recall, in this more general context, those of these results that resemble the respective statements from the cohomology of Kähler manifolds in §5.1. Except for the difference of complex-valuedness, the notation is kept from Chapter 1. In particular, we have the McMullen grading

$$\mathrm{Val}^\infty = \mathrm{Val}^\infty(\mathbb{R}^n) = \bigoplus_{k=0}^{n} \mathrm{Val}_k^\infty. \tag{5.18}$$

Theorem 5.7 (Alesker Product [8]). *Let* $A, B \in \mathcal{K}$ *have smooth boundaries with positive curvature. Then* (1.33) *defines a commutative associative distributive naturally continuous graded product on* Val^∞ *with unit* χ.

Theorem 5.8 (Alesker-Poincaré Duality [8]). *The pairing* $\mathrm{pd} : \mathrm{Val}^\infty \times \mathrm{Val}^\infty \to \mathbb{C}$ *given by*

$$\mathrm{pd}(\phi, \psi) = (\phi \cdot \psi)_n \tag{5.19}$$

is perfect, i.e. non-degenerate.

Let L be the linear operator on Val^∞ given by

$$L : \mathrm{Val}_k^\infty \to \mathrm{Val}_{k+1}^\infty : \phi \mapsto \phi \cdot \mu_1. \tag{5.20}$$

Theorem 5.9 (Hard Lefschetz Theorem [6,29]). *For* $0 \leq k < \frac{n}{2}$, *the map*

$$L^{n-2k} : \mathrm{Val}_k^\infty \to \mathrm{Val}_{n-k}^\infty$$

is an isomorphism.

Comparing this with §5.1, a *formal*, yet far-reaching analogy between the space Val^∞ and the *commutative* subring $\bigoplus_{k=0}^n H^{k,k} \subset H^\bullet$ in the cohomology of compact Kähler manifolds may be observed. In fact, one has the following *magical conversion table*:

	Cohomology	Valuations
Underlying space	Kähler manifold M $\dim_{\mathbb{C}} M = n$	Euclidean vector space V $\dim_{\mathbb{R}} V = n$
Graded ring	$\bigoplus_{k=0}^{n} H^{k,k}$	$\bigoplus_{k=0}^{n} \mathrm{Val}_k^\infty$
Canonical object	$[\omega] \in H^{1,1}$	$\mu_1 \in \mathrm{Val}_1^\infty$
Product	$[\alpha] \wedge [\beta] = [\alpha \wedge \beta]$	$\phi \cdot \psi$
Poincaré duality	$\mathrm{pd}\left([\alpha], [\beta]\right) = \int_M \alpha \wedge \beta$	$\mathrm{pd}(\phi, \psi) = (\phi \cdot \psi)_n$
Lefschetz map	$L : [\alpha] \mapsto [\alpha] \wedge [\omega]$	$L : \phi \mapsto \phi \cdot \mu_1$
Hard Lefschetz thm.	$L^{n-2k} : H^{k,k} \xrightarrow{\sim} H^{n-k,n-k}$	$L^{n-2k} : \mathrm{Val}_k^\infty \xrightarrow{\sim} \mathrm{Val}_{n-k}^\infty$

Table 5.1: Cohomology of Kähler manifolds vs. smooth valuations

5.3 Hodge-Riemann Bilinear Relations for Valuations

As it was pointed out to us by Semyon Alesker, there is still one more important result from the cohomology theory of compact Kähler manifolds listed in §5.1 that has not yet been included in the analogy with valuation algebras, namely, the Hodge-Riemann bilinear relations. To this end, the aim of this section is to conjecture a version of Theorem 5.5 for smooth valuations.

First of all, parallel to (5.11), let us denote the subspace of *primitive smooth valuations* of degree $0 \leq k \leq \lfloor \frac{n}{2} \rfloor$ by

$$\mathrm{P}_k^\infty = \mathrm{Val}_k^\infty \cap \ker(L^{n-2k+1}). \tag{5.21}$$

Let us also put

$$\mathrm{P}^\infty = \bigoplus_{k=0}^{\lfloor \frac{n}{2} \rfloor} \mathrm{P}_k^\infty . \tag{5.22}$$

Similarly, parallel to (5.14), we define the induced pairing $Q : \mathrm{Val}_k^\infty \times \mathrm{Val}_k^\infty \to \mathbb{C}$:

$$Q(\cdot, \cdot) = \mathrm{pd}\Big(L^{n-2k}(\cdot), \cdot\Big) = \mathrm{pd}\Big(\cdot, L^{n-2k}(\cdot)\Big) . \tag{5.23}$$

Explicitly, one has

$$Q(\phi, \psi) = (\phi \cdot \psi \cdot \mu_1^{n-2k})_n, \quad \phi, \psi \in \mathrm{Val}_k^\infty . \tag{5.24}$$

Then, pursuing the heurism of Table 5.1, one might expect that the counterpart to the Hodge-Riemann relations (5.16) in this setting is, according to Remark 5.6,

$$(-1)^k Q\big(\phi, \overline{\phi}\big) \overset{?}{>} 0, \quad \phi \in \mathrm{P}_k^\infty, \phi \neq 0. \tag{5.25}$$

Remark 5.10. Since one has

$$Q(\phi, \overline{\phi}) = Q(\mathrm{Re}\, \phi + i \,\mathrm{Im}\, \phi, \mathrm{Re}\, \phi - i \,\mathrm{Im}\, \phi) = Q(\mathrm{Re}\, \phi, \mathrm{Re}\, \phi) + Q(\mathrm{Im}\, \phi, \mathrm{Im}\, \phi),$$

(5.25) is clearly equivalent to

$$(-1)^k Q(\phi,\phi) \overset{?}{>} 0, \quad \phi \in \mathrm{Re}(\mathrm{P}_k^\infty) = \{\phi + \overline{\phi}\,;\, \phi \in \mathrm{P}_k^\infty\}, \phi \neq 0. \tag{5.26}$$

Observe that (5.25) is indeed true in the following special cases: Let us denote

$$\mathrm{P}^G = \mathrm{P}^\infty \cap \mathrm{Val}^G \quad \text{and} \quad \mathrm{P}_k^G = \mathrm{P}_k^\infty \cap \mathrm{Val}^G .$$

(a) $\mathrm{P}^{\mathrm{SO}(n)}$. Clearly, non-trivial primitive elements here only exist in degree 0, i.e are multiples of χ. It is easily seen from the Steiner formula 1.10 and Lemma 4.28 that $\mu_1^n = c\,\mathrm{vol}_n$ for some positive number $c \in \mathbb{R}$. Together, for any $0 \neq z \in \mathbb{C}$, one has

$$Q(z\chi, \bar{z}\chi) = |z|^2\,(c\,\mathrm{vol}_n)_n > 0.$$

(b) $\mathrm{P}^{\mathrm{U}(m)}$, for $n = 2m$. Similarly to the previous case, non-trivial primitive elements are of even degree entirely: Indeed, it follows at once from Theorem 1.54 that

$$\dim \mathrm{P}_k^{\mathrm{U}(n)} = \begin{cases} 1 & \text{if } k = 2l, \quad 0 \leq l \leq \lfloor \frac{n}{2} \rfloor, \\ 0 & \text{otherwise.} \end{cases}$$

Thus, take $0 \neq \phi \in \mathrm{P}_{2l}^{\mathrm{U}(m)}$. Since $\mathrm{Re}\,\phi, \mathrm{Im}\,\phi \in \mathrm{P}_{2l}^{\mathrm{U}(n)}$, according to Remark 5.10, we may assume $\phi = \mathrm{Re}\,\phi$. It was by shown Bernig and Fu in [31], Corollary 5.13, that

$$Q(\phi,\phi) = (\phi \cdot \phi \cdot \mu_1^{2n-4l})_n > 0.$$

(c) $\mathrm{P}^{\mathrm{Spin}(9)}$, for $n = 16$. It follows from Theorem 4.3 of Bernig and Voide that

k	0	1	2	3	4	5	6	7	8
$\dim \mathrm{P}_k^{\mathrm{Spin}(9)}$	1	0	1	1	3	4	5	5	7

.

From our Theorem 4.22, in particular, from its second part Appendix B, one can easily read a basis of $\mathrm{P}_{16-k}^{\mathrm{Spin}(9)}$ (in the convolution setting). For instance, for codegree $k = 4$, looking at the set of relations in codegree 13, one deduces at once

$$\begin{aligned} \mathrm{P}_{16-4}^{\mathrm{Spin}(9)} = \mathrm{span}\Big\{ & s^2 + \frac{9}{65}t^4 + \frac{6}{5}t^2 s + \frac{8}{5}tv, \\ & u_1 + \frac{1007}{36465}t^4 + \frac{131}{935}t^2 s + \frac{43}{85}v, \\ & u_2 + \frac{229}{1870}t^4 + \frac{783}{1870}t^9 s + \frac{62}{85}tv \Big\}. \end{aligned}$$

Then it is just a matter of computation (see the proof of Theorem 4.22) to verify that one indeed has the expected dependence on the (co-)degree:

$$(-1)^{16-k} Q(\phi,\phi) > 0, \quad \phi \in \mathrm{Re}\,\mathrm{P}_{16-k}^{\mathrm{Spin}(9)} . \tag{5.27}$$

The complex-valued version follows as before.

The general case of P^∞, however, turns out to be slightly more subtle. We shall see that this is due to appearance of *odd* valuations (observe that all the valuations in the previous three cases we treated are in fact even). It will be convenient to denote the subspaces of even and odd smooth valuations by $\mathrm{Val}^{\infty,0}$ and $\mathrm{Val}^{\infty,1}$, respectively, and similarly for Val_k^∞ as well as for P^∞ and P_k^∞. Then

$$\mathrm{P}^\infty = \mathrm{P}^{\infty,0} \oplus \mathrm{P}^{\infty,1} = \bigoplus_{\substack{k=0,\dots,n \\ s=0,1}} \mathrm{P}_k^{\infty,s} . \tag{5.28}$$

Theorem 5.11. *Let $n \geq 2$. Then, for $s = 0, 1$, and for any non-zero $\phi \in \mathrm{P}_1^{\infty,s}$, one has*

$$(-1)^{1+s}\, Q(\phi, \overline{\phi}) > 0. \tag{5.29}$$

Proof. All the necessary material is covered by [32] where we refer for details. Throughout the proof, we shall assume that c is a general *positive real* constant. The authors of [32] consider the so-called *spherical valuations*

$$\mu_{k,f}(K) = c \int_{S^{n-1}} f(y)\, \mathrm{d}S_k(K, y), \quad K \in \mathcal{K}, \tag{5.30}$$

where $S_k(K, \cdot)$ is the k-th surface area measure and $f \in C^\infty(S^{n-1})$. One has $\mu_{k,f} \in \mathrm{Val}_k^\infty$ and, moreover, each $\phi \in \mathrm{Val}_1^\infty$ is of this form. Further, the space $C^\infty(S^{n-1})$ decomposes orthogonally, with respect to the standard L^2-inner product, into eigenspaces of spherical Laplacian as $C^\infty(S^{n-1}) = \bigoplus_{q=0}^\infty \mathcal{H}_q^n$, where $\mathcal{H}_q^n$ is the space of spherical harmonics, i.e. restrictions of homogeneous harmonic polynomials of degree $q \in \mathbb{N}_0$ from $\mathbb{R}^n$. One has $\Delta_{S^{n-1}}|_{\mathcal{H}_q^n} = q(n+q-2)\,\mathrm{id}$. In the connection with (5.30), it is well known that $\mu_{k,f} = 0$ if $f \in \mathcal{H}_1^n$, that $\mu_{k,f} \in \mathrm{Val}_k^{\infty,s}$, where $s \equiv q \mod 2$, if $f \in \mathcal{H}_q^n$, and that $\mu_{k,1} = c\mu_k$. Now, Proposition 4.10 of [32] for $h \in \mathcal{H}_{q_1}^n$ and $g \in \mathcal{H}_{q_2}^n$ gives

$$(\mu_{k,h} \cdot \mu_{n-k,g})_n = \begin{cases} (-1)^{q_1} c \int_{S^{n-1}} (h - \frac{1}{n-1}\Delta_{S^{n-1}} h) g & \text{if } q_1 = q_2, \\ 0 & \text{otherwise.} \end{cases} \tag{5.31}$$

We shall also need one of the main results of [32], namely, that for $f \in \mathcal{H}_q^n$, $q \neq 0$, and $1 \leq k \leq n-1$, the Alesker-Fourier transform of these valuations is

$$\mathbb{F}(\mu_{k,f}) = i^q c \mu_{n-k,f}, \tag{5.32}$$

Together with $\mu_{n-1} * \mu_{k,f} = c\mu_{k-1,f}$, which follows easily from the considerations (4.30) in [32], and with standard properties of the Alesker-Fourier transform, (5.32) yields

$$\mu_1 \cdot \mu_{k,f} = c\mu_{k+1,f}. \tag{5.33}$$

Consider $\mu_{1,f} \in \mathrm{P}_1^{\infty,s}$. Then, according to (5.31), $f \perp 1$, i.e. $f = \sum_q f_q$, where $f_q \in \mathcal{H}_q^n$ and the sum runs over $q \geq 2$ with $q \equiv s \mod 2$. Consequently, using (5.31) again, together with (5.33), one has

$$\begin{aligned} Q(\mu_{1,f}, \overline{\mu_{1,f}}) &= (\mu_{1,f} \cdot \mu_1^{n-2} \cdot \overline{\mu_{1,f}})_n \\ &= c(\mu_{n-1,f} \cdot \overline{\mu_{1,f}})_n \\ &= (-1)^q c \sum_r \int_{S^{n-1}} \left(1 - \frac{q(n+q-2)}{n-1}\right) |f|^2. \end{aligned}$$

Finally, since $(-1)^q = (-1)^s$ and

$$1 - \frac{q(n+q-2)}{n-1} = \frac{(1-q)(n-1+q)}{n-1}$$

is always negative for $q, n \geq 2$, the claim follows. □

Corollary 5.12. *Let $1 \leq n \leq 3$. Then, for $0 \leq k \leq \lfloor \frac{n}{2} \rfloor$, for $s = 0, 1$, and for any non-zero primitive smooth valuation $\phi \in \mathrm{P}_k^{\infty,s}$, one has*

$$(-1)^{k+s}\, Q(\phi, \overline{\phi}) > 0. \tag{5.34}$$

Proof. This follows from the previous theorem and the discussion on $\mathrm{P}^{\mathrm{SO}(n)}$ above. □

To conclude, motivated by these observations, we propose the following:

Conjecture 5.13. *Let $n \in \mathbb{N}$. Then, for $0 \le k \le \lfloor \frac{n}{2} \rfloor$ and $s = 0, 1$, and for any non-zero primitive smooth valuation $\phi \in \mathrm{P}_k^{\infty,s}$, one has*

$$(-1)^{k+s}\, Q(\phi, \overline{\phi}) > 0. \tag{5.35}$$

Remark 5.14. In terms of the so-called *Euler-Verdier involution* σ (see [10]), defined by

$$\sigma(\phi) = (-1)^{k+s}\phi, \quad \phi \in \mathrm{Val}_k^{\infty,s},$$

(5.35) can be equivalently written as

$$Q\big(\sigma(\phi), \overline{\phi}\big) > 0. \tag{5.36}$$

Appendix A

A 702-Piece Puzzle

The Spin(9)-invariant 8-form Ψ_8 on $\mathbb{O}^2$ constructed in §3.4 will be now expressed in terms of the dual basis $\mathrm{d}x^0, \dots, \mathrm{d}x^7, \mathrm{d}y^0, \dots, \mathrm{d}y^7$ of $\bigwedge^1(\mathbb{O}^2)^*$ corresponding to the standard orthonormal basis $e_0, \dots, e_7$ of $\mathbb{O}$ (see also Example 3.4). Although the computations performed to this end are slightly more technical, they are based on elementary algebraic properties of the octonions entirely. Basically, we just use the identity (3.44) together with the rule

$$R_{\overline{e_i}} R_{e_j} = -R_{\overline{e_j}} R_{e_i}, \quad i \neq j, \tag{A.1}$$

following easily from (2.10).

We keep the notation from the proof of Theorem 3.18. In particular, let us recall that, in terms of the octonionic coordinate 1-forms $\mathrm{d}x, \mathrm{d}y$ on $\mathbb{O}^2$,

$$\Psi_8 = \Psi_{80} + 4\,\Psi_{62} + 6\,\Psi_{44} + 4\,\Psi_{26} + \Psi_{08}, \tag{A.2}$$

where

$$\begin{aligned}
\Psi_{80} &= \Psi_{40} \wedge \overline{\Psi_{40}}, \\
\Psi_{62} &= \Psi_{31} \wedge \overline{\Psi_{31}}, \\
\Psi_{44} &= -\frac{5}{3}\,\mathrm{Re}\,\Psi_{31} \wedge \Psi_{13}, \\
\Psi_{26} &= \Psi_{13} \wedge \overline{\Psi_{13}}, \\
\Psi_{08} &= \Psi_{04} \wedge \overline{\Psi_{04}},
\end{aligned}$$

where

$$\begin{aligned}
\Psi_{40} &= ((\overline{\mathrm{d}x} \wedge \mathrm{d}x) \wedge \overline{\mathrm{d}x}) \wedge \mathrm{d}x, \\
\Psi_{31} &= ((\overline{\mathrm{d}y} \wedge \mathrm{d}x) \wedge \overline{\mathrm{d}x}) \wedge \mathrm{d}x, \\
\Psi_{13} &= ((\overline{\mathrm{d}x} \wedge \mathrm{d}y) \wedge \overline{\mathrm{d}y}) \wedge \mathrm{d}y, \\
\Psi_{04} &= ((\overline{\mathrm{d}y} \wedge \mathrm{d}y) \wedge \overline{\mathrm{d}y}) \wedge \mathrm{d}y.
\end{aligned}$$

Finally, let us omit the wedge product symbol for the sake of brevity.

The Parts Ψ_{80} and Ψ_{08}

As already shown in Lemma 3.14, both these parts consist of one term each. Namely,

$$\Psi_{80} = 8!\,\mathrm{d}x^0 \cdots \mathrm{d}x^7 \tag{A.3}$$

and

$$\Psi_{08} = 8!\,\mathrm{d}y^0 \cdots \mathrm{d}y^7. \tag{A.4}$$

The Parts $4\,\Psi_{62}$ and $4\,\Psi_{26}$

Since $\Psi_{62} = \operatorname{Re}\Psi_{62}$, according to (3.23) we have

$$4\,\Psi_{62} = 4\sum\langle R_{e_{i_4}} R_{\overline{e_{i_5}}} R_{e_{i_6}} R_{\overline{e_{i_7}}} R_{e_{i_3}} R_{\overline{e_{i_2}}} R_{e_{i_1}} R_{\overline{e_{i_0}}}(1), 1\rangle\, \mathrm{d}y^{i_0}\mathrm{d}x^{i_1}\mathrm{d}x^{i_2}\mathrm{d}x^{i_3}\mathrm{d}y^{i_4}\mathrm{d}x^{i_5}\mathrm{d}x^{i_6}\mathrm{d}x^{i_7},$$

or, after reordering the canonical 1-forms,

$$4\,\Psi_{62} = -4\sum\langle R_{e_{i_7}} R_{\overline{e_{i_3}}} R_{e_{i_4}} R_{\overline{e_{i_5}}} R_{e_{i_2}} R_{\overline{e_{i_1}}} R_{e_{i_0}} R_{\overline{e_{i_6}}}(1), 1\rangle\, \mathrm{d}x^{i_0}\cdots\mathrm{d}x^{i_5}\mathrm{d}y^{i_6}\mathrm{d}y^{i_7}.$$

Clearly, a general term

$$-4\langle R_{e_{i_7}} R_{\overline{e_{i_3}}} R_{e_{i_4}} R_{\overline{e_{i_5}}} R_{e_{i_2}} R_{\overline{e_{i_1}}} R_{e_{i_0}} R_{\overline{e_{i_6}}}(1), 1\rangle\, \mathrm{d}x^{i_0}\cdots\mathrm{d}x^{i_5}\mathrm{d}y^{i_6}\mathrm{d}y^{i_7} \tag{A.5}$$

of this sum is possibly non-trivial only if $\#\{i_0,\dots,i_5\} = 6$ and $\#\{i_6,i_7\} = 2$. Hence, there are just three eventualities for $\#\{i_0,\dots,i_7\}$: 6, 7 or 8. Further necessary condition on non-triviality of (A.5) is obviously, in the sense of Remark 3.15,

$$\prod_{k=0}^{7} e_{i_k} = \pm 1. \tag{A.6}$$

First, suppose $\#\{i_0,\dots,i_7\} = 8$. This means all indices in (A.5) are distinct and the inner product there is thus totally skew-symmetric. Therefore, there are $\binom{8}{2} = 28$ distinct terms of this kind, each corresponding to a different set $\{i_6,i_7\}$, all with coefficients $\pm 4\cdot 2!\cdot 6! = \pm 8\cdot 6!$.

Second, let $\#\{i_0,\dots,i_7\} = 7$, i.e. precisely two indices coincide in (A.5). Then (A.6) requires that the product of six distinct basis vectors equals ± 1. According to (3.44), this would however mean that the product of the two remaining (and distinct) basis elements is also ± 1, which is impossible. We conclude, therefore, that there is no non-trivial term of this kind.

Finally, suppose $\#\{i_0,\dots,i_7\} = 6$, i.e. $\{i_6,i_7\} \subset \{i_0,\dots,i_5\}$. In particular, i_6 agrees with precisely one element in $\{i_0,\dots,i_5\}$ and thus, after commuting the operator $R_{\overline{e_{i_6}}}$ leftwards, (A.5) takes the form

$$+4\,\langle R_{e_{i_7}} R_{\overline{e_{i_6}}} R_{e_{i_3}} R_{\overline{e_{i_4}}} R_{e_{i_5}} R_{\overline{e_{i_2}}} R_{e_{i_1}} R_{\overline{e_{i_0}}}(1), 1\rangle\, \mathrm{d}x^{i_0}\cdots\mathrm{d}x^{i_5}\mathrm{d}y^{i_6}\mathrm{d}y^{i_7}$$

that is totally skew-symmetric in i_6, i_7, and $i_0,\dots,i_5$, respectively. The inner product is non-zero precisely when the product of the basis elements of indices $\{i_0,\cdots,i_5\}\backslash\{i_6,i_7\}$ is ± 1. So, as discussed in the proof of Lemma 3.16, there are 14 possibilities for the set $\{i_0,\cdots,i_5\}\backslash\{i_6,i_7\}$ and to each of them there are $\binom{4}{2} = 6$ choices of $\{i_6,i_7\}$. Therefore, there are $6\cdot 14 = 84$ terms of this kind, each with prefactor $\pm 4\cdot 2!\cdot 6! = \pm 8\cdot 6!$.

The case of Ψ_{26} is completely analogous.

The Part $6\,\Psi_{44}$

Now we have

$$6\,\Psi_{44} = -10\sum\langle((\overline{e_{i_0}}e_{i_1})\overline{e_{i_2}})e_{i_3}, \overline{((\overline{e_{i_4}}e_{i_5})\overline{e_{i_6}})e_{i_7}}\rangle\, \mathrm{d}y^{i_0}\mathrm{d}x^{i_1}\mathrm{d}x^{i_2}\mathrm{d}x^{i_3}\mathrm{d}x^{i_4}\mathrm{d}y^{i_5}\mathrm{d}y^{i_6}\mathrm{d}y^{i_7}.$$

After reordering, a general term takes the form

$$-10\langle((\overline{e_{i_4}}e_{i_0})\overline{e_{i_1}})e_{i_2}, \overline{((\overline{e_{i_3}}e_{i_5})\overline{e_{i_6}})e_{i_7}}\rangle\, \mathrm{d}x^{i_0}\cdots\mathrm{d}x^{i_3}\mathrm{d}y^{i_4}\cdots\mathrm{d}y^{i_7}, \tag{A.7}$$

that is only non-trivial if $\#\{i_0,\dots,i_3\} = \#\{i_4,\dots,i_7\} = 4$, i.e. $4 \le \#\{i_0,\dots,i_7\} \le 8$.

Due to the higher complexity of this case, we introduce a version of the product of indices considered in §2.4.4: for $0 \le i, j \le 7$, we define $i \cdot j = ij$ to be the (unique) element of $\{0, \dots, 7\}$ such that $e_{ij} = \pm e_i e_j$. This product is clearly commutative as well as associative (see Remark 3.15). The condition (A.6), which of course still applies, translates in this language as

$$\prod_{k=0}^{7} i_k = 0. \tag{A.8}$$

Let $\#\{i_0, \dots, i_7\} = 8$, i.e. $\{i_0, \dots, i_3\} \cap \{i_4, \dots, i_7\} = \emptyset$. We shall distinguish two cases here. First, suppose $i_0 i_1 i_2 i_3 = 0$. Then (A.8) is only fulfilled if $i_4 i_5 i_6 i_7 = 0$ too, i.e. if $i_5 i_6 i_7 = i_4$. Since $i_3 \neq i_4$, one has $i_5 i_6 i_7 \neq i_3$ and thus $i_3 i_5 i_6 i_7 \neq 0$. Therefore $\overline{((\overline{e_{i_3}} e_{i_5})\overline{e_{i_6}})e_{i_7}} = -((\overline{e_{i_3}} e_{i_5})\overline{e_{i_6}})e_{i_7}$ and so (A.7) takes the form

$$\begin{aligned} &+ 10\, \langle ((\overline{e_{i_4}} e_{i_0})\overline{e_{i_1}})e_{i_2}, ((\overline{e_{i_3}} e_{i_5})\overline{e_{i_6}})e_{i_7} \rangle \, \mathrm{d}x^{i_0} \cdots \mathrm{d}x^{i_3} \mathrm{d}y^{i_4} \cdots \mathrm{d}y^{i_7} \\ &\quad = +10 \langle R_{e_{i_3}} R_{\overline{e_{i_5}}} R_{e_{i_6}} R_{\overline{e_{i_7}}} R_{e_{i_2}} R_{\overline{e_{i_1}}} R_{e_{i_0}} R_{\overline{e_{i_4}}}(1), 1 \rangle \, \mathrm{d}x^{i_0} \cdots \mathrm{d}x^{i_3} \mathrm{d}y^{i_4} \cdots \mathrm{d}y^{i_7}, \end{aligned}$$

that is again totally skew-symmetric and thus the coefficient is $\pm 10 \cdot 4! \cdot 4! = \pm 8 \cdot 6!$. We have already shown that there exist 14 distinct sets $\{i_0, \dots, i_3\}$ with $i_0 i_1 i_2 i_3 = 0$, and there are therefore 14 terms of this kind. Second, if $i_0 i_1 i_2 i_3 \neq 0$ then, by (A.8), also $i_4 i_5 i_6 i_7 \neq 0$ and thus $i_5 i_6 i_7 \neq i_4$. If, for instance, $i_5 i_6 i_7 = i_5$, then $i_6 = i_7$, which is impossible. Similarly one shows that $i_5 i_6 i_7 \neq i_6$ and $i_5 i_6 i_7 \neq i_7$. It is therefore necessary that $i_5 i_6 i_7 \in \{i_0, i_1, i_2, i_3\}$. If $i_5 i_6 i_7 = i_3$, we have $\overline{((\overline{e_{i_3}} e_{i_5})\overline{e_{i_6}})e_{i_7}} = ((\overline{e_{i_3}} e_{i_5})\overline{e_{i_6}})e_{i_7}$ and (A.7) reads

$$-10\, \langle ((\overline{e_{i_4}} e_{i_0})\overline{e_{i_1}})e_{i_2}, ((\overline{e_{i_3}} e_{i_5})\overline{e_{i_6}})e_{i_7} \rangle \, \mathrm{d}x^{i_0} \cdots \mathrm{d}x^{i_3} \mathrm{d}y^{i_4} \cdots \mathrm{d}y^{i_7}.$$

In the three other cases $i_5 i_6 i_7 \in \{i_0, i_1, i_2\}$ one has $\overline{((\overline{e_{i_3}} e_{i_5})\overline{e_{i_6}})e_{i_7}} = -((\overline{e_{i_3}} e_{i_5})\overline{e_{i_6}})e_{i_7}$ and (A.7) equals

$$+10\, \langle ((\overline{e_{i_4}} e_{i_0})\overline{e_{i_1}})e_{i_2}, ((\overline{e_{i_3}} e_{i_5})\overline{e_{i_6}})e_{i_7} \rangle \, \mathrm{d}x^{i_0} \cdots \mathrm{d}x^{i_3} \mathrm{d}y^{i_4} \cdots \mathrm{d}y^{i_7}.$$

Altogether, the coefficient in front of such a term is $\pm \left(\frac{-1+3}{4}\right) \cdot 10 \cdot 4! \cdot 4! = \pm 4 \cdot 6!$. As discussed in the proof of Lemma 3.16, there are 56 sets $\{i_0, \dots, i_3\}$ with $i_0 i_1 i_2 i_3 \neq 0$ and so is the number of the corresponding terms.

If $\#\{i_0, \dots, i_7\} = 7$, then (A.8) could never be fulfil from exactly the same reason as in the case of Ψ_{62}. There is, hence, no such term again.

Let $\#\{i_0, \dots, i_7\} = 6$, and denote

$$\{j_0, \dots, j_3\} = \{i_0, \dots, i_3\} \quad \text{and} \quad \{j_2, \dots, j_5\} = \{i_4, \dots, i_7\}.$$

According to (A.8), we may assume $j_0 j_1 j_4 j_5 = 0$, i.e. $j_0 j_1 = j_4 j_5$. Let $\sigma_1, \sigma_2 = \pm 1$ be such that

$$\begin{aligned} \overline{((\overline{e_{i_3}} e_{i_5})\overline{e_{i_6}})e_{i_7}} &= \sigma_1 ((\overline{e_{i_3}} e_{i_5})\overline{e_{i_6}})e_{i_7}, \\ R_{e_{i_3}} R_{\overline{e_{i_5}}} R_{e_{i_6}} R_{\overline{e_{i_7}}} R_{e_{i_2}} R_{\overline{e_{i_1}}} R_{e_{i_0}} R_{\overline{e_{i_4}}} &= \sigma_2 R_{e_{i_5}} R_{\overline{e_{i_6}}} R_{e_{i_7}} R_{\overline{e_{i_4}}} R_{e_{i_3}} R_{\overline{e_{i_2}}} R_{e_{i_1}} R_{\overline{e_{i_0}}}. \end{aligned}$$

Using this notation, a general term (A.7) takes the form

$$6\,\Psi_{44} = -10\, \sigma_1 \sigma_2 \langle R_{e_{i_5}} R_{\overline{e_{i_6}}} R_{e_{i_7}} R_{\overline{e_{i_4}}} R_{e_{i_3}} R_{\overline{e_{i_2}}} R_{e_{i_1}} R_{\overline{e_{i_0}}}(1), 1 \rangle \, \mathrm{d}x^{i_0} \cdots \mathrm{d}x^{i_3} \mathrm{d}y^{i_4} \cdots \mathrm{d}y^{i_7}.$$

In what follows, we shall discuss how the sign $\sigma_1 \sigma_2$ alternates for different positions of i_3 within $\{j_0, \dots, j_3\}$, and of i_4 within $\{j_2, \dots, j_5\}$. Let us distinguish two separate cases.

First, assume $j_0 j_1 j_2 j_3 = 0$, or equivalently $j_0 j_1 = j_2 j_3$. Then, if $i_3 = j_0$ and $i_4 = j_2$, for instance, one has $\{i_5, i_6, i_7\} = \{j_3, j_4, j_5\}$ and $\{i_0, i_1, i_2\} = \{j_1, j_2, j_3\}$. Therefore,

$$i_3 i_5 i_6 i_7 = j_0 j_3 j_4 j_5 = j_0 j_3 j_2 j_3 = j_0 j_2 \neq 0,$$

meaning $((\overline{e_{i_3}} e_{i_5})\overline{e_{i_6}})e_{i_7} \neq \pm 1$ and thus $\sigma_1 = -1$. Further, since we have $i_3 \notin \{i_5, i_6, i_7\}$, $i_4 \notin \{i_0, i_1, i_2\}$, and $i_3 \neq i_4$, we can write

$$\begin{aligned} R_{e_{i_3}} R_{\overline{e_{i_5}}} R_{e_{i_6}} R_{\overline{e_{i_7}}} R_{e_{i_2}} R_{\overline{e_{i_1}}} R_{e_{i_0}} R_{\overline{e_{i_4}}} &= -R_{e_{i_5}} R_{\overline{e_{i_6}}} R_{e_{i_7}} R_{\overline{e_{i_3}}} R_{e_{i_2}} R_{\overline{e_{i_1}}} R_{e_{i_0}} R_{\overline{e_{i_4}}} \\ &= -R_{e_{i_5}} R_{\overline{e_{i_6}}} R_{e_{i_7}} R_{\overline{e_{i_3}}} R_{e_{i_4}} R_{\overline{e_{i_2}}} R_{e_{i_1}} R_{\overline{e_{i_0}}} \\ &= R_{e_{i_5}} R_{\overline{e_{i_6}}} R_{e_{i_7}} R_{\overline{e_{i_4}}} R_{e_{i_3}} R_{\overline{e_{i_2}}} R_{e_{i_1}} R_{\overline{e_{i_0}}}, \end{aligned}$$

and thus $\sigma_2 = +1$. The signs corresponding to the other positions of i_3 and i_4 are computed analogically and summarised in Table A.1. One can observe from the table that $\sigma_1\sigma_2$ equals +1 in precisely 8 cases and -1 in the 8 others, from which we conclude that the corresponding term is trivial in the end. We may thus assume $j_0 j_1 j_2 j_3 \neq 0$. Then it is easily seen that the eight indices j_0, j_1, j_2, j_3, $j_0 j_1 j_2$, $j_0 j_1 j_3$, $j_0 j_2 j_3$ and $j_1 j_2 j_3$ are all distinct. Therefore, j_4 and j_5 must be among the last four ones. The requirement $j_0 j_1 j_4 j_5 = 0$ however chooses the last two ones. Without loss of generality, we thus have $j_4 = j_0 j_2 j_3$ and $j_5 = j_1 j_2 j_3$. Now we investigate the behaviour of the sign $\sigma_1\sigma_2$ again, taking into account that $j_0 j_4 = j_1 j_5 = j_2 j_3$. The results are captured in Table A.2: Clearly, σ_2 stays the same as in the case $j_0 j_1 j_2 j_3 = 0$ but σ_1 alternates so that $\sigma_1\sigma_2$ is positive only in 4 cases and negative otherwise. This means that the corresponding term appears with the coefficient $\pm\left(\frac{-12+4}{16}\right) \cdot 10 \cdot 4! \cdot 4! = \pm 4 \cdot 6!$. Regarding the number of such terms, there are 56 options for $\{j_0, \dots, j_3\}$, $j_0 j_1 j_2 j_3 \neq 0$, and for each of them there are $\binom{4}{2} = 6$ possible partitions into $\{j_0, j_1\}$ and $\{j_2, j_3\}$. Since $\{j_4, j_5\}$ is then uniquely determined, there are altogether $56 \cdot 6 = 336$ terms of this kind.

i_3	j_0	j_0	j_0	j_0	j_1	j_1	j_1	j_1	j_2	j_2	j_2	j_2	j_3	j_3	j_3	j_3
i_4	j_2	j_3	j_4	j_5	j_2	j_3	j_4	j_5	j_2	j_3	j_4	j_5	j_2	j_3	j_4	j_5
σ_1	$-$	$-$	$-$	$-$	$-$	$-$	$-$	$-$	$+$	$-$	$-$	$-$	$-$	$+$	$-$	$-$
σ_2	$+$	$+$	$-$	$-$	$+$	$+$	$-$	$-$	$+$	$-$	$+$	$+$	$-$	$+$	$+$	$+$
$\sigma_1\sigma_2$	$-$	$-$	$+$	$+$	$-$	$-$	$+$	$+$	$+$	$+$	$-$	$-$	$+$	$+$	$-$	$-$

Table A.1: The signs $\sigma_1\sigma_2$ in the case $j_0 j_1 j_2 j_3 = 0$

i_3	j_0	j_0	j_0	j_0	j_1	j_1	j_1	j_1	j_2	j_2	j_2	j_2	j_3	j_3	j_3	j_3
i_4	j_2	j_3	j_4	j_5	j_2	j_3	j_4	j_5	j_2	j_3	j_4	j_5	j_2	j_3	j_4	j_5
σ_1	$-$	$-$	$-$	$+$	$-$	$-$	$+$	$-$	$-$	$-$	$-$	$-$	$-$	$-$	$-$	$-$
σ_2	$+$	$+$	$-$	$-$	$+$	$+$	$-$	$-$	$+$	$-$	$+$	$+$	$-$	$+$	$+$	$+$
$\sigma_1\sigma_2$	$-$	$-$	$+$	$-$	$-$	$-$	$-$	$+$	$-$	$+$	$-$	$-$	$+$	$-$	$-$	$-$

Table A.2: The signs $\sigma_1\sigma_2$ in the case $j_0 j_1 j_2 j_3 \neq 0$

Further, suppose that $\#\{i_0, \ldots, , i_7\} = 5$, i.e. that $\{i_0, \ldots, i_3\} \cap \{i_4, \ldots, i_7\}$ contains precisely three indices. (A.8) requires that the product of the two (distinct) elements of $\{i_0, \ldots, i_7\}$ that do not belong to this intersection is 0. This is again impossible and thus there are no terms here either.

Finally, let $\#\{i_0, \ldots, , i_7\} = 4$, i.e. $\{i_0, \ldots, i_3\} = \{i_4, \ldots, i_7\}$. First, assume $i_0 i_1 i_2 i_3 = 0$. If $i_3 = i_4$, it is easily seen that $\sigma_1 = \sigma_2 = 1$, whereas if $i_3 \neq i_4$, one has $\sigma_1 = \sigma_2 = -1$. In any case $\sigma_1 \sigma_2 = 1$ and so 14 these terms have all coefficients $\pm 10 \cdot 4! \cdot 4! = \pm 8 \cdot 6!$. Second, if $i_0 i_1 i_2 i_3 \neq 0$, then $\sigma_1 = -1$ regardless the relation between i_3 and i_4. Since σ_2 does not change from the previous case, for any i_3 we have $\sigma_1 \sigma_2 = -1$ if $i_4 = i_3$ and $\sigma_1 \sigma_2 = 1$ in the three other cases of $i_4 \neq i_3$. Altogether, the prefactors of these 56 terms are $\pm \left(\frac{-1+3}{4}\right) \cdot 10 \cdot 4! \cdot 4! = \pm 4 \cdot 6!$.

Summary

All in all, the expression of Ψ_8 in the standard basis possesses 702 non-trivial terms. They are summarized in Table A.3 below. Notice that we scaled the form Ψ_8 by $-\frac{1}{4 \cdot 6!}$ in order to adhere to the conventions of [110].

Each block of the table corresponds to one summand in (A.2). Each row of the table stands for a particular class of terms of $-\frac{1}{4 \cdot 6!}\Psi_8$. A general term of the class is stated in the second column and the class is further specified in the third column. In the first column, the coefficient standing in front of the terms from the respective class is given. Notice that the signs of the coefficients can be explicitly determined directly from the aforedescribed construction. Finally, the number of non-trivial terms within each class is given in the fourth column.

Throughout the table, we always assume $i_k \neq i_l$ if $k \neq l$. Recall also that the product of indices is taken in the following sense: $e_{ij} = \pm e_i e_j$.

Coefficient	Basis vector	Specification	Number
-14	$\mathrm{d}x^0 \cdots \mathrm{d}x^7$	$-$	1
± 2	$\mathrm{d}x^{i_0} \cdots \mathrm{d}x^{i_5} \mathrm{d}y^{i_6} \mathrm{d}y^{i_7}$	$i_0 < \cdots < i_5, \quad i_6 < i_7$	28
± 2	$\mathrm{d}x^{i_0} \cdots \mathrm{d}x^{i_5} \mathrm{d}y^{i_4} \mathrm{d}y^{i_5}$	$i_0 < \cdots < i_3, \quad i_4 < i_5,$ $i_0 i_1 i_2 i_3 = 0$	84
± 2	$\mathrm{d}x^{i_0} \cdots \mathrm{d}x^{i_3} \mathrm{d}y^{i_4} \cdots \mathrm{d}y^{i_7}$	$i_0 < \cdots < i_3, \quad i_4 < \cdots < i_7,$ $i_0 i_1 i_2 i_3 = 0$	14
± 1	$\mathrm{d}x^{i_0} \cdots \mathrm{d}x^{i_3} \mathrm{d}y^{i_4} \cdots \mathrm{d}y^{i_7}$	$i_0 < \cdots < i_3, \quad i_4 < \cdots < i_7,$ $i_0 i_1 i_2 i_3 \neq 0$	56
± 1	$\mathrm{d}x^{i_0} \cdots \mathrm{d}x^{i_3} \mathrm{d}y^{i_2} \cdots \mathrm{d}y^{i_5}$	$i_0 < i_1, \quad i_2 < i_3, \quad i_0 i_1 i_2 i_3 \neq 0,$ $i_4 = i_0 i_2 i_3, \quad i_5 = i_1 i_2 i_3$	336
± 1	$\mathrm{d}x^{i_0} \cdots \mathrm{d}x^{i_3} \mathrm{d}y^{i_0} \cdots \mathrm{d}y^{i_3}$	$i_0 < \cdots < i_3, \quad i_0 i_1 i_2 i_3 \neq 0$	56
± 2	$\mathrm{d}x^{i_0} \cdots \mathrm{d}x^{i_3} \mathrm{d}y^{i_0} \cdots \mathrm{d}y^{i_3}$	$i_0 < \cdots < i_3, \quad i_0 i_1 i_2 i_3 = 0$	14
± 2	$\mathrm{d}x^{i_0} \mathrm{d}x^{i_1} \mathrm{d}y^{i_0} \cdots \mathrm{d}y^{i_5}$	$i_0 < i_1, \quad i_2 < \cdots < i_5,$ $i_2 i_3 i_4 i_5 = 0$	84
± 2	$\mathrm{d}x^{i_0} \mathrm{d}x^{i_1} \mathrm{d}y^{i_2} \cdots \mathrm{d}y^{i_7}$	$i_0 < i_1, \quad i_2 < \cdots < i_7$	28
-14	$\mathrm{d}y^0 \cdots \mathrm{d}y^7$	$-$	1

Table A.3: Explicit expression of the form $-\frac{1}{4 \cdot 6!}\Psi_8$ in the standard basis

Appendix B

The Complete List of Relations in the Algebra $\mathrm{Val}(\mathbb{O}^2)^{\mathrm{Spin}(9)}$

Here we list all the relations defining the algebra of Spin(9)-invariant valuations on the octonionic plane, with respect to the Bernig-Fu convolution. This completes the statement of Theorem 4.22. Recall that the algebra is graded by the codegree as follows:

$$\mathrm{Val}(\mathbb{O}^2)^{\mathrm{Spin}(9)} = \mathrm{Val}^{\mathrm{Spin}(9)} = \bigoplus_{k=0}^{16} \mathrm{Val}_{16-k}^{\mathrm{Spin}(9)}. \tag{B.1}$$

Observe that all the listed relations are independent of each other since the monomials on the left-hand sides of all of the equations below occur exactly once.

Codegree 6

$$v^2 = -2tw_1 + 2su_1.$$

Codegree 7

$$\begin{aligned}
vu_1 &= -5tx_1 + 3sw_1, \\
vu_2 &= -\frac{109}{32}t^7 - \frac{23}{2}t^5 s - \frac{317}{48}t^4 v + \frac{125}{96}t^3 s^2 + \frac{317}{2}t^3 u_1 - \frac{161}{4}t^3 u_2 + 5t^2 sv + \frac{4835}{6}t^2 w_1 \\
&\quad - \frac{127}{4}t^2 w_2 - 291t^2 w_3 + \frac{1}{48}ts^3 + \frac{95}{12}tsu_1 - tsu_2 + 1440tx_1 - \frac{159}{4}tx_2 + \frac{1}{48}s^2 v \\
&\quad + \frac{3}{2}sw_1 - \frac{21}{4}sw_2 - 3sw_3.
\end{aligned}$$

Codegree 8

$$\begin{aligned}
sx_2 &= -\frac{67}{660}t^8 - \frac{151}{330}t^6 s - \frac{37}{99}t^5 v - \frac{65}{198}t^4 s^2 + \frac{233}{66}t^4 u_1 - t^4 u_2 - \frac{26}{99}t^3 sv + \frac{650}{33}t^3 w_1 \\
&\quad - \frac{2}{3}t^3 w_2 - \frac{62}{11}t^3 w_3 + \frac{1}{36}t^2 s^3 + \frac{122}{33}t^2 su_1 - t^2 su_2 + 36t^2 x_1 - t^2 x_2 + \frac{1}{9}ts^2 v \\
&\quad + \frac{650}{33}tsw_1 - \frac{2}{3}tsw_2 - \frac{62}{11}tsw_3 + \frac{1}{6}s^2 u_1 + 36sx_1, \\
vw_1 &= -9ty + 4sx_1, \\
u_1^2 &= -16ty + 6sx_1,
\end{aligned}$$

$$
\begin{aligned}
u_1u_2 = {} & \frac{2983}{880}t^8 + \frac{20051}{1760}t^6s + \frac{3365}{528}t^5v - \frac{1559}{1056}t^4s^2 - \frac{20939}{132}t^4u_1 + \frac{481}{12}t^4u_2 - \frac{1535}{264}t^3sv \\
& - \frac{318149}{396}t^3w_1 + \frac{379}{12}t^3w_2 + \frac{12769}{44}t^3w_3 - \frac{1}{144}t^2s^3 - \frac{2099}{198}t^2su_1 + \frac{1}{2}t^2su_2 \\
& - \frac{8573}{6}t^2x_1 + \frac{159}{4}t^2x_2 + \frac{1}{144}ts^2v - \frac{2867}{396}tsw_1 + \frac{59}{12}tsw_2 + \frac{9}{44}tsw_3 + 18ty \\
& + \frac{1}{36}s^2u_1 - 6sx_1 - \frac{1}{3}vw_2 - 4vw_3, \\
u_2^2 = {} & \frac{4199}{660}t^8 + \frac{46507}{2640}t^6s - \frac{2765}{99}t^5v - \frac{821}{396}t^4s^2 - \frac{17003}{33}t^4u_1 + \frac{571}{6}t^4u_2 - \frac{925}{132}t^3sv \\
& - \frac{430433}{198}t^3w_1 + \frac{250}{3}t^3w_2 + \frac{14731}{22}t^3w_3 + \frac{1}{18}t^2s^3 - \frac{41}{18}t^2su_1 + \frac{1}{3}t^2su_2 \\
& - \frac{38537}{11}t^2x_1 + 95t^2x_2 - \frac{1}{12}ts^2v + \frac{1807}{66}tsw_1 + 19tsw_2 - \frac{17}{22}tsw_3 + \frac{5616}{11}ty \\
& - \frac{1}{144}s^4 - \frac{1}{3}s^2u_1 + \frac{1}{6}s^2u_2 + \frac{210}{11}sx_1 - 2vw_2 + \frac{36}{11}vw_3.
\end{aligned}
$$

Codegree 9

$$
\begin{aligned}
ts^4 = {} & -\frac{25}{9}t^9 - \frac{52}{7}t^7s + 16t^6v + 10t^5s^2 + 64t^5u_1 + 32t^4sv - \frac{128}{3}t^4w_1 + \frac{28}{3}t^3s^3 \\
& - \frac{64}{3}t^3su_1 - \frac{16}{3}t^2s^2v, \\
ts^2u_1 = {} & -\frac{2}{1521}t^9 + \frac{580}{1183}t^7s + \frac{445}{169}t^6v + \frac{174}{169}t^5s^2 + \frac{2471}{169}t^5u_1 + \frac{534}{169}t^4sv + \frac{16208}{507}t^4w_1 \\
& - \frac{88}{507}t^3s^3 + \frac{1822}{507}t^3su_1 - \frac{600}{13}t^3x_1 - \frac{257}{507}t^2s^2v - \frac{152}{13}t^2sw_1, \\
ts^2u_2 = {} & -\frac{145939}{18252}t^9 - \frac{207881}{7098}t^7s - \frac{10897}{338}t^6v + \frac{471}{676}t^5s^2 + \frac{144604}{507}t^5u_1 - 79t^5u_2 \\
& + \frac{864}{169}t^4sv + \frac{2558476}{1521}t^4w_1 - 26t^4w_2 - 552t^4w_3 - \frac{167}{1521}t^3s^3 + \frac{6926}{1521}t^3su_1 \\
& + 2t^3su_2 + \frac{43200}{13}t^3x_1 - 90t^3x_2 - \frac{2531}{3042}t^2s^2v + \frac{180}{13}t^2sw_1 + 2t^2sw_2 + 24t^2sw_3, \\
tsx_1 = {} & \frac{25897}{8791380}t^9 + \frac{6180}{341887}t^7s + \frac{78332}{732615}t^6v - \frac{25481}{2930460}t^5s^2 + \frac{219157}{488410}t^5u_1 \\
& - \frac{12499}{293046}t^4sv + \frac{59099}{48841}t^4w_1 + \frac{7}{293046}t^3s^3 - \frac{16481}{97682}t^3su_1 + \frac{1575}{3757}t^3x_1 \\
& + \frac{7}{293046}t^2s^2v - \frac{1889}{3757}t^2sw_1 - \frac{168}{17}t^2y, \\
tvw_2 = {} & \frac{345859705}{464184864}t^9 + \frac{201050353}{85960160}t^7s + \frac{17103107}{96705180}t^6v - \frac{40427333}{77364144}t^5s^2 - \frac{329863723}{8058765}t^5u_1 \\
& + \frac{152}{15}t^5u_2 - \frac{78728239}{38682072}t^4sv - \frac{3748338689}{19341036}t^4w_1 + 13t^4w_2 + \frac{3111}{44}t^4w_3 \\
& - \frac{9205}{879138}t^3s^3 - \frac{81780211}{19341036}t^3su_1 + \frac{1}{2}t^3su_2 - \frac{2594261}{7514}t^3x_1 + \frac{19}{2}t^3x_2 \\
& + \frac{4007}{1172184}t^2s^2v - \frac{507563}{165308}t^2sw_1 + \frac{13}{6}t^2sw_2 + \frac{295}{44}t^2sw_3 - \frac{990}{17}t^2y,
\end{aligned}
$$

$$
\begin{aligned}
tvw_3 &= \frac{2269121}{7033104}t^9 + \frac{130671239}{123079320}t^7s + \frac{30778613}{52748280}t^6v - \frac{2169641}{10549656}t^5s^2 - \frac{21735751}{1465230}t^5u_1 \\
&\quad + \frac{176}{45}t^5u_2 - \frac{1002280}{1318707}t^4sv - \frac{65308993}{879138}t^4w_1 + \frac{11}{3}t^4w_2 + \frac{55}{2}t^4w_3 \\
&\quad + \frac{9869}{5274828}t^3s^3 - \frac{205373}{146523}t^3su_1 - \frac{498421}{3757}t^3x_1 + \frac{11}{3}t^3x_2 + \frac{9869}{5274828}t^2s^2v \\
&\quad - \frac{3014}{3757}t^2sw_1 + \frac{11}{9}t^2sw_2 - \frac{7}{3}t^2sw_3 - \frac{165}{17}t^2y, \\
tz &= -\frac{28331}{1107713880}t^9 - \frac{1249}{6153966}t^7s - \frac{87019}{61539660}t^6v + \frac{101}{123079320}t^5s^2 \\
&\quad - \frac{115473}{13675480}t^5u_1 + \frac{55}{24615864}t^4sv - \frac{388147}{9230949}t^4w_1 - \frac{1}{18461898}t^3s^3 \\
&\quad + \frac{43}{9230949}t^3su_1 - \frac{26449}{157794}t^3x_1 - \frac{1}{18461898}t^2s^2v + \frac{1}{157794}t^2sw_1 - \frac{179}{357}t^2y, \\
s^3v &= -\frac{2978}{507}t^9 - \frac{10272}{169}t^7s - \frac{32411}{169}t^6v - \frac{8254}{169}t^5s^2 - \frac{48612}{169}t^5u_1 - \frac{6479}{169}t^4sv \\
&\quad + \frac{111376}{169}t^4w_1 + \frac{3492}{169}t^3s^3 + \frac{51884}{169}t^3su_1 + \frac{4560}{13}t^3x_1 + \frac{8731}{169}t^2s^2v + \frac{864}{13}t^2sw_1, \\
s^2w_1 &= -\frac{423211}{2197845}t^9 - \frac{348476}{341887}t^7s - \frac{1288358}{244205}t^6v + \frac{203141}{244205}t^5s^2 - \frac{3244737}{244205}t^5u_1 \\
&\quad + \frac{199599}{48841}t^4sv + \frac{3397013}{146523}t^4w_1 - \frac{308}{146523}t^3s^3 + \frac{2370856}{146523}t^3su_1 + \frac{988500}{3757}t^3x_1 \\
&\quad - \frac{308}{146523}t^2s^2v + \frac{158718}{3757}t^2sw_1 + \frac{1320}{17}t^2y, \\
s^2w_2 &= \frac{1424473757}{580231080}t^9 + \frac{3491695369}{451290840}t^7s - \frac{99305936}{24176295}t^6v + \frac{148345751}{193410360}t^5s^2 \\
&\quad - \frac{1235838493}{8058765}t^5u_1 + \frac{544}{15}t^5u_2 + \frac{9534368}{4835259}t^4sv - \frac{3137856266}{4835259}t^4w_1 + 49t^4w_2 \\
&\quad + \frac{2610}{11}t^4w_3 - \frac{54913}{390728}t^3s^3 + \frac{38941391}{4835259}t^3su_1 + 2t^3su_2 - \frac{3601952}{3757}t^3x_1 + 34t^3x_2 \\
&\quad - \frac{55672}{439569}t^2s^2v + \frac{995606}{41327}t^2sw_1 + \frac{28}{3}t^2sw_2 + \frac{146}{11}t^2sw_3 + \frac{1584}{17}t^2y, \\
s^2w_3 &= \frac{271307}{450840}t^9 + \frac{9413891}{4733820}t^7s + \frac{228614}{507195}t^6v - \frac{68117}{4057560}t^5s^2 - \frac{1697401}{56355}t^5u_1 + \frac{352}{45}t^5u_2 \\
&\quad - \frac{7022}{101439}t^4sv - \frac{4702732}{33813}t^4w_1 + \frac{22}{3}t^4w_2 + 55t^4w_3 + \frac{193}{101439}t^3s^3 + \frac{3838}{3757}t^3su_1 \\
&\quad - \frac{63756}{289}t^3x_1 + \frac{22}{3}t^3x_2 + \frac{193}{101439}t^2s^2v + \frac{1188}{289}t^2sw_1 + \frac{22}{9}t^2sw_2 \\
&\quad - \frac{14}{3}t^2sw_3 + \frac{264}{17}t^2y, \\
sy &= \frac{28331}{19780605}t^9 + \frac{4996}{439569}t^7s + \frac{174038}{2197845}t^6v - \frac{101}{2197845}t^5s^2 + \frac{115473}{244205}t^5u_1 \\
&\quad - \frac{55}{439569}t^4sv + \frac{3105176}{1318707}t^4w_1 + \frac{4}{1318707}t^3s^3 - \frac{344}{1318707}t^3su_1 + \frac{105796}{11271}t^3x_1 \\
&\quad + \frac{4}{1318707}t^2s^2v - \frac{4}{11271}t^2sw_1 + \frac{1279}{51}t^2y, \\
vx_1 &= \frac{198317}{26374140}t^9 + \frac{8743}{146523}t^7s + \frac{609133}{1465230}t^6v - \frac{707}{2930460}t^5s^2 + \frac{2424933}{976820}t^5u_1 \\
&\quad - \frac{385}{586092}t^4sv + \frac{5434058}{439569}t^4w_1 + \frac{7}{439569}t^3s^3 - \frac{602}{439569}t^3su_1 + \frac{185143}{3757}t^3x_1 \\
&\quad + \frac{7}{439569}t^2s^2v - \frac{7}{3757}t^2sw_1 + \frac{2251}{17}t^2y,
\end{aligned}
$$

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\begin{aligned}
vx_2 = {} & \frac{1752942221}{2320924320}t^9 + \frac{12823165981}{2707745040}t^7 s + \frac{27192059569}{1160462160}t^6 v + \frac{1693971839}{2320924320}t^5 s^2 \\
& + \frac{590022453}{5372510}t^5 u_1 + \frac{391}{180}t^5 u_2 + \frac{102946787}{58023108}t^4 sv + \frac{2315759090}{4835259}t^4 w_1 - \frac{77}{12}t^4 w_2 \\
& + \frac{477}{22}t^4 w_3 - \frac{1397087}{21099312}t^3 s^3 + \frac{35304473}{19341036}t^3 su_1 + \frac{2}{3}t^3 su_2 + \frac{6613466}{3757}t^3 x_1 \\
& + \frac{49}{12}t^3 x_2 - \frac{2569271}{21099312}t^2 s^2 v - \frac{95061}{41327}t^2 sw_1 - \frac{221}{36}t^2 sw_2 + \frac{991}{66}t^2 sw_3 + \frac{67176}{17}t^2 y, \\
u_1 w_1 = {} & \frac{28331}{1861704}t^9 + \frac{6245}{51714}t^7 s + \frac{87019}{103428}t^6 v - \frac{101}{206856}t^5 s^2 + \frac{115473}{22984}t^5 u_1 - \frac{275}{206856}t^4 sv \\
& + \frac{1940735}{77571}t^4 w_1 + \frac{5}{155142}t^3 s^3 - \frac{215}{77571}t^3 su_1 + \frac{132245}{1326}t^3 x_1 + \frac{5}{155142}t^2 s^2 v \\
& - \frac{5}{1326}t^2 sw_1 + \frac{805}{3}t^2 y, \\
u_1 w_2 = {} & -\frac{1696241603}{2320924320}t^9 - \frac{3770703661}{1805163360}t^7 s + \frac{406638451}{193410360}t^6 v + \frac{24948791}{48352590}t^5 s^2 \\
& + \frac{875507521}{16117530}t^5 u_1 - \frac{32}{3}t^5 u_2 + \frac{77926367}{38682072}t^4 sv + \frac{4772084077}{19341036}t^4 w_1 - \frac{41}{3}t^4 w_2 \\
& - \frac{3279}{44}t^4 w_3 + \frac{4129}{390728}t^3 s^3 + \frac{79225709}{19341036}t^3 su_1 - \frac{1}{2}t^3 su_2 + \frac{10794811}{22542}t^3 x_1 \\
& - 10t^3 x_2 - \frac{1460}{439569}t^2 s^2 v + \frac{1338605}{495924}t^2 sw_1 - \frac{17}{6}t^2 sw_2 - \frac{287}{44}t^2 sw_3 + \frac{4082}{17}t^2 y, \\
u_1 w_3 = {} & -\frac{49421519}{158244840}t^9 - \frac{244104829}{246158640}t^7 s - \frac{232604}{1318707}t^6 v + \frac{2701219}{13187070}t^5 s^2 + \frac{1638935}{97682}t^5 u_1 \\
& - \frac{176}{45}t^5 u_2 + \frac{7994645}{10549656}t^4 sv + \frac{107533550}{1318707}t^4 w_1 - \frac{11}{3}t^4 w_2 - \frac{83}{3}t^4 w_3 \\
& - \frac{9583}{5274828}t^3 s^3 + \frac{1842208}{1318707}t^3 su_1 + \frac{1712623}{11271}t^3 x_1 - \frac{11}{3}t^3 x_2 - \frac{9583}{5274828}t^2 s^2 v \\
& + \frac{17941}{22542}t^2 sw_1 - \frac{11}{9}t^2 sw_2 + \frac{11}{6}t^2 sw_3 + \frac{1969}{51}t^2 y, \\
u_2 w_1 = {} & -\frac{10493459}{48352590}t^9 - \frac{124941559}{112822710}t^7 s - \frac{23738857}{6447012}t^6 v - \frac{30171637}{193410360}t^5 s^2 \\
& - \frac{181494101}{21490040}t^5 u_1 - \frac{14}{15}t^5 u_2 - \frac{1710261}{4298008}t^4 sv - \frac{179353831}{9670518}t^4 w_1 + \frac{10}{3}t^4 w_2 \\
& - \frac{119}{11}t^4 w_3 + \frac{2101}{439569}t^3 s^3 - \frac{2720161}{4835259}t^3 su_1 - \frac{771061}{7514}t^3 x_1 - 2t^3 x_2 \\
& + \frac{2101}{439569}t^2 s^2 v + \frac{118517}{123981}t^2 sw_1 + \frac{10}{3}t^2 sw_2 - \frac{64}{11}t^2 sw_3 - \frac{6858}{17}t^2 y, \\
u_2 w_2 = {} & \frac{3676123513}{2785109184}t^9 + \frac{5148236867}{676936260}t^7 s + \frac{5587823539}{154728288}t^6 v + \frac{2045441273}{1547282880}t^5 s^2 \\
& + \frac{33255409283}{193410360}t^5 u_1 + \frac{154}{15}t^5 u_2 + \frac{24508851}{4298008}t^4 sv + \frac{44687560949}{58023108}t^4 w_1 \\
& + \frac{335}{24}t^4 w_2 + \frac{1509}{22}t^4 w_3 + \frac{2651825}{42198624}t^3 s^3 + \frac{1075520785}{58023108}t^3 su_1 + \frac{13}{24}t^3 su_2 \\
& + \frac{19781025}{7514}t^3 x_1 + \frac{77}{8}t^3 x_2 + \frac{4800829}{42198624}t^2 s^2 v + \frac{5379211}{165308}t^2 sw_1 + \frac{77}{24}t^2 sw_2 \\
& + \frac{101}{22}t^2 sw_3 + \frac{79524}{17}t^2 y,
\end{aligned}
$$

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\begin{aligned}
u_2w_3 = {} & \frac{3676123513}{2785109184}t^9 + \frac{5148236867}{676936260}t^7s + \frac{5587823539}{154728288}t^6v + \frac{2045441273}{1547282880}t^5s^2 \\
& + \frac{33255409283}{193410360}t^5u_1 + \frac{154}{15}t^5u_2 + \frac{24508851}{4298008}t^4sv + \frac{44687560949}{58023108}t^4w_1 \\
& + \frac{335}{24}t^4w_2 + \frac{1509}{22}t^4w_3 + \frac{2651825}{42198624}t^3s^3 + \frac{1075520785}{58023108}t^3su_1 + \frac{13}{24}t^3su_2 \\
& + \frac{19781025}{7514}t^3x_1 + \frac{77}{8}t^3x_2 + \frac{4800829}{42198624}t^2s^2v + \frac{5379211}{165308}t^2sw_1 + \frac{77}{24}t^2sw_2 \\
& + \frac{101}{22}t^2sw_3 + \frac{79524}{17}t^2y.
\end{aligned}
$$

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\begin{aligned}
t^3s^2v = {} & \frac{97}{560}t^{10} + \frac{171}{112}t^8s + \frac{297}{70}t^7v + \frac{81}{80}t^6s^2 + \frac{21}{5}t^6u_1 + \frac{3}{10}t^5sv - \frac{112}{5}t^5w_1 \\
& - \frac{41}{80}t^4s^3 - \frac{38}{5}t^4su_1, \\
t^3sw_1 = {} & \frac{113}{17920}t^{10} + \frac{491}{17920}t^8s + \frac{59}{480}t^7v - \frac{53}{1536}t^6s^2 + \frac{303}{2240}t^6u_1 - \frac{171}{1120}t^5sv - \frac{53}{35}t^5w_1 \\
& + \frac{1}{17920}t^4s^3 - \frac{1129}{2240}t^4su_1 - \frac{55}{7}t^4x_1, \\
t^3sw_2 = {} & -\frac{7823}{134400}t^{10} - \frac{1091}{8960}t^8s + \frac{437}{840}t^7v + \frac{217}{3840}t^6s^2 + \frac{6171}{1120}t^6u_1 - \frac{101}{80}t^6u_2 + \frac{3}{14}t^5sv \\
& + \frac{1996}{105}t^5w_1 - \frac{13}{5}t^5w_2 - \frac{42}{5}t^5w_3 + \frac{23}{8960}t^4s^3 + \frac{293}{672}t^4su_1 - \frac{9}{80}t^4su_2 \\
& + \frac{804}{35}t^4x_1 - \frac{9}{10}t^4x_2, \\
t^3sw_3 = {} & -\frac{3499}{89600}t^{10} - \frac{2237}{17920}t^8s + \frac{1}{24}t^7v - \frac{5}{1536}t^6s^2 + \frac{16307}{6720}t^6u_1 - \frac{253}{480}t^6u_2 + \frac{103}{5040}t^5sv \\
& + \frac{1154}{105}t^5w_1 - \frac{11}{15}t^5w_2 - \frac{16}{5}t^5w_3 + \frac{107}{53760}t^4s^3 + \frac{451}{6720}t^4su_1 - \frac{11}{160}t^4su_2 \\
& + \frac{638}{35}t^4x_1 - \frac{11}{20}t^4x_2, \\
t^3y = {} & -\frac{769}{6451200}t^{10} - \frac{1961}{2150400}t^8s - \frac{271}{44800}t^7v + \frac{1}{307200}t^6s^2 - \frac{9061}{268800}t^6u_1 \\
& + \frac{1}{134400}t^5sv - \frac{423}{2800}t^5w_1 - \frac{1}{6451200}t^4s^3 + \frac{1}{89600}t^4su_1 - \frac{281}{560}t^4x_1, \\
s^5 = {} & -\frac{4699}{28}t^{10} - \frac{19701}{28}t^8s + \frac{468}{7}t^7v + \frac{593}{4}t^6s^2 + \frac{20718}{7}t^6u_1 + \frac{10940}{7}t^5sv \\
& + \frac{13696}{7}t^5w_1 + \frac{13373}{28}t^4s^3 + \frac{2606}{7}t^4su_1 + \frac{1920}{7}t^4x_1, \\
s^3u_1 = {} & -\frac{14219}{4032}t^{10} - \frac{1109}{2240}t^8s + \frac{5077}{140}t^7v + \frac{3757}{320}t^6s^2 + \frac{150531}{280}t^6u_1 + \frac{4435}{28}t^5sv \\
& + \frac{73704}{35}t^5w_1 - \frac{5837}{6720}t^4s^3 + \frac{98571}{280}t^4su_1 + \frac{5808}{7}t^4x_1, \\
s^3u_2 = {} & -\frac{605559}{11200}t^{10} - \frac{2365561}{11200}t^8s - \frac{903953}{4200}t^7v - \frac{89257}{4800}t^6s^2 + \frac{1275297}{700}t^6u_1 \\
& - \frac{24121}{40}t^6u_2 - \frac{32163}{1400}t^5sv + \frac{1903824}{175}t^5w_1 - \frac{1548}{5}t^5w_2 - 4788t^5w_3 \\
& + \frac{165853}{11200}t^4s^3 - \frac{146421}{700}t^4su_1 + \frac{2667}{40}t^4su_2 + \frac{835284}{35}t^4x_1 - \frac{3483}{5}t^4x_2,
\end{aligned}
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\begin{aligned}
s^2x_1 &= \frac{214609}{2419200}t^{10} + \frac{55729}{89600}t^8 s + \frac{64511}{16800}t^7 v - \frac{2107}{38400}t^6 s^2 + \frac{640487}{33600}t^6 u_1 - \frac{3547}{16800}t^5 sv \\
&\quad + \frac{25271}{350}t^5 w_1 + \frac{187}{806400}t^4 s^3 - \frac{5787}{11200}t^4 su_1 + \frac{11667}{70}t^4 x_1, \\
svw_2 &= \frac{6926111}{9676800}t^{10} + \frac{3964277}{1075200}t^8 s + \frac{615323}{33600}t^7 v - \frac{73879}{51200}t^6 s^2 + \frac{9098347}{134400}t^6 u_1 \\
&\quad + \frac{679}{640}t^6 u_2 - \frac{25661}{5600}t^5 sv + \frac{861193}{4200}t^5 w_1 - \frac{17}{4}t^5 w_2 + \frac{243}{20}t^5 w_3 + \frac{87677}{3225600}t^4 s^3 \\
&\quad - \frac{1113461}{134400}t^4 su_1 + \frac{19}{640}t^4 su_2 + \frac{122803}{280}t^4 x_1 + \frac{99}{80}t^4 x_2, \\
svw_3 &= -\frac{661805}{774144}t^{10} - \frac{860953}{307200}t^8 s - \frac{3863}{134400}t^7 v + \frac{9917}{102400}t^6 s^2 + \frac{13609109}{268800}t^6 u_1 \\
&\quad - \frac{3201}{320}t^6 u_2 + \frac{28457}{44800}t^5 sv + \frac{2085701}{8400}t^5 w_1 - \frac{11}{2}t^5 w_2 - \frac{349}{5}t^5 w_3 \\
&\quad + \frac{169}{6451200}t^4 s^3 + \frac{217613}{268800}t^4 su_1 + \frac{11}{320}t^4 su_2 + \frac{51733}{112}t^4 x_1 - \frac{429}{40}t^4 x_2, \\
sz &= -\frac{1987}{19353600}t^{10} - \frac{547}{716800}t^8 s - \frac{653}{134400}t^7 v + \frac{1}{307200}t^6 s^2 - \frac{6821}{268800}t^6 u_1 \\
&\quad + \frac{1}{134400}t^5 sv - \frac{283}{2800}t^5 w_1 - \frac{1}{6451200}t^4 s^3 + \frac{1}{89600}t^4 su_1 - \frac{141}{560}t^4 x_1, \\
vy &= -\frac{1987}{3225600}t^{10} - \frac{1641}{358400}t^8 s - \frac{653}{22400}t^7 v + \frac{1}{51200}t^6 s^2 - \frac{6821}{44800}t^6 u_1 \\
&\quad + \frac{1}{22400}t^5 sv - \frac{849}{1400}t^5 w_1 - \frac{1}{1075200}t^4 s^3 + \frac{3}{44800}t^4 su_1 - \frac{423}{280}t^4 x_1, \\
u_1x_1 &= -\frac{1987}{1290240}t^{10} - \frac{1641}{143360}t^8 s - \frac{653}{8960}t^7 v + \frac{1}{20480}t^6 s^2 - \frac{6821}{17920}t^6 u_1 \\
&\quad + \frac{1}{8960}t^5 sv - \frac{849}{560}t^5 w_1 - \frac{1}{430080}t^4 s^3 + \frac{3}{17920}t^4 su_1 - \frac{423}{112}t^4 x_1, \\
u_1x_2 &= \frac{24797}{28800}t^{10} + \frac{142987}{67200}t^8 s - \frac{11981}{1800}t^7 v - \frac{16741}{28800}t^6 s^2 - \frac{799313}{8400}t^6 u_1 + \frac{213}{20}t^6 u_2 \\
&\quad - \frac{28171}{12600}t^5 sv - \frac{454733}{1050}t^5 w_1 + \frac{88}{15}t^5 w_2 + \frac{66}{t}^5 w_3 + \frac{509}{67200}t^4 s^3 - \frac{29501}{8400}t^4 su_1 \\
&\quad - \frac{4}{15}t^4 su_2 - \frac{178733}{210}t^4 x_1 + \frac{737}{60}t^4 x_2, \\
u_2x_1 &= -\frac{1472719}{9676800}t^{10} - \frac{170223}{358400}t^8 s + \frac{12263}{67200}t^7 v + \frac{7109}{153600}t^6 s^2 + \frac{200273}{19200}t^6 u_1 - \frac{131}{80}t^6 u_2 \\
&\quad + \frac{649}{3200}t^5 sv + \frac{30877}{600}t^5 w_1 - \frac{4}{5}t^5 w_2 - \frac{49}{5}t^5 w_3 - \frac{49}{153600}t^4 s^3 + \frac{5161}{19200}t^4 su_1 \\
&\quad + \frac{1}{80}t^4 su_2 + \frac{4049}{40}t^4 x_1 - \frac{19}{10}t^4 x_2, \\
u_2x_2 &= -\frac{14018959}{1814400}t^{10} - \frac{740737}{28800}t^8 s - \frac{1200169}{151200}t^7 v + \frac{153107}{86400}t^6 s^2 + \frac{20598827}{50400}t^6 u_1 \\
&\quad - \frac{129703}{1440}t^6 u_2 + \frac{1101323}{151200}t^5 sv + \frac{1075106}{525}t^5 w_1 - \frac{667}{15}t^5 w_2 - \frac{635}{t}^5 w_3 \\
&\quad - \frac{68219}{604800}t^4 s^3 + \frac{131573}{16800}t^4 su_1 + \frac{2141}{1440}t^4 su_2 + \frac{131847}{35}t^4 x_1 - \frac{1961}{20}t^4 x_2, \\
w_1^2 &= -\frac{1987}{967680}t^{10} - \frac{547}{35840}t^8 s - \frac{653}{6720}t^7 v + \frac{1}{15360}t^6 s^2 - \frac{6821}{13440}t^6 u_1 + \frac{1}{6720}t^5 sv \\
&\quad - \frac{283}{140}t^5 w_1 - \frac{1}{322560}t^4 s^3 + \frac{1}{4480}t^4 su_1 - \frac{141}{28}t^4 x_1,
\end{aligned}
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\begin{aligned}
w_1w_2 &= \frac{667913}{4300800}t^{10} + \frac{661729}{1433600}t^8s - \frac{214337}{806400}t^7v - \frac{246481}{1843200}t^6s^2 - \frac{6013373}{537600}t^6u_1 \\
&\quad + \frac{7411}{3840}t^6u_2 - \frac{111437}{268800}t^5sv - \frac{2691721}{50400}t^5w_1 + \frac{41}{60}t^5w_2 + \frac{629}{40}t^5w_3 \\
&\quad + \frac{17849}{4300800}t^4s^3 - \frac{813583}{1612800}t^4su_1 - \frac{187}{1280}t^4su_2 - \frac{337997}{3360}t^4x_1 + \frac{333}{160}t^4x_2, \\
w_1w_3 &= \frac{2481497}{16588800}t^{10} + \frac{6598189}{12902400}t^8s + \frac{262457}{806400}t^7v - \frac{10301}{204800}t^6s^2 - \frac{11184631}{1612800}t^6u_1 \\
&\quad + \frac{935}{576}t^6u_2 - \frac{492167}{2419200}t^5sv - \frac{1828759}{50400}t^5w_1 + \frac{77}{90}t^5w_2 + \frac{1369}{120}t^5w_3 \\
&\quad + \frac{12389}{38707200}t^4s^3 - \frac{473407}{1612800}t^4su_1 - \frac{11}{960}t^4su_2 - \frac{224323}{3360}t^4x_1 + \frac{209}{120}t^4x_2, \\
w_2^2 &= -\frac{20627}{19353600}t^{10} - \frac{202963}{716800}t^8s - \frac{1116131}{403200}t^7v - \frac{98093}{921600}t^6s^2 - \frac{4410269}{268800}t^6u_1 \\
&\quad + \frac{737}{960}t^6u_2 - \frac{45331}{134400}t^5sv - \frac{489001}{8400}t^5w_1 + \frac{11}{30}t^5w_2 + \frac{11}{2}t^5w_3 \\
&\quad + \frac{4591}{6451200}t^4s^3 - \frac{143413}{268800}t^4su_1 - \frac{11}{960}t^4su_2 - \frac{55861}{560}t^4x_1 + \frac{33}{40}t^4x_2, \\
w_2w_3 &= \frac{2921}{1105920}t^{10} + \frac{14461}{860160}t^8s + \frac{14419}{161280}t^7v - \frac{7}{40960}t^6s^2 + \frac{5743}{15360}t^6u_1 \\
&\quad - \frac{1}{2560}t^5sv + \frac{179}{160}t^5w_1 + \frac{1}{122880}t^4s^3 - \frac{3}{5120}t^4su_1 + \frac{179}{96}t^4x_1, \\
w_3^2 &= -\frac{5159}{8294400}t^{10} - \frac{3157}{921600}t^8s - \frac{1001}{57600}t^7v + \frac{539}{921600}t^6s^2 - \frac{7777}{115200}t^6u_1 \\
&\quad + \frac{77}{57600}t^5sv - \frac{77}{400}t^5w_1 - \frac{77}{2764800}t^4s^3 + \frac{77}{38400}t^4su_1 - \frac{77}{240}t^4x_1.
\end{aligned}
$$

Codegree 11

$$
\begin{aligned}
t^5s^3 &= \frac{53}{147}t^{11} + \frac{73}{49}t^9s - \frac{24}{49}t^8v - \frac{15}{49}t^7s^2 - \frac{48}{7}t^7u_1 - \frac{24}{7}t^6sv, \\
t^5su_1 &= \frac{17}{1617}t^{11} - \frac{13}{1078}t^9s - \frac{95}{539}t^8v - \frac{159}{1078}t^7s^2 - \frac{135}{77}t^7u_1 - \frac{39}{77}t^6sv - \frac{64}{11}t^6w_1, \\
t^5su_2 &= -\frac{7439}{48510}t^{11} - \frac{10789}{16170}t^9s - \frac{3376}{2695}t^8v - \frac{18}{49}t^7s^2 + \frac{34}{77}t^7u_1 - \frac{3}{7}t^7u_2 - \frac{82}{105}t^6sv \\
&\quad + \frac{96}{11}t^6w_1 + \frac{96}{11}t^6w_3, \\
t^5x_1 &= -\frac{1759}{2457840}t^{11} - \frac{4149}{819280}t^9s - \frac{9281}{307230}t^8v + \frac{1}{61446}t^7s^2 - \frac{211}{1463}t^7u_1 \\
&\quad + \frac{1}{43890}t^6sv - \frac{105}{209}t^6w_1, \\
t^5x_2 &= -\frac{103727}{921690}t^{11} - \frac{76261}{153615}t^9s - \frac{1939876}{1382535} + \frac{761}{553014}t^7s^2 - \frac{25519}{13167}t^7u_1 - \frac{19}{21}t^7u_2 \\
&\quad + \frac{5501}{197505}t^6sv + \frac{96}{209}t^6w_1 - \frac{4}{9}t^6w_2 - \frac{200}{33}t^6w_3, \\
vz &= 0, \\
u_1y &= 0, \\
u_2y &= -\frac{4979}{110602800}t^{11} - \frac{52627}{13825350}t^9s - \frac{25226}{768075}t^8v + \frac{29}{614460}t^7s^2 - \frac{13933}{87780}t^7u_1 \\
&\quad + \frac{29}{438900}t^6sv - \frac{2447}{6270}t^6w_1 - \frac{1}{6}t^6w_3,
\end{aligned}
$$

$$
\begin{aligned}
w_1x_1 &= 0,\\
w_1x_2 &= \frac{15581}{9216900}t^{11} + \frac{1757219}{27650700}t^9 s + \frac{1294471}{2513700}t^8 v + \frac{6035}{1106028}t^7 s^2 + \frac{318643}{131670}t^7 u_1\\
&\quad - \frac{4}{105}t^7 u_2 + \frac{41879}{3950100}t^6 sv + \frac{17912}{3135}t^6 w_1 - \frac{1}{45}t^6 w_2 + \frac{262}{165}t^6 w_3,\\
w_2x_1 &= -\frac{138679}{126403200}t^{11} + \frac{82583}{126403200}t^9 s + \frac{648539}{15800400}t^8 v + \frac{367}{790020}t^7 s^2 + \frac{137509}{526680}t^7 u_1\\
&\quad - \frac{1}{30}t^7 u_2 - \frac{5633}{4514400}t^6 sv + \frac{8137}{12540}t^6 w_1 + \frac{37}{360}t^6 w_2 - \frac{98}{165}t^6 w_3,\\
w_2x_2 &= -\frac{361001}{13825350}t^{11} - \frac{881611}{4608450}t^9 s - \frac{617984}{768075}t^8 v - \frac{4376}{51205}t^7 s^2 - \frac{659159}{263340}t^7 u_1\\
&\quad - \frac{31}{252}t^7 u_2 - \frac{693521}{3950100}t^6 sv - \frac{11612}{3135}t^6 w_1 - \frac{17}{45}t^6 w_2 - \frac{22}{15}t^6 w_3,\\
w_3x_1 &= \frac{19129}{80438400}t^{11} + \frac{27149}{8937600}t^9 s + \frac{1331623}{60328800}t^8 v + \frac{124}{377055}t^7 s^2 + \frac{13759}{143640}t^7 u_1\\
&\quad - \frac{11}{630}t^7 u_2 - \frac{737}{2154600}t^6 sv + \frac{71}{380}t^6 w_1 - \frac{11}{540}t^6 w_2 - \frac{91}{360}t^6 w_3,\\
w_3x_2 &= -\frac{746203}{15800400}t^{11} - \frac{4453159}{15800400}t^9 s - \frac{3331721}{2633400}t^8 v - \frac{1493}{87780}t^7 s^2 - \frac{143107}{35112}t^7 u_1\\
&\quad - \frac{1}{40}t^7 u_2 - \frac{2971}{94050}t^6 sv - \frac{2023}{285}t^6 w_1 - \frac{21}{55}t^6 w_3.
\end{aligned}
$$

Codegree 12

$$
\begin{aligned}
t^7 sv &= -\frac{3}{224}t^{12} - \frac{41}{112}t^{10}s - \frac{59}{42}t^9 v - \frac{115}{224}t^8 s^2 - \frac{15}{4}t^8 u_1,\\
t^7 w_1 &= -\frac{221}{48384}t^{12} - \frac{229}{8064}t^{10}s - \frac{71}{504}t^9 v + \frac{1}{16128}t^8 s^2 - \frac{145}{288}t^8 u_1,\\
t^7 w_2 &= \frac{29}{2880}t^{12} + \frac{5}{96}t^{10}s + \frac{13}{72}t^9 v + \frac{1}{576}t^8 s^2 + \frac{1}{3}t^8 u_1 - \frac{1}{12}t^8 u_2,\\
t^7 w_3 &= -\frac{1313}{241920}t^{12} - \frac{169}{8064}t^{10}s - \frac{151}{3024}t^9 v + \frac{107}{48384}t^8 s^2 - \frac{11}{288}t^8 u_1 - \frac{11}{144}t^8 u_2,\\
u_1 z &= 0,\\
u_2 z &= -\frac{15671}{55883520}t^{12} - \frac{863}{846720}t^{10}s - \frac{1493}{635040}t^9 v + \frac{263}{5080320}t^8 s^2\\
&\quad - \frac{31}{15120}t^8 u_1 - \frac{11}{6048}t^8 u_2,\\
w_1 y &= 0,\\
w_2 y &= -\frac{31403}{111767040}t^{12} - \frac{10663}{8467200}t^{10}s - \frac{1033}{254016}t^9 v + \frac{299}{10160640}t^8 s^2\\
&\quad - \frac{31}{4320}t^8 u_1 - \frac{11}{12096}t^8 u_2,\\
w_3 y &= -\frac{40631}{243855360}t^{12} - \frac{136279}{203212800}t^{10}s - \frac{5687}{3048192}t^9 v + \frac{6017}{243855360}t^8 s^2\\
&\quad - \frac{3817}{1451520}t^8 u_1 - \frac{121}{145152}t^8 u_2,\\
x_1^2 &= 0,\\
x_1x_2 &= \frac{15671}{2794176}t^{12} + \frac{863}{42336}t^{10}s + \frac{1493}{31752}t^9 v - \frac{263}{254016}t^8 s^2 + \frac{31}{756}t^8 u_1 + \frac{55}{1512}t^8 u_2,\\
x_2^2 &= -\frac{815827}{3492720}t^{12} - \frac{11353}{10584}t^{10}s - \frac{13688}{3969}t^9 v + \frac{1943}{317520}t^8 s^2 - \frac{1609}{270}t^8 u_1 - \frac{356}{945}t^8 u_2.
\end{aligned}
$$

Codegree 13

$$
\begin{aligned}
t^9 s^2 &= -\frac{9}{65}t^{13} - \frac{6}{5}t^{11}s - \frac{8}{5}t^{10}v,\\
t^9 u_1 &= -\frac{1007}{36465}t^{13} - \frac{131}{935}t^{11}s - \frac{43}{85}t^{10}v,\\
t^9 u_2 &= -\frac{229}{1870}t^{13} - \frac{783}{1870}t^{11}s - \frac{62}{85}t^{10}v,\\
w_1 z &= 0,\\
w_2 z &= \frac{101}{61261200}t^{13} + \frac{23}{3534300}t^{11}s + \frac{1}{71400}t^{10}v,\\
w_3 z &= \frac{101}{267321600}t^{13} + \frac{23}{15422400}t^{11}s + \frac{11}{3427200}t^{10}v,\\
x_1 y &= 0,\\
x_2 y &= 0.
\end{aligned}
$$

Codegree 14

$$
\begin{aligned}
t^{11}v &= \frac{9}{64}t^{14} - \frac{33}{64}t^{12}s,\\
x_1 z &= 0,\\
x_2 z &= 0,\\
y^2 &= 0.
\end{aligned}
$$

Codegree 15

$$
\begin{aligned}
t^{13}s &= -\frac{7}{15}t^{15},\\
yz &= 0.
\end{aligned}
$$

Codegree 16

$$z^2 = 0.$$

Codegree 17

$$t^{17} = 0.$$

Appendix C

The Principal Kinematic Formula

In the last part of Appendix, the Principal kinematic formula on the octonionic plane is given in terms of the basis introduced in Theorem 4.22. This completes the statement of Theorem 4.25. Let us recall that, keeping the notation of §4.4.3, the Principal kinematic formula reads

$$\int_{\overline{\mathrm{Spin}(9)}} \chi(K \cap \bar{g}L) = \sum_{k=0}^{16} \sum_{i,j=1}^{d_k} (M_k^{-1})_{i,j} \Psi_k^{(i)}(K)\, \Psi_{16-k}^{(j)}(L), \quad K, L \in \mathcal{K}(\mathbb{O}^2).$$

Here we state explicitly the middle part corresponding to $k = 8$. Although we do not, for sake of space, list the rest, notice that it is computed in exactly the same way. We denote

$$\mu \odot \nu = \frac{1}{2}\left(\mu \otimes \nu + \nu \otimes \mu\right).$$

Then one has

$$\frac{806400^2}{5} \sum_{i,j=1}^{27} (M_8^{-1})_{i,j} \Psi_8^{(i)} \otimes \Psi_8^{(j)} =$$

$$\begin{aligned}
&\frac{101042723}{8448} t^8 \odot t^8 + \frac{41912485}{528} t^8 \odot t^6 s + \frac{131398835}{3168} t^8 \odot t^5 v - \frac{10178935}{792} t^8 \odot t^4 s^2 \\
&- \frac{600048805}{528} t^8 \odot t^4 u_1 + \frac{4538995}{16} t^8 \odot t^4 u_2 - \frac{37975765}{792} t^8 \odot t^3 sv \\
&- \frac{189207035}{33} t^8 \odot t^3 w_1 + \frac{5826055}{24} t^8 \odot t^3 w_2 + \frac{7976745}{4} t^8 \odot t^3 w_3 + \frac{133415}{352} t^8 \odot t^2 s^3 \\
&- \frac{10151365}{132} t^8 \odot t^2 s u_1 + \frac{8215}{8} t^8 \odot t^2 s u_2 - \frac{112512535}{11} t^8 \odot t^2 x_1 + \frac{2265005}{8} t^8 \odot t^2 x_2 \\
&+ \frac{2835155}{3168} t^8 \odot t s^2 v - \frac{79005}{44} t^8 \odot t s w_1 + \frac{397455}{8} t^8 \odot t s w_2 - \frac{360555}{11} t^8 \odot t s w_3 \\
&+ \frac{5465925}{22} t^8 \odot t y - \frac{190945}{12672} t^8 \odot s^4 + \frac{163545}{176} t^8 \odot s^2 u_1 + \frac{25135}{48} t^8 \odot s^2 u_2 \\
&+ \frac{882875}{11} t^8 \odot s x_1 - 9900 t^8 \odot v w_2 - \frac{661775}{22} t^8 \odot v w_3 + \frac{20210400}{11} t^8 \odot z \\
&+ \frac{17385985}{132} t^6 s \odot t^6 s + \frac{436979155}{3168} t^6 s \odot t^5 v - \frac{270203215}{6336} t^6 s \odot t^4 s^2 \\
&- \frac{662726705}{176} t^6 s \odot t^4 u_1 + \frac{45172565}{48} t^6 s \odot t^4 u_2 - \frac{31560245}{198} t^6 s \odot t^3 sv \\
&- \frac{7507001455}{396} t^6 s \odot t^3 w_1 + \frac{6451745}{8} t^6 s \odot t^3 w_2 + \frac{72690555}{11} t^6 s \odot t^3 w_3
\end{aligned}$$

$$
\begin{aligned}
&+\frac{999775}{792}t^6s\odot t^2s^3-\frac{101244185}{396}t^6s\odot t^2su_1+\frac{27955}{8}t^6s\odot t^2su_2\\
&-\frac{2196730045}{66}t^6s\odot t^2x_1+\frac{7511925}{8}t^6s\odot t^2x_2+\frac{9489955}{3168}t^6s\odot ts^2v\\
&+\frac{37990}{9}t^6s\odot tsw_1+\frac{4010285}{24}t^6s\odot tsw_2-\frac{5102905}{44}t^6s\odot tsw_3+\frac{113103315}{22}t^6s\odot ty\\
&-\frac{315575}{6336}t^6s\odot s^4+\frac{5247175}{1584}t^6s\odot s^2u_1+\frac{83515}{48}t^6s\odot s^2u_2+\frac{3741990}{11}t^6s\odot sx_1\\
&-\frac{93575}{3}t^6s\odot vw_2-\frac{2197925}{22}t^6s\odot vw_3+\frac{228977280}{11}t^6s\odot z+\frac{372273293}{9504}t^5v\odot t^5v\\
&-\frac{19067905}{864}t^5v\odot t^4s^2-\frac{493836119}{264}t^5v\odot t^4u_1+\frac{35130295}{72}t^5v\odot t^4u_2\\
&-\frac{66023681}{792}t^5v\odot t^3sv-\frac{10474501303}{1188}t^5v\odot t^3w_1+\frac{5095775}{12}t^5v\odot t^3w_2\\
&+\frac{148030505}{44}t^5v\odot t^3w_3+\frac{26071}{36}t^5v\odot t^2s^3-\frac{139961069}{1188}t^5v\odot t^2su_1\\
&+\frac{24175}{12}t^5v\odot t^2su_2-\frac{1569950995}{198}t^5v\odot t^2x_1+\frac{1943975}{4}t^5v\odot t^2x_2\\
&+\frac{358893}{176}t^5v\odot ts^2v+\frac{236431805}{1188}t^5v\odot tsw_1+\frac{416265}{4}t^5v\odot tsw_2-\frac{5516905}{44}t^5v\odot tsw_3\\
&+\frac{646168905}{11}t^5v\odot ty-\frac{195559}{9504}t^5v\odot s^4+\frac{11259665}{2376}t^5v\odot s^2u_1+\frac{65105}{72}t^5v\odot s^2u_2\\
&+\frac{12228860}{11}t^5v\odot sx_1-\frac{9665}{9}t^5v\odot vw_2-\frac{1898395}{33}t^5v\odot vw_3+\frac{2034789120}{11}t^5v\odot z\\
&+\frac{132412835}{38016}t^4s^2\odot t^4s^2+\frac{108459275}{176}t^4s^2\odot t^4u_1-\frac{21953365}{144}t^4s^2\odot t^4u_2\\
&+\frac{40775785}{1584}t^4s^2\odot t^3sv+\frac{7474927855}{2376}t^4s^2\odot t^3w_1-\frac{1031415}{8}t^4s^2\odot t^3w_2\\
&-\frac{94810205}{88}t^4s^2\odot t^3w_3-\frac{313285}{1584}t^4s^2\odot t^2s^3+\frac{98639645}{2376}t^4s^2\odot t^2su_1\\
&-\frac{6265}{24}t^4s^2\odot t^2su_2+\frac{2395429675}{396}t^4s^2\odot t^2x_1-\frac{1223425}{8}t^4s^2\odot t^2x_2\\
&-\frac{1392715}{3168}t^4s^2\odot ts^2v+\frac{2469905}{216}t^4s^2\odot tsw_1-\frac{194775}{8}t^4s^2\odot tsw_2+\frac{1053325}{88}t^4s^2\odot tsw_3\\
&+\frac{69259275}{22}t^4s^2\odot ty+\frac{76115}{9504}t^4s^2\odot s^4-\frac{1319135}{4752}t^4s^2\odot s^2u_1-\frac{38675}{144}t^4s^2\odot s^2u_2\\
&+\frac{172200}{11}t^4s^2\odot sx_1+\frac{106925}{18}t^4s^2\odot vw_2+\frac{1117375}{66}t^4s^2\odot vw_3+\frac{97070400}{11}t^4s^2\odot z\\
&+\frac{7302883913}{264}t^4u_1\odot t^4u_1-\frac{162129715}{12}t^4u_1\odot t^4u_2+\frac{305659649}{132}t^4u_1\odot t^3sv\\
&+\frac{56608773187}{198}t^4u_1\odot t^3w_1-\frac{69294665}{6}t^4u_1\odot t^3w_2-\frac{190666245}{2}t^4u_1\odot t^3w_3\\
&-\frac{3367217}{198}t^4u_1\odot t^2s^3+\frac{796349201}{198}t^4u_1\odot t^2su_1-\frac{295565}{6}t^4u_1\odot t^2su_2\\
&+\frac{19501574935}{33}t^4u_1\odot t^2x_1-\frac{26975625}{2}t^4u_1\odot t^2x_2-\frac{28250459}{792}t^4u_1\odot ts^2v\\
&+\frac{518106715}{198}t^4u_1\odot tsw_1-\frac{4543815}{2}t^4u_1\odot tsw_2+\frac{2273065}{2}t^4u_1\odot tsw_3\\
&+\frac{6008331870}{11}t^4u_1\odot ty+\frac{413467}{528}t^4u_1\odot s^4-\frac{5377685}{396}t^4u_1\odot s^2u_1-\frac{99455}{4}t^4u_1\odot s^2u_2\\
&+\frac{57302840}{11}t^4u_1\odot sx_1+\frac{1667170}{3}t^4u_1\odot vw_2+\frac{15188410}{11}t^4u_1\odot vw_3
\end{aligned}
$$

$$
\begin{aligned}
&+\frac{16174529280}{11}t^4u_1\odot z+\frac{40431875}{24}t^4u_2\odot t^4u_2-\frac{6793615}{12}t^4u_2\odot t^3sv\\
&-\frac{1226983165}{18}t^4u_2\odot t^3w_1+\frac{17297225}{6}t^4u_2\odot t^3w_2+\frac{47471025}{2}t^4u_2\odot t^3w_3\\
&+\frac{40190}{9}t^4u_2\odot t^2s^3-\frac{16143095}{18}t^4u_2\odot t^2su_1+\frac{74525}{6}t^4u_2\odot t^2su_2\\
&-\frac{366421225}{3}t^4u_2\odot t^2x_1+\frac{6726225}{2}t^4u_2\odot t^2x_2+\frac{761065}{72}t^4u_2\odot ts^2v\\
&+\frac{307835}{18}t^4u_2\odot tsw_1+\frac{1149775}{2}t^4u_2\odot tsw_2-\frac{655525}{2}t^4u_2\odot tsw_3-2063250t^4u_2\odot ty\\
&-\frac{25745}{144}t^4u_2\odot s^4+\frac{388055}{36}t^4u_2\odot s^2u_1+\frac{74525}{12}t^4u_2\odot s^2u_2+917000t^4u_2\odot sx_1\\
&-\frac{395450}{3}t^4u_2\odot vw_2-355350t^4u_2\odot vw_3+\frac{14256331}{297}t^3sv\odot t^3sv\\
&+\frac{108352963}{9}t^3sv\odot t^3w_1-\frac{2819225}{6}t^3sv\odot t^3w_2-\frac{132191920}{33}t^3sv\odot t^3w_3\\
&-\frac{222529}{297}t^3sv\odot t^2s^3+\frac{16188959}{99}t^3sv\odot t^2su_1-\frac{1675}{6}t^3sv\odot t^2su_2\\
&+\frac{275442210}{11}t^3sv\odot t^2x_1-\frac{3412525}{6}t^3sv\odot t^2x_2-\frac{3443051}{2376}t^3sv\odot ts^2v\\
&+\frac{10790080}{99}t^3sv\odot tsw_1-\frac{1582885}{18}t^3sv\odot tsw_2+\frac{1764145}{33}t^3sv\odot tsw_3+\\
&+\frac{254846310}{11}t^3sv\odot ty+\frac{52823}{1584}t^3sv\odot s^4-\frac{127885}{396}t^3sv\odot s^2u_1-\frac{12385}{12}t^3sv\odot s^2u_2\\
&+\frac{7738360}{33}t^3sv\odot sx_1+\frac{68660}{3}t^3sv\odot vw_2+\frac{2074690}{33}t^3sv\odot vw_3+\frac{629475840}{11}t^3sv\odot z\\
&+\frac{222457237834}{297}t^3w_1\odot t^3w_1-\frac{525452755}{9}t^3w_1\odot t^3w_2-\frac{5283356470}{11}t^3w_1\odot t^3w_3\\
&-\frac{23603398}{297}t^3w_1\odot t^2s^3+\frac{6703353544}{297}t^3w_1\odot t^2su_1-\frac{763135}{3}t^3w_1\odot t^2su_2\\
&+\frac{327601805680}{99}t^3w_1\odot t^2x_1-\frac{204106025}{3}t^3w_1\odot t^2x_2-\frac{58823117}{396}t^3w_1\odot ts^2v\\
&+\frac{6633686990}{297}t^3w_1\odot tsw_1-\frac{105045545}{9}t^3w_1\odot tsw_2+\frac{210155900}{33}t^3w_1\odot tsw_3\\
&+\frac{43860881220}{11}t^3w_1\odot ty+\frac{9913909}{2376}t^3w_1\odot s^4-\frac{1418575}{594}t^3w_1\odot s^2u_1\\
&-\frac{2256995}{18}t^3w_1\odot s^2u_2+\frac{1412144720}{33}t^3w_1\odot sx_1+\frac{23670160}{9}t^3w_1\odot vw_2\\
&+\frac{230312380}{33}t^3w_1\odot vw_3+\frac{104970055680}{11}t^3w_1\odot z+\frac{7244875}{6}t^3w_2\odot t^3w_2\\
&+20489175t^3w_2\odot t^3w_3+\frac{34565}{9}t^3w_2\odot t^2s^3-\frac{6291365}{9}t^3w_2\odot t^2su_1\\
&+\frac{24475}{3}t^3w_2\odot t^2su_2-\frac{314478350}{3}t^3w_2\odot t^2x_1+2880075t^3w_2\odot t^2x_2\\
&+\frac{323075}{36}t^3w_2\odot ts^2v+\frac{1721765}{9}t^3w_2\odot tsw_1+437525t^3w_2\odot tsw_2-149075t^3w_2\odot tsw_3\\
&-2223900t^3w_2\odot ty-\frac{1235}{8}t^3w_2\odot s^4+\frac{165245}{18}t^3w_2\odot s^2u_1+\frac{10725}{2}t^3w_2\odot s^2u_2\\
&+988400t^3w_2\odot sx_1-\frac{467500}{3}t^3w_2\odot vw_2-300500t^3w_2\odot vw_3+\frac{918940050}{11}t^3w_3\odot t^3w_3\\
&+\frac{93100}{3}t^3w_3\odot t^2s^3-\frac{70189910}{11}t^3w_3\odot t^2su_1+98525t^3w_3\odot t^2su_2
\end{aligned}
$$

$$
\begin{aligned}
&- 860220900t^3w_3 \odot t^2x_1 + 23687475t^3w_3 \odot t^2x_2 + \frac{9783655}{132}t^3w_3 \odot ts^2v \\
&+ \frac{103320}{11}t^3w_3 \odot tsw_1 + 4089225t^3w_3 \odot tsw_2 - \frac{23682750}{11}t^3w_3 \odot tsw_3 \\
&- \frac{158703300}{11}t^3w_3 \odot ty - \frac{330505}{264}t^3w_3 \odot s^4 + \frac{1660785}{22}t^3w_3 \odot s^2u_1 + \frac{86975}{2}t^3w_3 \odot s^2u_2 \\
&+ \frac{70534800}{11}t^3w_3 \odot sx_1 - 919800t^3w_3 \odot vw_2 - \frac{27461700}{11}t^3w_3 \odot vw_3 + \frac{26489}{4752}t^2s^3 \odot t^2s^3 \\
&- \frac{296414}{297}t^2s^3 \odot t^2su_1 + \frac{200}{9}t^2s^3 \odot t^2su_2 - \frac{7172180}{99}t^2s^3 \odot t^2x_1 + \frac{13375}{3}t^2s^3 \odot t^2x_2 \\
&+ \frac{36571}{1188}t^2s^3 \odot ts^2v + \frac{657680}{297}t^2s^3 \odot tsw_1 + \frac{8395}{9}t^2s^3 \odot tsw_2 - \frac{34750}{33}t^2s^3 \odot tsw_3 \\
&+ \frac{4812180}{11}t^2s^3 \odot ty - \frac{359}{1056}t^2s^3 \odot s^4 + \frac{37385}{594}t^2s^3 \odot s^2u_1 + 15t^2s^3 \odot s^2u_2 \\
&+ 8260t^2s^3 \odot sx_1 + \frac{40}{9}t^2s^3 \odot vw_2 - \frac{19780}{33}t^2s^3 \odot vw_3 + \frac{10926720}{11}t^2s^3 \odot z \\
&+ \frac{54046126}{297}t^2su_1 \odot t^2su_1 + 3905t^2su_1 \odot t^2su_2 + \frac{6185875640}{99}t^2su_1 \odot t^2x_1 \\
&- \frac{2724175}{3}t^2su_1 \odot t^2x_2 - \frac{94211}{396}t^2su_1 \odot ts^2v + \frac{222580090}{297}t^2su_1 \odot tsw_1 \\
&- \frac{1083055}{9}t^2su_1 \odot tsw_2 + \frac{3730420}{33}t^2su_1 \odot tsw_3 + \frac{1463127060}{11}t^2su_1 \odot ty \\
&+ \frac{178397}{2376}t^2su_1 \odot s^4 + \frac{3233635}{594}t^2su_1 \odot s^2u_1 - \frac{30865}{18}t^2su_1 \odot s^2u_2 + \frac{59595760}{33}t^2su_1 \odot sx_1 \\
&+ \frac{374480}{9}t^2su_1 \odot vw_2 + \frac{3496940}{33}t^2su_1 \odot vw_3 + \frac{3213181440}{11}t^2su_1 \odot z \\
&+ \frac{275}{6}t^2su_2 \odot t^2su_2 - 473750t^2su_2 \odot t^2x_1 + 12375t^2su_2 \odot t^2x_2 + \frac{725}{12}t^2su_2 \odot ts^2v \\
&+ \frac{54365}{3}t^2su_2 \odot tsw_1 - \frac{8525}{3}t^2su_2 \odot tsw_2 + 11275t^2su_2 \odot tsw_3 - 56700t^2su_2 \odot ty \\
&- \frac{95}{72}t^2su_2 \odot s^4 + \frac{395}{6}t^2su_2 \odot s^2u_1 + \frac{275}{6}t^2su_2 \odot s^2u_2 + 25200t^2su_2 \odot sx_1 \\
&- 5500t^2su_2 \odot vw_2 + 1500t^2su_2 \odot vw_3 + \frac{14477882600}{3}t^2x_1 \odot t^2x_1 \\
&- 121877050t^2x_1 \odot t^2x_2 + \frac{3463615}{66}t^2x_1 \odot ts^2v + \frac{13225952500}{99}t^2x_1 \odot tsw_1 \\
&- \frac{63199450}{3}t^2x_1 \odot tsw_2 + 11526800t^2x_1 \odot tsw_3 + \frac{242526403800}{11}t^2x_1 \odot ty \\
&+ \frac{3826345}{396}t^2x_1 \odot s^4 + \frac{6844375}{9}t^2x_1 \odot s^2u_1 - \frac{673175}{3}t^2x_1 \odot s^2u_2 \\
&+ \frac{3086557600}{11}t^2x_1 \odot sx_1 + \frac{13851200}{3}t^2x_1 \odot vw_2 + 12442600t^2x_1 \odot vw_3 \\
&+ \frac{522924595200}{11}t^2x_1 \odot z + \frac{918940050}{11}t^3w_3 \odot t^3w_3 + \frac{93100}{3}t^3w_3 \odot t^2s^3 \\
&- \frac{70189910}{11}t^3w_3 \odot t^2su_1 + 98525t^3w_3 \odot t^2su_2 - 860220900t^3w_3 \odot t^2x_1 \\
&+ 23687475t^3w_3 \odot t^2x_2 + \frac{9783655}{132}t^3w_3 \odot ts^2v + \frac{103320}{11}t^3w_3 \odot tsw_1 \\
&+ 4089225t^3w_3 \odot tsw_2 - \frac{23682750}{11}t^3w_3 \odot tsw_3 - \frac{158703300}{11}t^3w_3 \odot ty \\
&- \frac{330505}{264}t^3w_3 \odot s^4 + \frac{1660785}{22}t^3w_3 \odot s^2u_1 + \frac{86975}{2}t^3w_3 \odot s^2u_2
\end{aligned}
$$

$$
\begin{aligned}
&+\frac{70534800}{11}t^3w_3\odot sx_1-919800t^3w_3\odot vw_2-\frac{27461700}{11}t^3w_3\odot vw_3+\frac{26489}{4752}t^2s^3\odot t^2s^3\\
&-\frac{296414}{297}t^2s^3\odot t^2su_1+\frac{200}{9}t^2s^3\odot t^2su_2-\frac{7172180}{99}t^2s^3\odot t^2x_1+\frac{13375}{3}t^2s^3\odot t^2x_2\\
&+\frac{36571}{1188}t^2s^3\odot ts^2v+\frac{657680}{297}t^2s^3\odot tsw_1+\frac{8395}{9}t^2s^3\odot tsw_2-\frac{34750}{33}t^2s^3\odot tsw_3\\
&+\frac{4812180}{11}t^2s^3\odot ty-\frac{359}{1056}t^2s^3\odot s^4+\frac{37385}{594}t^2s^3\odot s^2u_1+15t^2s^3\odot s^2u_2\\
&+8260t^2s^3\odot sx_1+\frac{40}{9}t^2s^3\odot vw_2-\frac{19780}{33}t^2s^3\odot vw_3+\frac{10926720}{11}t^2s^3\odot z\\
&+\frac{54046126}{297}t^2su_1\odot t^2su_1+3905t^2su_1\odot t^2su_2+\frac{6185875640}{99}t^2su_1\odot t^2x_1\\
&-\frac{2724175}{3}t^2su_1\odot t^2x_2-\frac{94211}{396}t^2su_1\odot ts^2v+\frac{222580090}{297}t^2su_1\odot tsw_1\\
&-\frac{1083055}{9}t^2su_1\odot tsw_2+\frac{3730420}{33}t^2su_1\odot tsw_3+\frac{1463127060}{11}t^2su_1\odot ty\\
&+\frac{178397}{2376}t^2su_1\odot s^4+\frac{3233635}{594}t^2su_1\odot s^2u_1-\frac{30865}{18}t^2su_1\odot s^2u_2+\frac{59595760}{33}t^2su_1\odot sx_1\\
&+\frac{374480}{9}t^2su_1\odot vw_2+\frac{3496940}{33}t^2su_1\odot vw_3+\frac{3213181440}{11}t^2su_1\odot z+\frac{275}{6}t^2su_2\odot t^2su_2\\
&-473750t^2su_2\odot t^2x_1+12375t^2su_2\odot t^2x_2+\frac{725}{12}t^2su_2\odot ts^2v+\frac{54365}{3}t^2su_2\odot tsw_1\\
&-\frac{8525}{3}t^2su_2\odot tsw_2+11275t^2su_2\odot tsw_3-56700t^2su_2\odot ty-\frac{95}{72}t^2su_2\odot s^4\\
&+\frac{395}{6}t^2su_2\odot s^2u_1+\frac{275}{6}t^2su_2\odot s^2u_2+25200t^2su_2\odot sx_1-5500t^2su_2\odot vw_2\\
&+1500t^2su_2\odot vw_3+\frac{14477882600}{3}t^2x_1\odot t^2x_1-121877050t^2x_1\odot t^2x_2\\
&+\frac{3463615}{66}t^2x_1\odot ts^2v+\frac{13225952500}{99}t^2x_1\odot tsw_1-\frac{63199450}{3}t^2x_1\odot tsw_2\\
&+11526800t^2x_1\odot tsw_3+\frac{242526403800}{11}t^2x_1\odot ty+\frac{3826345}{396}t^2x_1\odot s^4\\
&+\frac{6844375}{9}t^2x_1\odot s^2u_1-\frac{673175}{3}t^2x_1\odot s^2u_2+\frac{3086557600}{11}t^2x_1\odot sx_1\\
&+\frac{13851200}{3}t^2x_1\odot vw_2+12442600t^2x_1\odot vw_3+\frac{522924595200}{11}t^2x_1\odot z\\
&+\frac{3356925}{2}t^2x_2\odot t^2x_2+\frac{126575}{12}t^2x_2\odot ts^2v-\frac{37325}{3}t^2x_2\odot tsw_1+578875t^2x_2\odot tsw_2\\
&-325875t^2x_2\odot tsw_3-1984500t^2x_2\odot ty-\frac{1425}{8}t^2x_2\odot s^4+\frac{64525}{6}t^2x_2\odot s^2u_1\\
&+\frac{12375}{2}t^2x_2\odot s^2u_2+882000t^2x_2\odot sx_1-126500t^2x_2\odot vw_2-343500t^2x_2\odot vw_3\\
&+\frac{532517}{9504}ts^2v\odot ts^2v+\frac{5089175}{396}ts^2v\odot tsw_1+\frac{68795}{36}ts^2v\odot tsw_2-\frac{217055}{132}ts^2v\odot tsw_3\\
&+\frac{19873065}{11}ts^2v\odot ty-\frac{4633}{9504}ts^2v\odot s^4+\frac{166205}{792}ts^2v\odot s^2u_1+\frac{2735}{72}ts^2v\odot s^2u_2\\
&+\frac{1034740}{33}ts^2v\odot sx_1-\frac{935}{3}ts^2v\odot vw_2-\frac{38915}{33}ts^2v\odot vw_3+\frac{39432960}{11}ts^2v\odot z\\
&+\frac{502001260}{297}tsw_1\odot tsw_1+\frac{590395}{9}tsw_1\odot tsw_2+\frac{4781930}{33}tsw_1\odot tsw_3\\
&+\frac{5760977100}{11}tsw_1\odot ty+\frac{254365}{2376}tsw_1\odot s^4+\frac{1659235}{54}tsw_1\odot s^2u_1-\frac{3995}{18}tsw_1\odot s^2u_2
\end{aligned}
$$

$$
\begin{aligned}
&+ \frac{245058800}{33} tsw_1 \odot sx_1 + \frac{59800}{9} tsw_1 \odot vw_2 + \frac{876100}{33} tsw_1 \odot vw_3 \\
&+ \frac{11292825600}{11} tsw_1 \odot z + \frac{241175}{6} tsw_2 \odot tsw_2 - 129025 tsw_2 \odot tsw_3 \\
&- 434700 tsw_2 \odot ty - \frac{2375}{72} tsw_2 \odot s^4 + \frac{35785}{18} tsw_2 \odot s^2u_1 + \frac{6875}{6} tsw_2 \odot s^2u_2 \\
&+ 193200 tsw_2 \odot sx_1 - \frac{73700}{3} tsw_2 \odot vw_2 - 94100 tsw_2 \odot vw_3 + \frac{3656700}{11} tsw_3 \odot tsw_3 \\
&- \frac{4932900}{11} tsw_3 \odot ty + \frac{6365}{264} tsw_3 \odot s^4 - \frac{114535}{66} tsw_3 \odot s^2u_1 - \frac{1675}{2} tsw_3 \odot s^2u_2 \\
&+ \frac{2192400}{11} tsw_3 \odot sx_1 - 75200 tsw_3 \odot vw_2 + \frac{2222700}{11} tsw_3 \odot vw_3 \\
&+ \frac{457398003600}{11} ty \odot ty + \frac{252315}{22} ty \odot s^4 + \frac{44199750}{11} ty \odot s^2u_1 - 3150 ty \odot s^2u_2 \\
&+ \frac{11863958400}{11} ty \odot sx_1 + 126000 ty \odot vw_2 + \frac{1512000}{11} ty \odot vw_3 + \frac{1784556748800}{11} ty \odot z \\
&+ \frac{1109}{76032} s^4 \odot s^4 - \frac{365}{4752} s^4 \odot s^2u_1 - \frac{95}{144} s^4 \odot s^2u_2 + \frac{4445}{33} s^4 \odot sx_1 + \frac{95}{18} s^4 \odot vw_2 \\
&+ \frac{1235}{66} s^4 \odot vw_3 + \frac{211680}{11} s^4 \odot z + \frac{408305}{2376} s^2u_1 \odot s^2u_1 + \frac{1465}{36} s^2u_1 \odot s^2u_2 \\
&+ \frac{1913800}{33} s^2u_1 \odot sx_1 - \frac{2650}{9} s^2u_1 \odot vw_2 - \frac{38650}{33} s^2u_1 \odot vw_3 + \frac{76204800}{11} s^2u_1 \odot z \\
&+ \frac{275}{24} s^2u_2 \odot s^2u_2 + 1400 s^2u_2 \odot sx_1 - \frac{550}{3} s^2u_2 \odot vw_2 - 650 s^2u_2 \odot vw_3 \\
&+ \frac{76574400}{11} sx_1 \odot sx_1 - 56000 sx_1 \odot vw_2 - \frac{672000}{11} sx_1 \odot vw_3 + \frac{20727705600}{11} sx_1 \odot z \\
&+ \frac{17600}{3} vw_2 \odot vw_2 - 400 vw_2 \odot vw_3 + \frac{433200}{11} vw_3 \odot vw_3 + \frac{1575305625600}{11} z \odot z.
\end{aligned}
$$

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