Czech Technical University in Prague
Faculty of Nuclear Sciences and Physical Engineering

SVOČ Competition Work

Automatic Identification of Low-Dimensional Lie Algebras

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Academic Year: **2014/2015**

Acknowledgments

I would like to express my gratitude to doc. Severin Pošta for supervising my work on this text, for his time, patience, and for his valuable advice throughout the year.

Furthermore, I would like to thank to my family for their unflagging support and love.

Last but not least, I am grateful to prof. Miloslav Havlíček for lending me the crucial publication [13], as well as for his distinctive lectures that have been always inspiring, certainly not only for me.

Abstract

This work is devoted to the problem of identifying a given Lie algebra in the classification list. Namely, we consider real and complex indecomposable Lie algebras up to dimension four. The main identification tool we apply for this purpose is the use of invariants. This method was widely discussed by prof. P. Winternitz and doc. L. Šnobl in their monograph *Classification and identification of Lie algebras*. In our work, we make practical use of the techniques described in their book.

First, the theory necessary for establishing various invariants is introduced. Second, the classification of considered algebras is presented a finally, the identification process itself is demonstrated. The result of our work is represented by a simple automatic online identificator, recognising any finite-dimensional real or complex Lie algebra that is directly composed from indecomposable ideals of dimension at most four.

Contents

List of Notations

0	trivial vector space $\{0\}$ (when used for vector spaces)
$\mathbb{1}$	identity map
arg	argument of a complex number; convention: $\arg(\mathbb{C}) = [0, 2\pi), \arg(0) = 0$
B_1	$\{z \in \mathbb{C} \mid \lvert z\rvert < 1\}$
$\mathbb{C}$	field of complex numbers
$\mathbb{C}^+$	$\{z \in \mathbb{C} \mid \operatorname{Im}(z) > 0\}$
dim	dimension
$f\vert_V$	restriction of a map f to V
$f^{(-1)}(A)$	preimage of $A \subset \operatorname{Ran} f$
$f^{(-1)}(a)$	preimage of $a \in \operatorname{Ran} f$
Im	imaginary part of a complex number
Ker	kernel of a map
$\widehat{n}$	$\{1, 2, \ldots, n\}$
$\underline{\widehat{n}}$	$\{0, 1, 2, \ldots, n\}$
$\mathbb{N}$	set of natural numbers
$\mathbb{N}_0$	set of natural numbers with zero
$\mathbb{R}$	field of real numbers
$\mathbb{R}^{n,n}$	vector space of all $n \times n$ real matrices
Ran	range of a map
rank	rank of a matrix
Re	real part of a complex number
S_1	$\{z \in \mathbb{C} \mid \lvert z\rvert = 1\}$
Span	linear span
$\operatorname{Span}_{\mathbb{R}}$	real linear span
Tr	trace form
V/W	quotient (factor) space of V with respect to W
$x + W$	element (class of isomorphism) of a quotient space V/W containing $x \in V$
$V \times W$	Cartesian product of V and W
$V^{\times k}$	$\underbrace{V \times \cdots \times V}_{k \text{ times}}$
$\subset\subset$	linear subspace
$\emptyset$	empty set
$\dot{+}$	direct sum of vector spaces

Introduction

The following problem arises naturally in various areas of mathematical physics. When one works with an object having a Lie algebra structure, it is often useful, or sometimes even necessary, to identify it with some known abstract Lie algebra. In other words, one aims to recognize which item in the list of all possible non-isomorphic Lie algebras is isomorphic to the Lie algebra one working with.

The first difficulty lies in obtaining such a list. This represents a very hard challenge that seems to be even unsolvable in general. However, as we shall discuss in the second chapter of our work, for particular classes of Lie algebras the solution has been found and the lists were already prepared. All complex and real Lie algebras up to dimension six are such a case. Thus, an interesting task to deal with is to identify at least these low-dimensional Lie algebras. The question that remains is how to do this.

In principle, one way of identification would be to start directly with the definition of an isomorphism. By the definition, two Lie algebras are isomorphic if and only if a bijective map between them preserving their Lie bracket structures exists. If we assign this abstract bijection to its matrix with respect to some chosen bases of considered Lie algebras and apply the conditions on preserving Lie brackets to all pairs of basis vectors, we obtain a system of quadratic equations for elements of the matrix. It is well-known that such a system is very difficult to solve in general. Moreover, the number of equations increases as a cubic function of the dimension of considered Lie algebras. Another problem is that we would have to conduct these computations repeatedly, until we find the right Lie algebra in the list. Therefore even the use of computer algebra systems is highly time-demanding and hence improper for this purpose in general.

Instead, the way that is more suitable for practical usage is to compute and to compare the so-called *invariants* of Lie algebras. Roughly speaking, an invariant is such a property of Lie algebras that is common to any two isomorphic Lie algebras. Of coarse, one cannot decide whether two Lie algebras sharing the same value of an invariant are isomorphic or not, however if an invariant differs between two Lie algebras, then they may not be related with each other through an isomorphism. If the number of available invariants is high enough, comparing their values on the identifying Lie algebra with values on the Lie algebras in the list, we can restrict the list in successive steps until only one eventuality remains and that one has to be isomorphic to the identifying Lie algebra. This method has two great advantages. First, the invariants are developed in such a way that their computation is a linear problem, hence they could be computed in a real time using computer algebra systems. Second, when a proper set of invariants is found for a particular list of Lie algebras, i.e. such that each two items are distinguished from each other by these invariants, it suffices to compute the invariants only for the identifying Lie algebras.

In [13], the method of identification of Lie algebras through the use of invariants was discussed in some detail and a lot of invariants were described as well. Furthermore, the same publication also contains the list of all complex and real indecomposable Lie algebras up to dimension six. In our work, we restricted ourselves to the dimension four or less and for each item in this list, we chose and computed the unique sets of invariants sufficient for identification of any such a Lie algebra. As a result of our work, we present a simple internet application "LIEIDENTIFICATOR", which is able to identify any finite-dimensional complex or real Lie algebra that is the direct sum of at most four-dimensional indecomposable ideals.

The text of our work is organized as follows. In the first chapter we introduce the fundamentals of Lie algebras theory necessary in further parts for defining various invariants and using them for identification of Lie algebras. In the second chapter we describe the process of classifying Lie algebras, we discuss the methods of classification and the current results. Further, we present the list of those Lie algebras we focus on in our work, i.e. complex and real indecomposable Lie algebras up to dimension four. Finally, the third chapter is devoted to the identification of a given Lie algebra in the presented lists. First, the invariants needed for this purpose are established and invariance of all of them is verified and second, the computed values of the invariants for all considered Lie algebras are written out in a tabular manner.

Chapter 1

Basic Theory of Lie Algebras

The aim of the first chapter of this work is to provide a brief introduction to the theory of Lie algebras. As announced, we introduce mainly the results necessary for defining various invariants in further text. The presented theory is mostly the "classical" one (cf. [5], [8], [10], [11]), except where otherwise emphasized.

1.1 Lie Algebras, Subalgebras and Ideals

Definition 1.1. Let $\mathbb{F}$ be a field. A *Lie algebra* over $\mathbb{F}$ is an $\mathbb{F}$-vector space L together with a bilinear map (the so-called *Lie bracket*) $[\ ,\ ]\colon L \times L \to L$ fulfilling for any $x, y, z \in L$ the two following conditions:

$$[x, x] = 0, \tag{1.1}$$

$$[x, [y, z]] + [y, [z, x]] + [z, [x, y]] = 0. \tag{1.2}$$

Remark 1.1.
(a) We shall consider finite-dimensional vector spaces and Lie algebras entirely.[1] Furthermore, we shall restrict ourselves only to the fields of real and complex numbers.
(b) If $\mathbb{F} \in \{\mathbb{R}, \mathbb{C}\}$, the condition (1.1) is equivalent to *antisymmetry* of the Lie bracket: for all y and z from a Lie algebra L we have

$$0 = [y + z, y + z] = [y, y] + [y, z] + [z, y] + [z, z] = [y, z] + [z, y]$$

and hence $[y, z] = -[z, y]$. The condition (1.2) is usually called the *Jacobi identity*.

Example 1.2. Let V be a real or complex vector space. It is an easy exercise to check that the vector space $\mathfrak{gl}(V)$ of all linear transformations of V becomes a Lie algebra over $\mathbb{R}$ or $\mathbb{C}$, respectively, with the Lie bracket $[\ ,\ ]$ defined for all $x, y \in \mathfrak{gl}(V)$ as follows:

$$[x, y] := x \circ y - y \circ x. \tag{1.3}$$

Similarly, one can verify that the vector space $\mathfrak{gl}(n, \mathbb{F}) \equiv \mathbb{F}^{n,n}$, $n \in \mathbb{N}$, $\mathbb{F} \in \{\mathbb{R}, \mathbb{C}\}$, of all $n \times n$ matrices over $\mathbb{F}$ together with the Lie bracket defined for all $x, y \in \mathfrak{gl}(n, \mathbb{F})$ by

$$[x, y] := xy - yx \tag{1.4}$$

represents another example of a real or complex, respectively, Lie algebra (cf. [12]).

[1] Talking about such features of a Lie algebra as *basis*, *dimension*, etc., one always means the respective properties of the underlying vector space.

Definition 1.2. Let L be a Lie algebra over $\mathbb{F}$ and let $\mathcal{B} = (x_1, \ldots, x_n)$ be a basis for L. The *structure constants* of L with respect to the basis $\mathcal{B}$ are numbers $a_{ij}^k \in \mathbb{F}$; $i, j, k \in \widehat{n}$; such that for all $i, j \in \widehat{n}$ we can write

$$[x_i, x_j] = \sum_{k=1}^{n} a_{ij}^k x_k. \tag{1.5}$$

Definition 1.3. A Lie algebra L is said to be *abelian* if for any $x, y \in L$ it holds that

$$[x, y] = 0. \tag{1.6}$$

Definition 1.4. A *(Lie) subalgebra* of a Lie algebra L is a vector subspace $K \subset\subset L$ such that for all $x, y \in K$ it is satisfied that

$$[x, y] \in K. \tag{1.7}$$

Definition 1.5. An *ideal* of a Lie algebra L is a vector subspace $I \subset\subset L$ such that for all $x \in L$ and $y \in I$ it is satisfied that

$$[x, y] \in I. \tag{1.8}$$

Remark 1.3. Let L be a Lie algebra. The following assertions are obvious:
(a) a subalgebra K of L becomes a Lie algebra in its own right with the restriction of the original Lie bracket on L to $K \times K$;
(b) any ideal is a subalgebra;
(c) both 0 and L are ideals of L (an ideal I of L is said to be *non-trivial* if $0 \neq I \neq L$);
(d) the intersection of two ideals of L is an ideal of L as well.

Definition 1.6. Let L be a Lie algebra. The *center* of L is defined as

$$Z(L) := \left\{ x \in L \;\middle|\; \text{for all } y \in L, [x, y] = 0 \right\}. \tag{1.9}$$

Remark 1.4. Since $0 \in Z(L)$, is is clear that $Z(L)$ is an ideal of L.

Definition 1.7. Let I and J be ideals of a Lie algebra L. We define

$$[I, J] := \operatorname{Span} \left\{ [x, y] \;\middle|\; x \in I,\ y \in J \right\}. \tag{1.10}$$

In particular, $L' := [L, L]$ denotes the so-called *derived algebra* of L.

Proposition 1.8. *Let I and J be ideals of a Lie algebra L. Then $[I, J]$ is an ideal of L as well.*

Proof. For all $x \in L$, $y \in I$ and $z \in J$ we have $[x, [y, z]] = [[x, y], z] + [y, [x, z]] \in [I, J]$. Clearly, this implies that $[x, y] \in [I, J]$ for all $x \in L$ and $y \in [I, J]$. □

Definition 1.9. Let L be a Lie algebra and let I be an ideal of L. The quotient vector space L/I together with a map $[\ ,\ ]_q : L/I \times L/I \to L/I : (x + I, y + I) \mapsto [x, y] + I$, where $[\ ,\ ]$ is the Lie bracket on L, is called the *quotient Lie algebra* of L by I.

Remark 1.5. One has to verify, that the map $[\ ,\ ]_q$ from the previous definition is a well-defined Lie bracket on L/I and therefore the quotient Lie algebra L/I is a Lie algebra indeed. First, the definition is unambiguous since for arbitrary $x, \tilde{x}, y, \tilde{y} \in L$ such that $y_I := y - \tilde{y} \in I$ and $x_I := x - \tilde{x} \in I$ we have (using I is an ideal)

$$[x + I, y + I] = [x, y] + I = [x_I + \tilde{x}, y_I + \tilde{y}] + I = [\tilde{x}, \tilde{y}] + I = [\tilde{x} + I, \tilde{y} + I].$$

Second, it is obvious that the definitory properties of the Lie bracket are all immediately implied by the respective properties of the Lie bracket on L.

1.2 Lie Algebra Homomorphisms

Definition 1.10. Let L_1 and L_2 be Lie algebras. A linear map $\varphi : L_1 \to L_2$ is called a *homomorphism* if for all $x, y \in L_1$ it is satisfied that

$$\varphi([x,y]_1) = [\varphi(x), \varphi(y)]_2, \tag{1.11}$$

where $[\ ,\ ]_i$ is the Lie bracket on L_i, $i = 1, 2$. Furthermore,
(a) if φ is bijective, then it is called an *isomorphism*;
(b) if $L_1 = L_2$, then φ is called an *endomorphism* (of L_1);
(c) if φ is bijective and $L_1 = L_2$, then φ is called an *automorphism* (of L_1).

Remark 1.6. We say that two Lie algebras L_1 and L_2 are *isomorphic* if there exists an isomorphism $\varphi : L_1 \to L_2$. We denote this fact $L_1 \cong L_2$. It is not hard to see that $\cong$ is an equivalence relation: an identity is an obvious isomorphism (reflexivity); each isomorphism is invertible and its inversion is an isomorphism as well (symmetry); the composition of two isomorphisms is an isomorphism again (transitivity). Thus, all Lie algebras are divided into *classes of isomorphism* through this relation.

Example 1.7. Let L be a Lie algebra. For all $x, y \in L$ we define the *adjoint homomorphism* $\mathrm{ad} : L \to \mathrm{gl}(L)$ as follows:

$$(\mathrm{ad}\, x)(y) := [x, y]. \tag{1.12}$$

ad is a homomorphism indeed since for all $x, \tilde{x}, y \in L$ we have

$$(\mathrm{ad}[x,\tilde{x}])(y) = [[x,\tilde{x}], y] = [x, [\tilde{x}, y]] - [\tilde{x}, [x, y]] = ([\mathrm{ad}\, x, \mathrm{ad}\, \tilde{x}])(y).$$

Definition 1.11. Let L be a Lie algebra over $\mathbb{F}$. We define the *Killing form* on L to be the form $\kappa : L \times L \to \mathbb{F}$ sending all $x, y \in L$ to

$$\kappa(x, y) := \mathrm{Tr}(\mathrm{ad}\, x \circ \mathrm{ad}\, y). \tag{1.13}$$

Proposition 1.12. *The Killing form on a Lie algebra L is bilinear and symmetric.*

Proof. First, bilinearity follows from linearity of Tr and ad and from bilinearity of the composition of two linear maps. Second, symmetry is an immediate consequence of cyclicity of the trace form. □

Proposition 1.13. *Let L_1 and L_2 be Lie algebras and let $\varphi : L_1 \to L_2$ be a homomorphism. Then* $\mathrm{Ker}\,\varphi$ *is an ideal of L_1 and* $\mathrm{Ran}\,\varphi$ *is a subalgebra of L_2.*

Proof. First of all, it is well-known from linear algebra that both kernel and range of a linear map are always subspaces. Second, for any $x \in L_1$ and $y \in \mathrm{Ker}\,\varphi$ we have $\varphi([x,y]) = [\varphi(x), \varphi(y)] = [\varphi(x), 0] = 0$ and hence $[x, y] \in \mathrm{Ker}\,\varphi$. Third, given $u, v \in \mathrm{Ran}\,\varphi$, there exist $x, y \in L_1$ such that $u = \varphi(x)$ and $v = \varphi(y)$. Then $[u, v] = [\varphi(x), \varphi(y)] = \varphi([x, y]) \in \mathrm{Ran}\,\varphi$. □

Proposition 1.14. *Let L_1 and L_2 be Lie algebras and let $\varphi : L_1 \to L_2$ be a homomorphism. If I is an ideal of L_1, then $\varphi(I)$ is an ideal of* $\mathrm{Ran}\,\varphi \equiv \varphi(L_1)$.

Proof. Let I be an ideal of L_1. For any $u \in \varphi(I)$ and $v \in \mathrm{Ran}\,\varphi$ there are $x \in I$ and $y \in L_1$ such that $u = \varphi(x)$ and $v = \varphi(y)$. Then $[u, v] = [\varphi(x), \varphi(y)] = \varphi([x, y]) \in \varphi(I)$ proves $\varphi(I)$ to be an ideal. □

Definition 1.15. Let $\mathbb{F}$ be a field. Suppose that L is a Lie algebra over $\mathbb{F}$ and V is an $\mathbb{F}$-vector space. A *representation* of L on V is a homomorphism $\rho : L \to \mathrm{gl}(V)$.

Definition 1.16. Let L_1 and L_2 be Lie algebras over $\mathbb{F}$, let V_1 and V_2 be $\mathbb{F}$-vector spaces and let $\rho_i : L_i \to \mathrm{gl}(V_i)$, $i = 1, 2$, be representations of L_1 and L_2, respectively. We say that ρ_1 and ρ_2 are *equivalent* and we denote $\rho_1 \sim \rho_2$, if there exist an isomorphism $\varphi : L_1 \to L_2$ and a bijection $\phi : V_1 \to V_2$ such that for all $x \in L_1$ it holds

$$\phi \circ \rho_1(x) = \rho_2(\varphi(x)) \circ \phi. \tag{1.14}$$

Remark 1.8. Considering analogical arguments as in Remark 1.6, one can easily verify that also $\sim$ is en equivalence relation indeed.

1.3 Direct Sum Decomposition

Definition 1.17. Let L be a Lie algebra and let $I_1, \ldots, I_k$ be ideals of L such that, as vector spaces, $L = I_1 \dot{+} \cdots \dot{+} I_k$. We say that L, as a Lie algebra, is the *direct sum* of ideals $I_1, \ldots, I_k$ and we write $L = I_1 \oplus \cdots \oplus I_k$.

Remark 1.9. Given $L = I_1 \oplus \cdots \oplus I_k$, any $x, y \in L$ decompose uniquely as $x = \sum_{i=1}^{k} x_i$ and $y = \sum_{i=1}^{k} y_i$, where $x_i, y_i \in I_i$, $i \in \widehat{k}$. Since for all $i \neq j$ we have $[x_i, y_j] \in I_i \cap I_j = 0$ (both I_i and I_j are ideals), we can write

$$[x, y] = [\sum_{i=1}^{k} x_i, \sum_{j=1}^{k} y_j] = \sum_{i,j=1}^{k} [x_i, y_j] = \sum_{i=1}^{k} [x_i, y_i].$$

Definition 1.18. Let L be a Lie algebra. If there exist non-trivial ideals I and J of L such that $L = I \oplus J$, L is said to be *decomposable*. Otherwise L is said to be *indecomposable*.

Proposition 1.19. *Let L be a Lie algebra, $L \neq 0$. There exist indecomposable non-zero ideals $I_1, \ldots, I_k$ of L such that $L = I_1 \oplus \cdots \oplus I_k$.*

Proof. We will proceed by induction on $\dim L$. First, each one-dimensional Lie algebra is obviously indecomposable. For the inductive step, suppose that we proved the assertion for $\dim L = n - 1$ and assume $\dim L = n$. If L is indecomposable, we are done. Otherwise, there are non-zero ideals I, J of L such that $L = I \oplus J$. Since $\dim I, \dim J \geq 1$, it follows that $\dim I, \dim J \leq n - 1$ and hence we can apply our inductive hypothesis to both I and J in order to obtain the desired decomposition of L. $\square$

Remark 1.10. A non-zero abelian Lie algebra is indecomposable precisely when it is one-dimensional: if it had dimension greater then one, then it would be decomposable into any one-dimensional subspace and its complement.

Proposition 1.20. *Let L_1 and L_2 be isomorphic Lie algebras. L_1 is indecomposable precisely when L_2 is so.*

Proof. Let $\varphi : L_1 \to L_2$ be an isomorphism. If L_1 was decomposable via $L_1 = I \oplus J$, then both $\varphi(I)$ and $\varphi(J)$ would be non-trivial ideals of L_2 (cf. Proposition 1.14). L_2 would be therefore decomposable via $L_2 = \varphi(I) \oplus \varphi(J)$. And vice versa. $\square$

In Proposition 1.19, we proved the existence of the direct sum decomposition. However, we still do not know anything about the uniqueness. Maybe surprisingly, this topic is not discussed in the standard literature (referred at the beginning of this chapter) at all. The ideas presented it the rest of this section were performed in [6].

Definition 1.21. Let L be a Lie algebra. An endomorphism $\varphi : L \to L$ is said to be *normal* if for all $x \in L$ it holds that $\varphi \circ \operatorname{ad} x = \operatorname{ad} x \circ \varphi$.

Remark 1.11.
(a) Clearly, the composition of two (or finitely many) normal endomorphisms is a normal endomorphism again.
(b) Let L be a Lie algebra and let K be a subalgebra of L. If φ is a normal endomorphism of L such that $\varphi(K) \subset K$, then the restriction of φ to K is a normal endomorphism of K.

Example 1.12. Let us keep the notation from Remark 1.9. For all $i \in \widehat{k}$ we define the projection $\pi_i : L \to L : x \mapsto x_i$ onto I_i. Then for any $x, y \in L$ and $i \in \widehat{k}$ we can write

$$\pi_i([x,y]) = \pi_i\left(\sum_{j=1}^{k}[x_j, y_j]\right) = [x_i, y_i] = [\pi_i(x), \pi_i(y)] = \sum_{j=1}^{k}[\pi_j(x), \pi_i(y)] = [x, \pi_i(y)],$$

from which one can deduce that all π_i, $i \in \widehat{k}$, are normal endomorphisms.

Lemma 1.22. *Let L be a Lie algebra and let $\varphi : L \to L$ be a normal endomorphism such that* $\operatorname{Ker}\varphi = \operatorname{Ker}\varphi^2$ *and* $\operatorname{Ran}\varphi = \operatorname{Ran}\varphi^2$. *Then* $L = \operatorname{Ker}\varphi \oplus \operatorname{Ran}\varphi$.

Proof. According to Proposition 1.13, $\operatorname{Ker}\varphi$ is an ideal of L. We must verify that $\operatorname{Ran}\varphi$ is so: for any $x \in \operatorname{Ran}\varphi$ and $y \in L$, there is $u \in L$ such that $x = \varphi(u)$ and we have

$$[y, x] = [y, \varphi(u)] = \varphi([y, u]) \in \operatorname{Ran}\varphi,$$

provided φ is normal. Now it remains to show that $L = \operatorname{Ker}\varphi \dotplus \operatorname{Ran}\varphi$. First, $\operatorname{Ker}\varphi + \operatorname{Ran}\varphi \subset L$, obviously. Second, given any $x \in L$, $\varphi(x) \in \operatorname{Ran}\varphi = \operatorname{Ran}\varphi^2$, there is $y \in \operatorname{Ran}\varphi$ such that $\varphi(y) = \varphi(x)$. Then $\varphi(x-y) = \varphi(x) - \varphi(y) = 0$ and hence we have $x = (x-y) + y \in \operatorname{Ker}\varphi + \operatorname{Ran}\varphi$. Finally, let us take any $x \in \operatorname{Ker}\varphi \cap \operatorname{Ran}\varphi$. Then $x = \varphi(y)$ for some $y \in L$. Since $\varphi^2(y) = \varphi(x) = 0$, it follows $y \in \operatorname{Ker}\varphi^2 = \operatorname{Ker}\varphi$ which means $x = \varphi(y) = 0$. □

Corollary 1.23. *Let L be an indecomposable Lie algebra and let $\varphi : L \to L$ be a normal endomorphism. Then φ is either bijective or nilpotent.*

Proof. For all $n \in \mathbb{N}$, $\operatorname{Ran}\varphi^n$ and $\operatorname{Ker}\varphi^n$ are ideals of L, moreover $\operatorname{Ran}\varphi^{n+1} \subset \operatorname{Ran}\varphi^n$ and $\operatorname{Ker}\varphi^{n+1} \supset \operatorname{Ker}\varphi^n$. Since L is finite-dimensional, there exists $n_0 \in \mathbb{N}$ such that for any $n \geq n_0$ it holds $\operatorname{Ran}\varphi^n = \operatorname{Ran}\varphi^{n_0}$ and $\operatorname{Ker}\varphi^n = \operatorname{Ker}\varphi^{n_0}$. In particular, $\operatorname{Ran}\varphi^{2n_0} = \operatorname{Ran}\varphi^{n_0}$ and $\operatorname{Ker}\varphi^{2n_0} = \operatorname{Ker}\varphi^{n_0}$. Thus, by the previous lemma, $L = \operatorname{Ker}\varphi^{n_0} \oplus \operatorname{Ran}\varphi^{n_0}$. But as L is indecomposable, it follows that either $\operatorname{Ker}\varphi^{n_0} = 0$ and hence $\operatorname{Ker}\varphi = 0$ or $\operatorname{Ran}\varphi^{n_0} = 0$ implying $\varphi^{n_0} = 0$. □

Theorem 1.24. *Let L be a Lie algebra, $L \neq 0$, and let $L = I_1 \oplus \cdots \oplus I_k = J_1 \oplus \cdots \oplus J_l$ be decompositions of L into the direct sum of indecomposable ideals. Then $k = l$ and the summands can be numbered such that for each $i \in \widehat{k}$ it holds that $I_i \cong J_i$. Furthermore, if $Z(L) = 0$ or $L' = L$, then the numbering could be done in such a way that for each $i \in \widehat{k}$ it holds that $I_i = J_i$.*

Proof. Let π_i and ψ_j be projections onto I_i and J_j, respectively, $i \in \widehat{k}$, $j \in \widehat{l}$, as defined in Example 1.12. According to Remark 1.11, all $\pi_i\psi_j|_{I_i}$ and $\psi_j\pi_i|_{J_j}$, $i \in \widehat{k}$, $j \in \widehat{l}$, are normal endomorphisms and hence, by Corollary 1.23, they are either bijective or nilpotent. Furthermore, since $(\pi_i\psi_j|_{I_i})^n = \pi_i(\psi_j\pi_i|_{J_j})^{n-1}\psi_j|_{I_i}$ and $(\psi_j\pi_i|_{J_j})^n = \psi_j(\pi_i\psi_j|_{I_i})^{n-1}\pi_i|_{J_j}$ hold for all $n \in \mathbb{N}$, $\pi_i\psi_j|_{I_i}$ and $\psi_j\pi_i|_{J_j}$ are either both bijective or both nilpotent.

Take any $i \in \widehat{k}$. If I_i is abelian then $\dim I_i = 1$. Because

$$0 \neq I_i = \pi_i(I_i) = (\pi_i\mathbb{1})(I_i) = \left(\pi_i\sum_{j=1}^{l}\psi_j\right)(I_i) = \pi_i(\psi_1(I_i)) + \cdots + \pi_i(\psi_l(I_i)),$$

there exists $j \in \widehat{l}$ such that $\pi_i(\psi_j(I_i)) \neq 0$, but this is enough to prove that $\pi_i\psi_j|_{I_i}$ is a bijection. Now assume that I_i is non-abelian. Choose an arbitrary $j \in \widehat{l}$ and suppose that $\pi_i\psi_j|_{I_i}$ is nilpotent. Take any $x \in I_i$ such that $(\mathbb{1} - \pi_i\psi_j)(x) = 0$. Then $(\pi_i\psi_j)(x) = x$ and consequently $(\pi_i\psi_j)^n(x) = x$ for all $n \in \mathbb{N}$. It follows that $x = 0$ and hence $(\mathbb{1} - \pi_i\psi_j)|_{I_i}$ is bijective. Therefore we have

$$I_i = (\mathbb{1} - \pi_i\psi_j)(I_i) = \pi_i(\mathbb{1} - \psi_j)(I_i) \subset \pi_i(\hat{J}_j), \quad \text{where } \hat{J}_j := (\mathbb{1} - \psi_j)(L) = \bigoplus_{q \neq j} J_q.$$

Thus

$$I_i' = [I_i, I_i] \subset [\pi_i(\hat{J}_j), I_i] = \pi_i([\hat{J}_j, I_i]) = [\hat{J}_j, \pi_i(I_i)] = [\hat{J}_j, I_i] \subset \hat{J}_j$$

and hence $\psi_j(I_i') = 0$. If this hold for all $j \in \widehat{l}$, then we would reach a contradiction:

$$0 \neq I_i' = \mathbb{1}(I_i') = \left(\sum_{j=1}^{l} \psi_j\right)(I_i') = \psi_1(I_i') + \cdots + \psi_l(I_i') = 0.$$

Accordingly, independently of abelianness of I_i, there exists $j \in \widehat{l}$ such that $\pi_i\psi_j|_{I_i}$ is a bijection, so is $\psi_j\pi_i|_{J_j}$ and thus it follows that all the restrictions $\psi_j : I_i \to \psi_j(I_i)$, $\pi_i : \psi_j(I_i) \to I_i$, $\pi_i : J_j \to \pi_i(J_j)$ and $\psi_j : \pi_i(J_j) \to J_j$ are bijections. Moreover from

$$I_i = (\pi_i\psi_j)(I_i) \subset \pi_i(J_j) \subset I_i$$

we obtain $\pi_i(J_j) = I_i$, similarly we could get $\psi_j(I_i) = J_j$ and hence $\pi_i|_{J_j} : J_j \to I_i$ and $\psi_j|_{I_i} : I_i \to J_j$ are isomorphisms. All in all, for any $i \in \widehat{k}$, there is $j \in \widehat{l}$ such that $I_i \cong J_j$.

Furthermore,

$$I_i' = [I_i, I_i] = [I_i, \pi_i(J_j)] = [\pi_i(I_i), J_j] = [I_i, \psi_j(J_j)] = [\psi_j(I_i), J_j] = [J_j, J_j] = J_j' \tag{1.15}$$

and since $J_q' \subset J_q$ for all $q \in \widehat{l}$, it is clear that if I_i is not abelian, then such $j \in \widehat{l}$ is determined uniquely. We could repeat the whole process for an arbitrary $j \in \widehat{l}$ and we would obtain an analogous result, naturally. Thus, we conclude that all non-abelian ideals among $I_1, \dots, I_k$ and $J_1, \dots, J_l$, respectively, are in one-to-one correspondence and that the corresponding ideals are mutually isomorphic. Comparing the sums of dimensions, we deduce that the same holds also for (one-dimensional) abelian ideals. The first part of the theorem is now obvious.

For the second part, assume that we have already denote the ideals such that $I_i \cong J_i$, $i \in \widehat{k}$. First, if $L' = L$, then for all $i \in \widehat{k}$ we have $I_i' = I_i$ and $J_i' = J_i$ and hence, by (1.15), $I_i = J_i$. Second, take $i \in \widehat{k}$ and suppose that $I_i \neq J_i$. Thus assume, without loss of generality, that there exists a non-zero $x \in I_i$ such that $x \notin J_i$. Since $\pi_i : J_i \to I_i$ is a bijection, there is $z \in J_i$ such that $\pi_i(z) = x$. Then $\tilde{x} := x - \psi_i(x) \in \hat{J}_i$ is not equal to the zero vector and for all $y \in \hat{J}_i$ we have

$$[\tilde{x}, y] = [x, y] - [\psi_i(x), y] = [x, y] = [\pi_i(z), y] = \pi_i[z, y] = \pi_i(0) = 0.$$

Therefore $0 \neq \tilde{x} \in Z(\hat{J}_i)$ and, as the center of the direct sum of ideals is the direct sum of their centers, obviously, we conclude that $Z(L) \neq 0$. All in all, if $Z(L) = 0$, then the equality $I_i = J_i$ holds for all $i \in \widehat{k}$.[2] □

[2] For brevity, we omit the sign "$\circ$" for composition of two maps in this proof. It is, however, always clear from context that we mean the map composition rather than any kind of multiplication.

1.4 Solvable and Nilpotent Lie Algebras

Definition 1.25. Let L be a Lie algebra.
(a) The *derived series* of L is the sequence $\left(L^{(n)}\right)_{n=0}^{\infty}$ with terms

$$L^{(0)} = L \quad \text{and} \quad L^{(n)} = [L^{(n-1)}, L^{(n-1)}], \text{ for } n \geq 1. \tag{1.16}$$

If there exists $n \in \mathbb{N}$ such that $L^{(n)} = 0$, then L is said to be *solvable*.
(b) The *lower central series* of L is the sequence $(L^n)_{n=0}^{\infty}$ with terms

$$L^0 = L \quad \text{and} \quad L^n = [L, L^{n-1}], \text{ for } n \geq 1. \tag{1.17}$$

If there exists $n \in \mathbb{N}$ such that $L^n = 0$, then L is said to be *nilpotent*.
(c) The *upper central series* of L is the sequence $(Z_n(L))_{n=0}^{\infty}$ with terms

$$Z_0(L) = 0 \quad \text{and} \quad Z_n(L) = \left\{x \in L \mid \text{for all } y \in L, [x,y] \in Z_{n-1}(L)\right\}, \text{ for } n \geq 1. \tag{1.18}$$

Remark 1.13.
(a) Notice that $L^{(1)} = L^1 = L'$ and $Z_1(L) = Z(L)$.
(b) Since each Lie algebra is an ideal of itself, it is obvious from Proposition 1.8 that $L^{(n)}$ and L^n are ideals of L for all $n \in \mathbb{N}_0$. One can easily show this by induction on n.
(c) We show that also $Z_n(L)$, $n \in \mathbb{N}_0$, are ideals of L. Again we use induction on n. When $n = 0$, $Z_0(L) \equiv 0$ and there is nothing to prove. For the inductive step, assume $Z_{n-1}(L)$ to be an ideal and take arbitrary $x \in Z_n(L)$ and $y, z \in L$. Then we have $[[x,z],y] = [x,[z,y]] + [[x,y],z] \in Z_{n-1}(L)$, since $[z,y] \in L$ and $[x,y] \in Z_{n-1}(L)$, and hence $[x,z] \in Z_n(L)$, as desired.
(d) In view of the fact that all terms of these so-called *characteristic series of ideals* are ideals indeed, one can easily see that for all $n \in \mathbb{N}_0$ it holds that $L^{(n)} \supset L^{(n+1)}$ and $L^n \supset L^{n+1}$, while $Z_n(L) \subset Z_{n+1}(L)$.
(e) It is easily seen by induction that for all $n \in \mathbb{N}_0$ it holds $L^{(n)} \subset L^n$ and hence each nilpotent Lie algebra is solvable.

Lemma 1.26. *Let L and M be Lie algebras and let $\varphi : L \to M$ be a homomorphism. Then for all $n \in \mathbb{N}_0$ it holds that*
(a) $\varphi(L^{(n)}) = \varphi(L)^{(n)}$,
(b) $\varphi(L^n) = \varphi(L)^n$.

Proof. We use induction on n.
(a) First, for $n = 0$ we have $L^{(0)} = L$ and the assertion is a tautology. For the inductive step, suppose that $\varphi(L^{(n-1)}) = \varphi(L)^{(n-1)}$. Then we can write

$$\begin{aligned}\varphi(L^{(n)}) &= \varphi\left(\operatorname{Span}\left\{[x,y] \mid x,y \in L^{(n-1)}\right\}\right) = \operatorname{Span}\left\{[\varphi(x),\varphi(y)] \mid x,y \in L^{(n-1)}\right\} \\ &= \operatorname{Span}\left\{[u,v] \mid u,v \in \varphi(L)^{(n-1)}\right\} = \varphi(L)^{(n)}.\end{aligned}$$

(b) Analogically to (a). □

Corollary 1.27. *A homomorphic image of a solvable / nilpotent Lie algebra is solvable / nilpotent Lie algebra as well.*

Proof. It follows immediately from parts (a) and (b), respectively, of Lemma 1.26. □

Proposition 1.28. *Let I be an ideal of a Lie algebra L. Then for all $n \in \mathbb{N}_0$, $I^{(n)}$ and I^n are ideals of L as well.*

Proof. We use induction on n to show the statement for $I^{(n)}$. For I^n, the proceeding is completely analogous. First, $I \equiv I^{(0)}$ is an ideal. For the inductive step, assume that $I^{(n-1)}$ is an ideal and take arbitrary $x \in I^{(n)}$ and $y \in L$. Then there exist $k \in \mathbb{N}$ and $x_1, \ldots, x_k, y_1, \ldots, y_k \in I^{(n-1)}$ such that $x = \sum_{i=1}^{k}[x_i, y_i]$ and therefore

$$[x, y] = \sum_{i=1}^{k}[[x_i, y_i], y] = -\sum_{i=1}^{k}\left([[y, x_i], y_i] + [[y_i, y], x_i]\right) \in I^{(n)},$$

since for all $i \in \widehat{k}$, both $[y, x_i]$ and $[y_i, y]$ belong to $I^{(n-1)}$. $\square$

Lemma 1.29. *Let I and J be ideals of a Lie algebra L. Then*
(a) $I + J$ is an ideal of L;
(b) if I and J are solvable, then $I + J$ is solvable;
(c) if I and J are nilpotent, then $I + J$ is nilpotent.

Proof.
(a) Each $x \in I + J$ can be written as $x = x_I + x_J$, where $x_I \in I$ and $x_J \in J$. Then for all $y \in L$ we have $[x, y] = [x_I, y] + [x_J, y] \in I + J$.
(b) Obviously, J is an ideal of $I + J$. We define a map $\varphi\colon I \to (I + J)/J : x \mapsto x + J$. Since

$$\varphi([x, y]) = [x, y] + I = [x + I, y + I] = [\varphi(x), \varphi(y)]$$

holds for all $x, y \in I$, φ is a homomorphism. Further for any $x_I + x_J + J = x_I + J \in (I + J)/J$ we have $\varphi(x_I) = x_I + J$ and hence $\varphi(I) = (I + J)/J$. Now Corollary 1.27 gives that $(I + J)/J$ is solvable.

Next, we claim that $((I + J)/J)^{(k)} = ((I + J)^{(k)} + J)/J$, $k \in \mathbb{N}_0$. For $k = 0$ it is nothing to prove, thus suppose for the inductive step that this assertion holds for $k - 1$. Then

$$\begin{aligned}((I + J)/J)^{(k)} &= \left(((I + J)/J)^{(k-1)}\right)' = \left(((I + J)^{(k-1)} + J)/J\right)' \\ &= \operatorname{Span}\left\{[x + y + J, \tilde{x} + \tilde{y} + J] \mid x, \tilde{x} \in (I + J)^{(k-1)}; y, \tilde{y} \in J\right\} \\ &= \operatorname{Span}\left\{[x, \tilde{x}] + \tilde{\tilde{y}} + J \mid x, \tilde{x} \in (I + J)^{(k-1)}, \tilde{\tilde{y}} \in J\right\} = ((I + J)^{(k)} + J)/J.\end{aligned}$$

Since both J and $(I + J)/J$ are solvable, there exist $m, n \in \mathbb{N}$ such that $((I + J)/J)^{(m)} = 0$ and $J^{(n)} = 0$. Therefore $(I + J)^{(m)} \in J$ and finally $(I + J)^{(m+n)} = ((I + J)^{(m)})^{(n)} = 0$.
(c) Surely there exists $n \in \mathbb{N}$ such that $I^n = J^n = 0$. By part (a) and Proposition 1.28, $(I + J)^{2n}$ is an ideal in L. Each element of this ideal has to be a (finite) sum of terms having the following form:

$$[x_{2n} + y_{2n}, [x_{2n-1} + y_{2n-1}, \cdots [x_1 + y_1, x_0 + y_0] \cdots]], \tag{1.19}$$

where for all $i \in \widehat{2n}$, $x_i \in I$ and $y_i \in J$. Such a term (1.19) can be further expressed as a (finite) sum of multiple commutators in the form $a := [a_{2n}, [a_{2n-1}, \cdots [a_1, a_0] \cdots]]$, where for all $i \in \widehat{2n}$, $a_i = x_i \in I$ or $a_i = y_i \in J$. It is obvious that there are at least $n + 1$ vectors from I or at least $n + 1$ vectors from J among $2n + 1$ vectors $a_0, \ldots, a_{2n}$. Since both I and J are ideals, it follows that either $a \in I^n$ or $a \in J^n$. In both cases we have $a = 0$ and consequently $(I + J)^{2n} = 0$. $\square$

Corollary 1.30. *Let L be a Lie algebra. There exists a unique solvable / nilpotent ideal of L which is maximal, i.e. containing any other solvable / nilpotent ideal of L.*

Proof. We show the "solvable" case. For the other part, it suffices to replace the word "solvable" by "nilpotent". Thus, let R be a solvable ideal of L having maximal dimension and let I be any other solvable ideal of L. By part (b) of the previous lemma, $R + I$ is a solvable ideal again. But $\dim R \geq \dim(R + I)$ and hence $I \subset R$. $\square$

Definition 1.31. Let L be a Lie algebra. The maximal solvable / nilpotent ideal of L is called the *radical* / *nilradical* of L and is denoted by $\mathrm{Rad}(L)$ / $\mathrm{NilRad}(L)$.[3]

Proposition 1.32. *Let L_1 and L_2 be Lie algebras and let $\varphi : L_1 \to L_2$ be an isomorphism. Then $\varphi(\mathrm{Rad}(L_1)) = \mathrm{Rad}(L_2)$.*

Proof. According to Corollary 1.27, $\varphi(\mathrm{Rad}(L_1))$ is a solvable ideal of L_2. If a solvable ideal I of L_2 existed such that $I \not\subset \varphi(\mathrm{Rad}(L_1))$, then $\varphi^{-1}(I)$ would be a solvable ideal of L_1 not included in $\mathrm{Rad}(L_1)$, a contradiction. $\square$

1.5 Semisimple and Simple Lie Algebras

Definition 1.33. A Lie algebra L is said to be *semisimple* if $L \neq 0$ and $\mathrm{Rad}(L) = 0$.

Definition 1.34. A Lie algebra L is said to be *simple* if it is not abelian and its only ideals are 0 and L.

Remark 1.14. Clearly, any simple Lie algebra is indecomposable as well as semisimple.

The following theorem put simple and semisimple Lie algebras into the relationship. Its proof is fairly long since a lot of auxiliary assertions are needed (cf. [10]), so we omit it here.

Theorem 1.35. *Let L be a Lie algebra. Then L is semisimple if and only if there are simple ideals $I_1, \dots, I_k$ of L such that $L = I_1 \oplus \cdots \oplus I_k$.*

Remark 1.15. It is obvious directly from the definition of the center that it is a solvable (even abelian) ideal. Hence, $Z(L) = 0$ for each semisimple Lie algebra L. Now it follows from Remark 1.14 and from Theorem 1.24 that the decomposition in the previous theorem is unique up to the ordering of ideals.

At this place we state a crucial theorem describing the structure of a general Lie algebra. For its statement and proof see [2]. Before, we establish notation of the so-called "*semidirect sum*" (cf. [4]).

Definition 1.36. Let L be a Lie algebra, let I be an ideal of L and let K be a subalgebra of L such that, as vector spaces, $L = I \dotplus K$. We say that L, as a Lie algebra, is the *semidirect sum* of I and K and we write $L = I \inplus K$.

Theorem 1.37 (Levi - Malcev)**.** *Let L be a Lie algebra. If $\mathrm{Rad}(L) \neq L$, then there exists a semisimple subalgebra K of L such that*

$$L = K \niplus \mathrm{Rad}(L). \tag{1.20}$$

Moreover, if $\tilde{K}$ is any other semisimple subalgebra of L satisfying (1.20)*, then there exists an automorphism $\varphi : L \to L$ such that $\varphi(K) = \tilde{K}$.*[4]

Remark 1.16. According to Proposition 1.32 and uniqueness of the radical, if φ is an automorphism of a Lie algebra L, one has $\varphi(\mathrm{Rad}(L)) = \mathrm{Rad}(L)$.

[3] To avoid any misunderstanding, note that the slash does not mean the factor vector space here at all.

[4] In its original version, the theorem even further specify such an automorphism, namely to be a so-called *special* automorphism of L (cf. [2]).

1.6 Derivations and Their Generalizations

Definition 1.38. Let L be a Lie algebra. A linear map $D: L \to L$ is called a *derivation* of L if the following identity is satisfied for all $x, y \in L$:

$$D([x,y]) = [D(x), y] + [x, D(y)]. \tag{1.21}$$

We denote the set of all derivations of L as $\mathrm{Der}(L)$.

Proposition 1.39. *Let L be a Lie algebra.* $\mathrm{Der}(L)$ *is a subalgebra of* $\mathrm{gl}(L)$*.*

Proof. It is easily seen that the condition (1.21) is linear in D, hence $\mathrm{Der}\, L \subset\subset \mathrm{gl}(L)$. Further, one can readily check that for all $D, E \in \mathrm{Der}\, L$ and $x, y \in L$ it holds

$$[D,E]([x,y]) = [[D,E](x), y] + [x, [D,E](y)]$$

(cf. [12]). Thus $[D,E] \in \mathrm{Der}\, L$ and $\mathrm{Der}\, L$ is a subalgebra of $\mathrm{gl}(L)$ indeed.[5] □

In contrast with the above part of the first chapter, the following definition is not included in the "classical" theory of Lie algebras. This concept that generalize the definition of a derivation of a Lie algebra by adding three scalar parameters was introduced quite recently in [14] and it turns out to be extremely useful for the identification of Lie algebras, at least in low dimensions.

Definition 1.40. Let L be a Lie algebra over $\mathbb{F}$ and let $\alpha, \beta, \gamma \in \mathbb{F}$. A linear map $D: L \to L$ is called an (α, β, γ)*-derivation* of L if the following identity is satisfied for all $x, y \in L$:

$$\alpha D([x,y]) = \beta[D(x), y] + \gamma[x, D(y)]. \tag{1.22}$$

We denote the set of all (α, β, γ)-derivations of L as $\mathrm{Der}_{(\alpha,\beta,\gamma)}(L)$.

Remark 1.17. It is obvious from the bilinearity of the Lie bracket that for any $\alpha, \beta, \gamma \in \mathbb{F}$, equality (1.22) is satisfied for each linear combination of operators from $\mathrm{Der}_{(\alpha,\beta,\gamma)}(L)$ and therefore it holds $\mathrm{Der}_{(\alpha,\beta,\gamma)}(L) \subset\subset \mathrm{gl}(L)$. However, this subspace do not need to became a Lie subalgebra in general. In fact, this happens in very few cases (cf. [14]), one example of such a case we have already introduced above, namely $\alpha = \beta = \gamma = 1$.

Yet further generalization of (α, β, γ)-derivations is possible. In [15], the authors of [14] generalized the concept of the so-called cohomology cocycles of Lie algebras (cf. [3]) in the analogical way as in the case of derivations, i.e. through consideration of additional parameters. In a special case, this generalization coincides with (α, β, γ)-derivations precisely (cf. Remark 1.18 below). Also this tool turns out to be very useful for the identification, particularly in the cases that are resistant to (α, β, γ)-derivations.

Definition 1.41. Let L be a Lie algebra over $\mathbb{F}$, let $k \in \mathbb{N}$ and let $\mathcal{K}$ be a $(k+1) \times (k+1)$ symmetric matrix over $\mathbb{F}$. A totally antisymmetric multilinear map $c: L^{\times k} \to L$ is called a $\mathcal{K}$*-twisted cocycle* on L if for any $x_1, \ldots, x_{k+1} \in L$ it holds that

$$\begin{aligned} 0 = &\sum_{i=1}^{k+1} (-1)^{i+1} \mathcal{K}_{ii} [x_i, c(x_1, \ldots, x_{i-1}, x_{i+1} \ldots, x_{k+1})] \\ &+ \sum_{1 \le i < j \le k+1} (-1)^{i+j} \mathcal{K}_{ij} c([x_i, x_j], x_1, \ldots, x_{i-1}, x_{i+1} \ldots, x_{j-1}, x_{j+1} \ldots, x_{k+1}). \end{aligned} \tag{1.23}$$

We denote the set of all $\mathcal{K}$-twisted cocycles on L as $Z^k(L, \mathrm{ad}, \mathcal{K})$.[6]

[5] We do not discriminate between the Lie brackets on L and $\mathrm{gl}(L)$ here, but it is clear from the context which one we mean at the moment.

[6] Here "ad" denotes the adjoint representation of L. The reason for us not to omit it here is that one could define twisted cocycles for general representation $\rho : L \to \mathrm{gl}(V)$ of L and the definition differs among various representations (cf. [15]). The definition we present here is a special case $V = L$ and $\rho = \mathrm{ad}$.

Proposition 1.42. *Let L be a Lie algebra over $\mathbb{F}$, let $k \in \mathbb{N}$ and let $\mathcal{K}$ be a $(k+1) \times (k+1)$ symmetric matrix over $\mathbb{F}$. Then $Z^k(L, \mathrm{ad}, \mathcal{K})$ forms a vector space over $\mathbb{F}$, with addition and scalar multiplication defined pointwise.*

Proof. Given any $c_1, c_2 \in Z^k(L, \mathrm{ad}, \mathcal{K})$ and $\alpha \in \mathbb{F}$, it is clear directly from the definition of addition and scalar multiplication on $Z^k(L, \mathrm{ad}, \mathcal{K})$ that $\alpha c_1 + c_2$ is multilinear and totally antisymmetric. Furthermore, the second sum in (1.23) is obviously linear in c, the first one is so since the Lie bracket is bilinear, and hence (1.23) holds for $\alpha c_1 + c_2$. All in all, $\alpha c_1 + c_2 \in Z^k(L, \mathrm{ad}, \mathcal{K})$ and thus $Z^k(L, \mathrm{ad}, \mathcal{K})$ is a subspace of the vector space of all multilinear mappings from: $L^{\times k} \to L$. $\square$

Remark 1.18. For $k = 1$ and $\mathcal{K} = \begin{pmatrix} \gamma & \alpha \\ \alpha & \beta \end{pmatrix}$ the condition (1.23) is precisely (1.22) and hence for this $\mathcal{K}$ one has $Z^1(L, \mathrm{ad}, \mathcal{K}) = \mathrm{Der}_{(\alpha,\beta,\gamma)}(L)$.

Chapter 2

Classification of Lie Algebras

Naturally, as for any other mathematical structures, one aims to classify all Lie algebras that may occur. Of coarse, it only makes sense to execute the classification up to isomorphism, i.e. to classify all possible classes of isomorphism.

In other words, the final goal of the classification process is to provide a list of mutually non-isomorphic Lie algebras such that any Lie algebra is isomorphic to precisely one item in the list. This task turns out, however, to be very complicated problem in general and thus nowadays only certain classes of Lie algebras are completely classified, despite the great interest of many mathematicians in this topic. Recall that we are still talking about finite-dimensional real or complex Lie algebras entirely.

In the second chapter we present list of those indecomposable Lie algebras which we shall be able to identify later using methods that we shall introduce in Chapter 3. Namely, we list all indecomposable Lie algebras up to dimension four. Before, we explain why it is sufficient to take only indecomposable Lie algebras into consideration and we briefly sketch out the ways in which the listings were obtained.

2.1 Decomposition into the Direct Sum

We have seen that each Lie algebra decomposes into the direct sum of its indecomposable ideals and that the decomposition is unique up to isomorphisms of particular ideals and up to the ordering of ideals in the sum (cf. Section 1.3).

Consider two isomorphic Lie algebras, say L and M, and $\varphi : L \to M$, an isomorphism between them. Let $L = I_1 \oplus \cdots \oplus I_k$ be a decomposition of L into the direct sum of its indecomposable ideals. It follows from Proposition 1.14 and from bijectivity of φ that $M = \varphi(L) = \varphi(I_1) \oplus \cdots \oplus \varphi(I_k)$ is a direct sum decomposition into ideals of M. Moreover, for all $i \in \widehat{k}$ it holds $\varphi(I_i) \cong I_i$ and hence, according to Proposition 1.20, all these ideals are indecomposable. The uniqueness of such a decomposition, as described above, then guarantees that isomorphic Lie algebras are directly composed of respectively isomorphic indecomposable ideals.

Conversely, let $L = I_1 \oplus \cdots \oplus I_k$ and $M = J_1 \oplus \cdots \oplus J_k$ be decompositions of Lie algebras L and M into their indecomposable ideals such that, for all $i \in \widehat{k}$, there exists an isomorphism $\varphi_i : I_i \to J_i$. Then the map $\varphi : L \to M$ defined for all $x = \sum_{i=1}^{k} x_i \in L$, $x_i \in I_i$, as $\varphi(x) := \sum_{i=1}^{k} \varphi_i(x_i)$ is an isomorphism obviously and hence $L \cong M$.

All in all, the following holds: *two Lie algebras are isomorphic if and only if they are directly composed of respectively isomorphic indecomposable ideals*. This fact allows us to restrict ourselves only to the classification of indecomposable Lie algebras.

2.2 Methods of Classification

2.2.1 General Structure of Indecomposable Lie Algebras

Let L be a (finite-dimensional real or complex) Lie algebra assumed from now on to be moreover indecomposable. Depending on the radical of L, one distinguishes three general cases of structure of L. First, if $\mathrm{Rad}(L) = 0$, then L is semisimple and therefore *simple*, since it is indecomposable (cf. Remark 1.14). Second, if $\mathrm{Rad}(L) = L$, then L is obviously *solvable*. Finally, consider the case $0 \neq \mathrm{Rad}(L) \neq L$. Such a Lie algebra is neither semisimple nor solvable and, according to the theorem of Levi and Malcev, L decomposes non-trivially as $L = K \niplus \mathrm{Rad}(L)$, where K is a semisimple subalgebra of L. In this case, L is said to be *Levi decomposable*. Let us discussed this last possibility in some detail now.

Thus, assume L to be non-trivially Levi decomposable, i.e. $L = K \dotplus R$, where K is semisimple, $R := \mathrm{Rad}(L)$, $[R,R] \subset R$, $[K,K] \subset K$ and $[K,R] \subset R$. In view of the last inclusion, we may associate to L a representation $\rho : K \to \mathfrak{gl}(R)$ of the chosen subalgebra K on the radical R: for all $k \in K$ and $r \in R$ we define

$$(\rho(k))(r) := [k,r]. \tag{2.1}$$

It follows easily from the Jacobi identity on L that ρ is a representation indeed and also that $\rho(K) \subset \mathrm{Der}(R)$.

Contrariwise, consider a semisimple Lie algebra K, a solvable Lie algebra R and a representation $\rho : K \to \mathfrak{gl}(R)$ such that $\rho(K) \subset \mathrm{Der}(R)$. Then we may define a Lie algebra L to be the vector space $K \dotplus R$ with the Lie bracket $[\ ,\ ]$ given for all $k_1, k_2 \in K$ and $r_1, r_2 \in R$ as follows:

$$[k_1 + r_1, k_2 + r_2] := [k_1,k_2]_K + (\rho(k_1))(r_2) - (\rho(k_2))(r_1) + [r_1,r_2]_R, \tag{2.2}$$

where $[\ ,\ ]_K$ and $[\ ,\ ]_R$ are the Lie brackets on K and R, respectively.[7] It is an easy exercise to verify that (2.2) defines the Lie bracket indeed (here one needs the definitional property (1.11) of a representation as well as the fact that $\rho(K) \subset \mathrm{Der}(R)$). Furthermore, it is clear that $R = \mathrm{Rad}(L)$ and hence $L = K \niplus R$ is a Levi decomposition of L. Finally, comparing (2.1) with (2.2), one can obviously identify L with the Levi decomposable Lie algebra considered in the previous paragraph. Thus, each Levi decomposition $K \niplus R$ is completely characterized by a triplet K, R, ρ satisfying the above properties.

2.2.2 Current Results on Classifying Lie Algebras

According to the previous subsection, the classification of all indecomposable Lie algebras requires solution of the following tasks (cf. [13]):

(I) classifying simple Lie algebras;

(II) classifying indecomposable solvable Lie algebras;

(III) classifying possible triplets (K, R, ρ), where R is a solvable Lie algebra, K is a semisimple Lie algebra and ρ is a representation of K on R compatible (in the sense described above) with the Lie algebra structure of R.

Up to now, only the first task is solved completely. The classification of complex simple Lie algebras were obtained already at the end of the nineteenth century by

[7] More precisely, we first define $L := K \times R$. Then it is clear that $(K,0)$ and $(0,R)$ are subspaces of L such that $L = (K,0) \dotplus (0,R)$. Thus we may identify $K \equiv (K,0)$ and $R \equiv (0,R)$ and we may regard elements from L to be sums of two parts from K and R, respectively, rather then ordered pairs.

W. Killing and E. Cartan. Their work is based on classifying the so-called *root systems*, i.e. certain subsets of the vector space dual to the so-called *Cartan subalgebra*. Nowadays, this achievement is a part of the standard Lie theory and one can found it e.g. in [11], Chap. 4. Over $\mathbb{R}$, the classification problem of simple Lie algebras was solved a few years later, at the beginning of the twentieth century, again by E. Cartan. His approach lay in studying of involutive automorphisms, i.e. those automorphisms whose square is the identity, of complex simple Lie algebras (cf. [7]).

In contrast, (indecomposable) solvable Lie algebras over both $\mathbb{R}$ and $\mathbb{C}$, respectively, are completely classified only in low dimensions, namely up to dimension six. Only certain classes of solvable Lie algebras are classified in higher or even in all dimensions. This is the case of solvable Lie algebras with a given nilpotent Lie algebra as the non-trivial nilradical. For summary of results on this topic as well as for classification of low-dimensional nilpotent Lie algebras, see [13].

Although all possible (non-equivalent) representations of semisimple Lie algebras are known, the complete classification of all Levi decomposable Lie algebras cannot be obtained, since we have not uncovered all solvable Lie algebras, i.e. all potential radicals. However, also in this case, the classification problem is solved at least in low dimensions. Namely all Levi decomposable Lie algebras up to dimension nine (over $\mathbb{R}$ as well as over $\mathbb{C}$) were found in [16] and [17]. Another approach to the same results was described in [13], Chap. 14.

2.3 List of Indecomposable Lie Algebras up to Dimension Four

In this section we list all complex and real, respectively, indecomposable Lie algebras up to dimension four. These listings were adopted from [13] together with the notation used there.

Items independent of any parameters denote single Lie algebras whereas items depending on one or two parameters denote one- or two-, respectively, parametric class of mutually non-isomorphic Lie algebras. In the second case, specification of parameters is always attached in order to avoid any repetitions in the list. Precisely as in [13], Latin letters denote the complex parameters while Greek letters stand for the real ones.

2.3.1 Lie Algebras over $\mathbb{C}$

Nilpotent one-dimensional Lie algebra:

- $\mathfrak{n}_{1,1}$ abelian.

Solvable two-dimensional Lie algebra with the nilradical $\mathfrak{n}_{1,1} \equiv \mathrm{Span}\{e_1\}$:

- $\mathfrak{s}_{2,1}$ $[e_2, e_1] = e_1.$

Nilpotent three-dimensional Lie algebra:

- $\mathfrak{n}_{3,1}$ $[e_2, e_3] = e_1.$

Solvable three-dimensional Lie algebras with the nilradical $2\mathfrak{n}_{1,1} \equiv \mathrm{Span}\{e_1, e_2\}$:

- $\mathfrak{s}_{3,1}(a)^{\mathbb{C}\text{-}\mathrm{I}}$ $[e_3, e_1] = e_1,\ [e_3, e_2] = ae_2;$
- $\mathfrak{s}_{3,2}$ $[e_3, e_1] = e_1,\ [e_3, e_2] = e_1 + e_2.$

Simple three-dimensional Lie algebra:

- $\mathfrak{sl}(2, \mathbb{C})$ $[e_1, e_2] = 2e_1,\ [e_1, e_3] = -e_2,\ [e_2, e_3] = 2e_3.$

Nilpotent four-dimensional Lie algebra:

- $\mathsf{n}_{4,1}$ $[e_2, e_4] = e_1,\ [e_3, e_4] = e_2.$

Solvable four-dimensional Lie algebras with the nilradical $3\mathsf{n}_{1,1} \equiv \mathrm{Span}\{e_1, e_2, e_3\}$:

- $\mathsf{s}_{4,1}$ $[e_4, e_2] = e_1\ , [e_4, e_3] = e_3;$
- $\mathsf{s}_{4,2}$ $[e_4, e_1] = e_1,\ [e_4, e_2] = e_1 + e_2,\ [e_4, e_3] = e_2 + e_3;$
- $\mathsf{s}_{4,3}(a, b)^{\mathbb{C}\text{-II}}$ $[e_4, e_1] = e_1,\ [e_4, e_2] = ae_2,\ [e_4, e_3] = be_3;$
- $\mathsf{s}_{4,4}(a)^{\mathbb{C}\text{-III}}$ $[e_4, e_1] = e_1,\ [e_4, e_2] = e_1 + e_2,\ [e_4, e_3] = ae_3.$

Solvable four-dimensional Lie algebras with the nilradical $\mathsf{n}_{3,1} \equiv \mathrm{Span}\{e_1, e_2, e_3\}$:

- $\mathsf{s}_{4,6}$ $[e_4, e_2] = e_2,\ [e_4, e_3] = -e_3,\ [e_2, e_3] = e_1;$
- $\mathsf{s}_{4,8}(a)^{\mathbb{C}\text{-IV}}$ $[e_4, e_1] = (1 + a)e_1,\ [e_4, e_2] = e_2,\ [e_4, e_3] = ae_3,\ [e_2, e_3] = e_1;$
- $\mathsf{s}_{4,10}$ $[e_4, e_1] = 2e_1,\ [e_4, e_2] = e_2,\ [e_4, e_3] = e_2 + e_3,\ [e_2, e_3] = e_1;$
- $\mathsf{s}_{4,11}$ $[e_4, e_1] = e_1,\ [e_4, e_2] = e_2,\ [e_2, e_3] = e_1.$

Parameters specification:

($\mathbb{C}$-I) $0 < |a| \leq 1$, if $|a| = 1$ then $\arg(a) \leq \pi$;

($\mathbb{C}$-II) $0 < |b| \leq |a| \leq 1$,
if $|b| = |a| < 1$ then $\arg(a) \leq \arg(b)$,
if $|b| < |a| = 1$ then $\arg(a) \leq \pi$ and if $\arg(a) = \pi$ then $\arg(b) < \pi$,
if $|b| = |a| = 1$ then $\arg(a) \leq \frac{2}{3}\pi$, $\arg(a) \leq \arg(b) \leq \frac{4}{3}\pi$ and
if $\arg(a) = \frac{2}{3}\pi$ then $\arg(b) = \frac{4}{3}\pi$;

($\mathbb{C}$-III) $a \neq 0$;

($\mathbb{C}$-IV) $0 < |a| \leq 1$, if $|a| = 1$ then $\arg(a) < \pi$.

2.3.2 Lie Algebras over $\mathbb{R}$

Nilpotent one-dimensional Lie algebra:

- $\mathsf{n}_{1,1}$ abelian.

Solvable two-dimensional Lie algebra with the nilradical $\mathsf{n}_{1,1} \equiv \mathrm{Span}\{e_1\}$:

- $\mathsf{s}_{2,1}$ $[e_2, e_1] = e_1.$

Nilpotent three-dimensional Lie algebra:

- $\mathsf{n}_{3,1}$ $[e_2, e_3] = e_1.$

Solvable three-dimensional Lie algebras with the nilradical $2\mathsf{n}_{1,1} \equiv \mathrm{Span}\{e_1, e_2\}$:

- $\mathsf{s}_{3,1}(\alpha)^{\mathbb{R}\text{-I}}$ $[e_3, e_1] = e_1,\ [e_3, e_2] = \alpha e_2;$
- $\mathsf{s}_{3,2}$ $[e_3, e_1] = e_1,\ [e_3, e_2] = e_1 + e_2;$
- $\mathsf{s}_{3,3}(\alpha)^{\mathbb{R}\text{-II}}$ $[e_3, e_1] = \alpha e_1 - e_2,\ [e_3, e_2] = e_1 + \alpha e_2.$

Simple three-dimensional Lie algebras:

- $\mathsf{sl}(2, \mathbb{R})$ $[e_1, e_2] = 2e_1,\ [e_1, e_3] = -e_2,\ [e_2, e_3] = 2e_3;$
- $\mathsf{so}(3, \mathbb{R})$ $[e_1, e_2] = e_3,\ [e_1, e_3] = -e_2,\ [e_2, e_3] = e_1.$

Nilpotent four-dimensional Lie algebra:

- $\mathsf{n}_{4,1}$ $[e_2, e_4] = e_1,\ [e_3, e_4] = e_2.$

Solvable four-dimensional Lie algebras with the nilradical $3\mathfrak{n}_{1,1} \equiv \text{Span}\{e_1, e_2, e_3\}$:

- $\mathfrak{s}_{4,1}$ $[e_4, e_2] = e_1$, $[e_4, e_3] = e_3$;
- $\mathfrak{s}_{4,2}$ $[e_4, e_1] = e_1,\ [e_4, e_2] = e_1 + e_2,\ [e_4, e_3] = e_2 + e_3$;
- $\mathfrak{s}_{4,3}(\alpha, \beta)^{\mathbb{R}\text{-III}}$ $[e_4, e_1] = e_1,\ [e_4, e_2] = \alpha e_2,\ [e_4, e_3] = \beta e_3$;
- $\mathfrak{s}_{4,4}(\alpha)^{\mathbb{R}\text{-IV}}$ $[e_4, e_1] = e_1,\ [e_4, e_2] = e_1 + e_2,\ [e_4, e_3] = \alpha e_3$;
- $\mathfrak{s}_{4,5}(\alpha, \beta)^{\mathbb{R}\text{-V}}$ $[e_4, e_1] = \alpha e_1,\ [e_4, e_2] = \beta e_2 - e_3,\ [e_4, e_3] = e_2 + \beta e_3$.

Solvable four-dimensional Lie algebras with the nilradical $\mathfrak{n}_{3,1} \equiv \text{Span}\{e_1, e_2, e_3\}$:

- $\mathfrak{s}_{4,6}$ $[e_4, e_2] = e_2,\ [e_4, e_3] = -e_3,\ [e_2, e_3] = e_1$;
- $\mathfrak{s}_{4,7}$ $[e_4, e_2] = -e_3,\ [e_4, e_3] = e_2,\ [e_2, e_3] = e_1$;
- $\mathfrak{s}_{4,8}(\alpha)^{\mathbb{R}\text{-VI}}$ $[e_4, e_1] = (1 + \alpha)e_1,\ [e_4, e_2] = e_2,\ [e_4, e_3] = \alpha e_3,\ [e_2, e_3] = e_1$;
- $\mathfrak{s}_{4,9}(\alpha)^{\mathbb{R}\text{-VII}}$ $[e_4, e_1] = 2\alpha e_1,\ [e_4, e_2] = \alpha e_2 - e_3,\ [e_4, e_3] = e_2 + \alpha e_3,\ [e_2, e_3] = e_1$;
- $\mathfrak{s}_{4,10}$ $[e_4, e_1] = 2e_1,\ [e_4, e_2] = e_2,\ [e_4, e_3] = e_2 + e_3,\ [e_2, e_3] = e_1$;
- $\mathfrak{s}_{4,11}$ $[e_4, e_1] = e_1,\ [e_4, e_2] = e_2,\ [e_2, e_3] = e_1$.

Solvable four-dimensional Lie algebra with the nilradical $2\mathfrak{n}_{1,1} \equiv \text{Span}\{e_1, e_2\}$:

- $\mathfrak{s}_{4,12}$ $[e_3, e_1] = e_1,\ [e_3, e_2] = e_2,\ [e_4, e_1] = -e_2,\ [e_4, e_2] = e_1$.

Parameters specification:

(ℝ-I) $0 < |\alpha| \leq 1$;
(ℝ-II) $\alpha \geq 0$;
(ℝ-III) $0 < |\beta| \leq |\alpha| \leq 1$,
if $|\beta| = |\alpha| < 1$ then $\beta \leq \alpha$,
if $\alpha = -1$ and $|\beta| < 1$ then $\beta > 0$,
if $|\beta| = |\alpha| = 1$ then $\alpha = 1$;
(ℝ-IV) $\alpha \neq 0$;
(ℝ-V) $\alpha > 0$;
(ℝ-VI) $0 < |\alpha| \leq 1, \alpha \neq -1$;
(ℝ-VII) $\alpha > 0$.

Chapter 3

Invariants of Lie Algebras

Given a real or complex indecomposable Lie algebra of dimension less or equal four, it must be isomorphic to precisely one item in the list of all such Lie algebras presented in Section 2.3. In this final chapter we describe the way of recognising such a Lie algebra among those in the list.

As anticipated, we shall use various *invariants* for this purpose. We shall proceed separately for complex and real Lie algebras, the identification process is, however, exactly the same. First, we have to find a set of invariants such that the sets of their values are pairwise distinct for any two Lie algebras from the lists in Section 2.3. Such a set of invariants is said to be *complete*. Then, given a Lie algebra to identify, we compute the values of invariants from the complete set on it and we restrict the list of eventually isomorphic Lie algebras until the last possibility remains.

3.1 Introduction of Invariants

First, we establish the invariants needed for identification. The invariants introduced here were chosen among those described in [13], Chapter 4. To make the identification process rigorous, we prove here that each single invariant stays unchanged under the action of an isomorphism of Lie algebras, i.e. that it is "invariant" indeed.

3.1.1 Dimensions of Ideals

Lemma 3.1. *Let L_1 and L_2 be isomorphic Lie algebras. Then* $\dim L_1 = \dim L_2$*.*

Proof. Since L_1 and L_2 are isomorphic, there exists a linear bijection between them. Consequently, they must have the same dimensions. $\square$

Lemma 3.2. *Let L_1 and L_2 be isomorphic Lie algebras. Then for all $n \in \mathbb{N}_0$ it holds that*
(a) $\dim L_1^{(n)} = \dim L_2^{(n)}$*;*
(b) $\dim L_1^{n} = \dim L_2^{n}$*;*
(c) $\dim Z_n(L_1) = \dim Z_n(L_2)$*.*

Proof. Let $\varphi : L_1 \to L_2$ be an isomorphism.
(a) According to part (a) of Lemma 1.26, even $\varphi(L_1^{(n)}) = \varphi(L_1)^{(n)} = L_2^{(n)}$ holds for all $n \in \mathbb{N}_0$. The equality of dimensions is now obvious since the dimension of a subspace is preserved by a linear bijection.
(b) Analogically to (a).

(c) Again it suffices to show that $\varphi(Z_n(L_1)) = Z_n(L_2)$ holds for all $n \in \mathbb{N}_0$. We use induction on n. First, for $n = 0$ we have $Z_0(L_i) = 0$, $i = 1,2$, and the assertion is obvious. Second, assume that $\varphi(Z_{n-1}(L_1)) = Z_{n-1}(L_2)$. Then for any $x \in L_1$ it holds

$$\begin{aligned} x \in Z_n(L_1) &\iff \text{for all } y \in L_1, [x,y] \in Z_{n-1}(L_1) \\ &\iff \text{for all } y \in L_1, [\varphi(x), \varphi(y)] \in \varphi(Z_{n-1}(L_1)) \\ &\iff \text{for all } z \in L_2, [\varphi(x), z] \in Z_{n-1}(L_2) \\ &\iff \varphi(x) \in Z_n(L_2). \end{aligned}$$

and it follows that $\varphi(Z_n(L_1)) = Z_n(L_2)$. □

3.1.2 Signature of the Killing Form

Let us recall some well-known results from linear algebra at this place (cf. [9]).

Remark 3.1. Let V be an n-dimensional real vector space, let $\mathcal{B} := (b_1, \ldots, b_n)$ be a basis of V and let $f : V \times V \to \mathbb{R}$ be a symmetric bilinear form. To the form f, one can associate the (symmetric) matrix $A \in \mathbb{R}^{n,n}$ with entries $A_{ij} = f(b_i, b_j)$, $i, j \in \hat{n}$. Then the *signature* of the form f is defined to be the ordered triplet of non-negative integers (π, ν, δ), where π, ν and δ denote the numbers of positive, negative and zero, respectively, eigenvalues (including repetitions) of A. Since A is symmetric and hence diagonalisable, it holds that $\pi + \nu + \delta = n$. Furthermore, it is a theorem (due to Sylvester, cf. [1]) that the signature does not depend on the choice of basis $\mathcal{B}$.

Lemma 3.3. *Let L_1 and L_2 be isomorphic real Lie algebras. Let κ_i be the Killing form on L_i, i=1,2. Then the signatures of κ_1 and κ_2 are the same.*

Proof. Let us choose a basis of L_1, say $\mathcal{X} := (x_1, \ldots, x_n)$ and let us denote the structure constants of L with respect to $\mathcal{X}$ as a_{ij}^k; $i, j, k \in \hat{n}$. Thus for all $i, j \in \hat{n}$ we have

$$[x_i, x_j] = \sum_{k=1}^{n} a_{ij}^k x_k.$$

Further, let $\varphi : L_1 \to L_2$ be an isomorphism and consider $\mathcal{Y} := (y_1, \ldots, y_n)$, where $y_i := \varphi(x_i)$, $i \in \hat{n}$. $\mathcal{Y}$ forms a basis of L_2 obviously. Moreover, for all $i, j \in \hat{n}$ we have

$$[y_i, y_j] = [\varphi(x_i), \varphi(x_j)] = \varphi([x_i, x_j]) = \sum_{k=1}^{n} a_{ij}^k \varphi(x_k) = \sum_{k=1}^{n} a_{ij}^k y_k.$$

Now it is clear that the matrices of κ_1 and κ_2 with respect to bases $\mathcal{X}$ and $\mathcal{Y}$, respectively, are identical. It follows that the signatures of κ_1 and κ_2 are equal. □

Lemma 3.4. *Let L_1 and L_2 be isomorphic real Lie algebras. Let κ_i be the Killing form on* $\mathrm{Der}(L_i)$*, i=1,2. Then the signatures of κ_1 and κ_2 are the same.*

Proof. Let $\varphi : L_1 \to L_2$ be an isomorphism. We define

$$\phi : \mathrm{Der}(L_1) \to \mathrm{Der}(L_2) : D \mapsto \varphi \circ D \circ \varphi^{-1}.$$

First, we must check that ϕ is well-defined. For any $x, y \in L_2$ and $D \in \mathrm{Der}(L_1)$ we have

$$\begin{aligned} \phi(D)[x,y] &= (\varphi \circ D \circ \varphi^{-1})[x,y] \\ &= (\varphi \circ D)[\varphi^{-1}(x), \varphi^{-1}(y)] \\ &= \varphi[(D \circ \varphi^{-1})(x), \varphi^{-1}(y)] + \varphi[\varphi^{-1}(x), (D \circ \varphi^{-1})(y)] \end{aligned}$$

$$
\begin{aligned}
&= [(\varphi \circ D \circ \varphi^{-1})(x), y] + [x, (\varphi \circ D \circ \varphi^{-1})(y)] \\
&= [\phi(D)(x), y] + [x, \phi(D)(y)].
\end{aligned}
$$

As a composition of linear maps, $\phi(D)$ is linear as well and thus $\phi(D) \in \mathrm{Der}(L_2)$.

Since φ is a bijection, it is clear that $\phi(D) = 0$ if and only if $D = 0$ and that the map $\phi^{-1} : \mathrm{Der}(L_2) \to \mathrm{Der}(L_1) : D \mapsto \varphi^{-1} \circ D \circ \varphi$ is the inverse of ϕ. Thus ϕ is a bijection. Further, one can easily see that ϕ is linear. Finally, for any $D, E \in \mathrm{Der}(L_1)$ we can write (the commutators are taken on $\mathrm{Der}(L_1)$ and $\mathrm{Der}(L_2)$, respectively)

$$
\begin{aligned}
\phi([D, E]) &= \varphi \circ D \circ \varphi^{-1} \circ \varphi \circ E \circ \varphi^{-1} - \varphi \circ E \circ \varphi^{-1} \circ \varphi \circ D \circ \varphi^{-1} \\
&= \phi(D) \circ \phi(E) - \phi(E) \circ \phi(D) \\
&= [\phi(D), \phi(E)]
\end{aligned}
$$

to show that ϕ is a homomorphism. All in all, we have proved that $\mathrm{Der}(L_1) \cong \mathrm{Der}(L_2)$ and hence we can use Lemma 3.3 in order to get the statement. □

3.1.3 Invariant Functions Ψ and Φ

Definition 3.5. Let L be a complex Lie algebra. We define the function $\Psi_L : \mathbb{C} \to \mathbb{N}_0$ associated to the Lie algebra L as follows:

$$
\Psi_L(\alpha) := \dim \mathrm{Der}_{(\alpha,1,1)}(L), \quad \alpha \in \mathbb{C}. \tag{3.1}
$$

Lemma 3.6. *Let L_1 and L_2 be isomorphic complex Lie algebras. Then $\Psi_{L_1} = \Psi_{L_2}$.*

Proof. We must show that for all $\alpha \in \mathbb{C}$ it holds $\dim \mathrm{Der}_{(\alpha,1,1)}(L_1) = \dim \mathrm{Der}_{(\alpha,1,1)}(L_2)$. Let $\varphi : L_1 \to L_2$ be an isomorphism. Take an arbitrary $\alpha \in \mathbb{C}$ and let us define

$$
\phi : \mathrm{Der}_{(\alpha,1,1)}(L_1) \to \mathrm{Der}_{(\alpha,1,1)}(L_2) : D \mapsto \varphi \circ D \circ \varphi^{-1}.
$$

We must check that ϕ is well-defined . For any $x, y \in L_2$ and $D \in \mathrm{Der}_{(\alpha,1,1)}(L_1)$ we have

$$
\begin{aligned}
\alpha\phi(D)[x, y] &= \alpha(\varphi \circ D \circ \varphi^{-1})[x, y] \\
&= \alpha(\varphi \circ D)[\varphi^{-1}(x), \varphi^{-1}(y)] \\
&= \varphi[(D \circ \varphi^{-1})(x), \varphi^{-1}(y)] + \varphi[\varphi^{-1}(x), (D \circ \varphi^{-1})(y)] \\
&= [(\varphi \circ D \circ \varphi^{-1})(x), y] + [x, (\varphi \circ D \circ \varphi^{-1})(y)] \\
&= [\phi(D)(x), y] + [x, \phi(D)(y)].
\end{aligned}
$$

As a composition of linear maps, $\phi(D)$ is linear as well and thus $\phi(D) \in \mathrm{Der}_{(\alpha,1,1)}(L_2)$.

Since φ is a bijection, it is clear that $\phi(D) = 0$ if and only if $D = 0$ and that the map $\phi^{-1} : \mathrm{Der}(L_2) \to \mathrm{Der}(L_1) : D \mapsto \varphi^{-1} \circ D \circ \varphi$ is the inverse of ϕ. Thus, ϕ is a bijection and the equality between dimensions holds. □

Definition 3.7. Let L be a complex Lie algebra. For each $\alpha \in \mathbb{C}$, let

$$
\mathcal{K}_\alpha = \begin{pmatrix} \alpha & 1 & 1 \\ 1 & \alpha & 1 \\ 1 & 1 & \alpha \end{pmatrix}. \tag{3.2}
$$

We define the function $\Phi_L : \mathbb{C} \to \mathbb{N}_0$ associated to the Lie algebra L as follows:

$$
\Phi_L(\alpha) := \dim Z^2(L, \mathrm{ad}, \mathcal{K}_\alpha), \quad \alpha \in \mathbb{C}. \tag{3.3}
$$

Remark 3.2. For $k = 2$ and $\mathcal{K}_\alpha$ established by (3.2), the condition (1.23) takes the form

$$\begin{aligned} 0 = \alpha\{[x_1, c(x_2, x_3)] - [x_2, c(x_1, x_3)] + [x_3, c(x_1, x_2)]\} \\ - c([x_1, x_2], x_3) + c([x_1, x_3], x_2) - c([x_2, x_3], x_1). \end{aligned} \tag{3.4}$$

Lemma 3.8. *Let L_1 and L_2 be isomorphic complex Lie algebras. Then $\Phi_{L_1} = \Phi_{L_2}$.*

Proof. We must show that for all $\alpha \in \mathbb{C}$ it holds $\dim Z^2(L_1, \mathrm{ad}, \mathcal{K}_\alpha) = \dim Z^2(L_2, \mathrm{ad}, \mathcal{K}_\alpha)$. Let $\varphi : L_1 \to L_2$ be an isomorphism. Take an arbitrary $\alpha \in \mathbb{C}$ and let us define

$$\phi : Z^2(L_1, \mathrm{ad}, \mathcal{K}_\alpha) \to Z^2(L_2, \mathrm{ad}, \mathcal{K}_\alpha) : c \mapsto \phi(c),$$

where $\phi(c)$ is for any $x, y \in L_2$ defined as $\phi(c)(x, y) := (\varphi \circ c)(\varphi^{-1}(x), \varphi^{-1}(y))$. We must check that ϕ is well-defined . For any $x, y, z \in L_2$ and $c \in Z^2(L_1, \mathrm{ad}, \mathcal{K}_\alpha)$ we have

$$\begin{aligned} \alpha[x, \phi(c)(y, z)] &= \alpha[(\varphi \circ \varphi^{-1})(x), (\varphi \circ c)(\varphi^{-1}(y), \varphi^{-1}(z))] \\ &= \varphi\left(\alpha[\varphi^{-1}(x), c(\varphi^{-1}(y), \varphi^{-1}(z))]\right) \end{aligned}$$

and similarly

$$-\phi(c)([x, y], z) = -(\varphi \circ c)(\varphi^{-1}([x, y]), \varphi^{-1}(z)) = \varphi\left(-c([\varphi^{-1}(x), \varphi^{-1}(y)], \varphi^{-1}(z))\right).$$

Now one can easily see that the condition (3.4) is satisfied. Further, as a composition of (multi-)linear maps, $\phi(c)$ is multilinear as well and therefore $\phi(c) \in Z^2(L_2, \mathrm{ad}, \mathcal{K}_\alpha)$.

Since φ is a bijection, it is clear that $\phi(c) = 0$ if and only if $c = 0$ and that the map

$$\phi^{-1} : Z^2(L_2, \mathrm{ad}, \mathcal{K}_\alpha) \to Z^2(L_1, \mathrm{ad}, \mathcal{K}_\alpha) : c \mapsto \phi^{-1}(c),$$

where $\phi(c)$ is for any $x, y \in L_1$ defined as $\phi^{-1}(c)(x, y) := (\varphi^{-1} \circ c)(\varphi(x), \varphi(y))$ is the inverse of ϕ. Thus, ϕ is a bijection and the equality between dimensions holds. □

Definition 3.9. Let L be a real Lie algebra. The *complexification* of L is the complex vector space $L_\mathbb{C} := \mathrm{Span}_\mathbb{R}\left\{x_1 + ix_2 \mid x_1, x_2 \in L\right\}$ with addition and scalar multiplication defined for all $x = x_1 + ix_2, y = y_1 + iy_2 \in L_\mathbb{C}$ and $\alpha = \mathrm{Re}(\alpha) + i\,\mathrm{Im}(\alpha) \in \mathbb{C}$ as

$$x + y := (x_1 + y_1) + i(x_2 + y_2), \tag{3.5}$$

$$\alpha x := (\mathrm{Re}(\alpha)x_1 - \mathrm{Im}(\alpha)x_2) + i(\mathrm{Re}(\alpha)x_2 + \mathrm{Im}(\alpha)x_1), \tag{3.6}$$

equipped with the map $[\ ,\ ]_\mathbb{C} : L_\mathbb{C} \times L_\mathbb{C} \to L_\mathbb{C}$ defined for all $x = x_1 + ix_2, y = y_1 + iy_2 \in L_\mathbb{C}$ as

$$[x, y]_\mathbb{C} := ([x_1, y_1] - [x_2, y_2]) + i([x_1, y_2] + [x_2, y_1]), \tag{3.7}$$

where $[\ ,\ ]$ is the Lie bracket on L.[8]

Remark 3.3. It is a routine matter to verify that $L_\mathbb{C}$ has a vector space structure indeed. Furthermore, one can readily check, that the map $[\ ,\ ]_\mathbb{C}$ fulfills all conditions on the Lie bracket and hence that the complexification of L is a (complex) Lie algebra.

One can easily see that if $(x_i)_{i=1}^n$ is a basis of L, then $(x_i + i \cdot 0)_{i=1}^n$ is a basis of $L_\mathbb{C}$. In particular, $\dim L = \dim L_\mathbb{C}$.

[8]Here i stands for the imaginary unit. Note that all sums of the type $x + iy$ in this definition are only formal. More precisely, we should define $L_\mathbb{C}$ to be the space of ordered pairs of elements from L. However, the notation we used is more illustrative since the same formalism applies to the notation of complex numbers.

Proposition 3.10. *Let L_1 and L_2 be real Lie algebras. If $L_1 \cong L_2$ then $(L_1)_\mathbb{C} \cong (L_2)_\mathbb{C}$.*

Proof. Let $\varphi : L_1 \to L_2$ be an isomorphism. We show that the map $\phi : (L_1)_\mathbb{C} \to (L_2)_\mathbb{C}$ sending $x_1 + ix_2$ to $\varphi(x_1) + i\varphi(x_2)$ is an isomorphism as well. First, the zero vector in $L_\mathbb{C}$ is $0 + i \cdot 0$ obviously and the condition $\phi(x_1 + ix_2) = 0$ imply $x_1 = x_2 = 0$. As $\dim(L_1)_\mathbb{C} = \dim L_1 = \dim L_2 = \dim(L_2)_\mathbb{C}$, ϕ is a bijection. Second, for arbitrary $x = x_1 + ix_2, y = y_1 + iy_2 \in L_\mathbb{C}$ we have

$$\begin{aligned}\phi([x,y]_\mathbb{C}) &= \varphi([x_1,y_1] - [x_2,y_2]) + i\varphi([x_1,y_2] + [x_2,y_1]) \\ &= ([\varphi(x_1),\varphi(y_1)] - [\varphi(x_2),\varphi(y_2)]) + i([\varphi(x_1),\varphi(y_2)] + [\varphi(x_2),\varphi(y_1)]) \\ &= [\varphi(x_1) + i\varphi(x_2), \varphi(y_1) + i\varphi(y_2)]_\mathbb{C} \\ &= [\phi(x),\phi(y)]_\mathbb{C}.\end{aligned}$$

It remains to verify linearity of ϕ but this can be done in the very same manner as above, using just the definitions (3.5) and (3.6) together with linearity of φ. □

Lemma 3.11. *Let L_1 and L_2 be isomorphic real Lie algebras. Then $\Psi_{(L_1)_\mathbb{C}} = \Psi_{(L_2)_\mathbb{C}}$ and $\Phi_{(L_1)_\mathbb{C}} = \Phi_{(L_2)_\mathbb{C}}$.*

Proof. Trivial consequence of Lemmas 3.6, 3.8, respectively, and Proposition 3.10. □

3.2 Complete Sets of Invariants for Indecomposable Lie Algebras up to Dimension Four

In this section we present the computed complete sets of invariants for all Lie algebras from the list in Section 2.3. As anticipated, we proceed separately for complex and real Lie algebras. For both fields, Lie algebras are further divided by their dimensions, which is the first very rough invariant in fact (cf. Lemma 3.1). Then, for each dimension, concrete invariants are established and tables with their computed values follow. The mark "$-$" in the tables means that the value of the respective invariant is irrelevant.

For the sake of clarity, if the number of needed invariants is too large to comprise all of them in one table, then the references to appendix tables are attached. In these appendices, some other invariants or invariant properties occur in columns denoted "Further specification" in addition to those established at the beginning of each part.

All invariants were computed using MAPLE 18 computer algebra system. More precisely, we used procedures contained in DifferentialGeometry package and its subpackage LieAlgebras, namely Series for computing the dimensions of ideals in the characteristic series, Killing for obtaining the matrix of the Killing form and Derivations in order to compute the basis of the Lie algebra of derivations of a Lie algebra.

For computing the invariant functions Ψ and Φ, we had to implement our own procedures. The principle is as folows. Given a Lie algebra L with a basis $\mathcal{B} = (x_1, \dots, x_n)$, for any $\alpha \in \mathbb{C}$ we have a system of linear equations for n^2 independent elements of an $(\alpha, 1, 1)$-derivation, with respect to $\mathcal{B}$. If A denotes the matrix of such a system, then

$$\Psi_L(\alpha) = n^2 - \operatorname{rank}(A). \tag{3.8}$$

Analogously, for any $\alpha \in \mathbb{C}$ we may consider a system of equations for $\frac{n^2(n-1)}{2}$ independent elements of a $\mathcal{K}_\alpha$-twisted cocycle, where $\mathcal{K}_\alpha$ is defined by (3.2), with respect to $\mathcal{B}$. If A denotes the matrix of this system now, it holds

$$\Phi_L(\alpha) = \frac{n^2(n-1)}{2} - \operatorname{rank}(A). \tag{3.9}$$

Here we used MAPLE procedures Rank, Matrix and GaussianElimination, all included in package LinearAlgebra.

3.2.1 Lie Algebras over $\mathbb{C}$

Dimension one - only one possibility: $\mathfrak{n}_{1,1}$.

Dimension two - only one possibility: $\mathfrak{s}_{2,1}$.

Dimension three - the following invariants are used for identification:

$$\begin{aligned}
\mathcal{I}_1(L) &:= \dim(L'),\\
\mathcal{I}_2(L) &:= \text{number of singularities of } \Psi_L,\\
\mathcal{I}_3(L) &:= \Psi_L(1),\\
\mathcal{I}_4(L) &:= \Psi_L^{(-1)}(4) \cap B_1,\\
\mathcal{I}_5(L) &:= \Psi_L^{(-1)}(4) \cap \mathbb{C}^+.
\end{aligned}$$

L	$\mathcal{I}_1(L)$	$\mathcal{I}_2(L)$	$\mathcal{I}_3(L)$	$\mathcal{I}_4(L)$	$\mathcal{I}_5(L)$
$\mathfrak{n}_{3,1}$	1	–	–	–	–
$\mathfrak{s}_{3,2}$	2	1	4	–	–
$\mathfrak{s}_{3,1}(1)$	2	1	6	–	–
$\mathfrak{s}_{3,1}(-1)$	2	2	–	–	–
$\mathfrak{s}_{3,1}(a), \lvert a\rvert \neq 1$	2	3	–	$\{a\}$	–
$\mathfrak{s}_{3,1}(a), \lvert a\rvert = 1, a \neq \pm 1$	2	3	–	$\emptyset$	$\{a\}$
$\mathfrak{sl}(2,\mathbb{C})$	3	–	–	–	–

Table 3.1: Three-dimensional complex Lie algebras

Dimension four - the following invariants are used for identification:

$$\begin{aligned}
\mathcal{I}_1(L) &:= \dim(Z_3(L)),\\
\mathcal{I}_2(L) &:= \left(\dim(L^{(1)}), \dim(L^{(2)})\right),\\
\mathcal{I}_3(L) &:= \text{number of singularities of } \Psi_L,\\
\mathcal{I}_4(L) &:= \Psi_L(1),\\
\mathcal{I}_5(L) &:= \Phi_L^{(-1)}(15),\\
\mathcal{I}_6(L) &:= \Psi_L^{(-1)}(6)/\{1\},\\
\mathcal{I}_7(L) &:= \Phi_L^{(-1)}(13)+1,\\
\mathcal{I}_8(L) &:= \Psi_L^{(-1)}(5) \cap B_1,\\
\mathcal{I}_9(L) &:= \Psi_L^{(-1)}(4) \cap S_1 \cap \mathbb{C}^+,\\
\mathcal{I}_{10}(L) &:= \Psi_L^{(-1)}(4) \cap B_1.
\end{aligned}$$

L	$\mathcal{I}_1(L)$	$\mathcal{I}_2(L)$	$\mathcal{I}_3(L)$	$\mathcal{I}_4(L)$	Appendix
$\mathsf{s}_{4,11}$	0	$(2,0)$	–	–	–
$\mathsf{s}_{4,2}$	0	$(3,0)$	1	6	–
$\mathsf{s}_{4,4}(1)$	0	$(3,0)$	1	8	–
$\mathsf{s}_{4,3}(1,1)$	0	$(3,0)$	1	12	–
$\mathsf{s}_{4,4}(-1)$	0	$(3,0)$	2	5	–
$\mathsf{s}_{4,3}(1,-1)$	0	$(3,0)$	2	8	–
$\mathsf{s}_{4,3}(-\frac{1}{2}+i\frac{\sqrt{3}}{2},-\frac{1}{2}-i\frac{\sqrt{3}}{2})$	0	$(3,0)$	3	6	Table 3.3
$\mathsf{s}_{4,4}(a), a \neq \pm 1$	0	$(3,0)$	3	6	
$\mathsf{s}_{4,3}(1,b), b \neq \pm 1$	0	$(3,0)$	3	8	Table 3.4
$\mathsf{s}_{4,3}(a,a), a \neq 1$	0	$(3,0)$	3	8	
$\mathsf{s}_{4,3}(i,-1)$	0	$(3,0)$	4	–	–
$\mathsf{s}_{4,3}(b^2,b), b \notin \{\pm 1, -\frac{1}{2}-i\frac{\sqrt{3}}{2}\}$	0	$(3,0)$	5	–	Table 3.5
$\mathsf{s}_{4,3}(a,a^2), a \notin \{\pm 1, -\frac{1}{2}+i\frac{\sqrt{3}}{2}\}$	0	$(3,0)$	5	–	
$\mathsf{s}_{4,3}(-1,b)$	0	$(3,0)$	6	–	Table 3.6
$\mathsf{s}_{4,3}(a,-1), a \notin \{1,i\}$	0	$(3,0)$	6	–	
$\mathsf{s}_{4,3}(a,-a), a \neq 1$	0	$(3,0)$	6	–	
$\mathsf{s}_{4,3}(a,b), a \notin \{\pm 1, \pm b, b^2\},$ $b \notin \{\pm 1, a^2\}$	0	$(3,0)$	7	–	Table 3.7
$\mathsf{s}_{4,10}$	0	$(3,1)$	2	5	–
$\mathsf{s}_{4,8}(1)$	0	$(3,1)$	2	7	–
$\mathsf{s}_{4,8}(\frac{1}{2})$	0	$(3,1)$	3	–	–
$\mathsf{s}_{4,8}(a), a \notin \{\frac{1}{2},1\}$	0	$(3,1)$	4	–	Table 3.9
$\mathsf{s}_{4,6}$	1	–	–	–	–
$\mathsf{s}_{4,1}$	2	–	–	–	–
$\mathsf{n}_{4,1}$	4	–	–	–	–

Table 3.2: Four-dimensional complex Lie algebras

L	Further specification
$\mathsf{s}_{4,3}(-\frac{1}{2}+i\frac{\sqrt{3}}{2},-\frac{1}{2}-i\frac{\sqrt{3}}{2})$	$7 \in \Psi_L(\mathbb{C})$
$\mathsf{s}_{4,4}(a), a \neq \pm 1$	$7 \notin \Psi_L(\mathbb{C})$; $\Psi_L^{(-1)}(5) = \{a, \frac{1}{a}\}$, $\{a+1, \frac{1}{a}+1\} \cap \Phi_L^{(-1)}(13) - 1 = \{a\}$

Table 3.3: Four-dimensional complex Lie algebras - appendix 1

L	$\mathcal{I}_5(L)$	Further spec.
$\mathsf{s}_{4,3}(1,b), \lvert b\rvert < 1$	$\{b\}$	$\lvert b\rvert < 1$
$\mathsf{s}_{4,3}(1,b), \lvert b\rvert = 1, b \neq \pm 1$	$\{b\}$	$\lvert b\rvert = 1, \arg(b) \in (0, \frac{4}{3}\pi)$
$\mathsf{s}_{4,3}(a,a), \lvert a\rvert < 1$	$\{\frac{1}{a}\}$	$\lvert\frac{1}{a}\rvert > 1$
$\mathsf{s}_{4,3}(a,a), \lvert a\rvert = 1, a \neq 1$	$\{\frac{1}{a}\}$	$\lvert\frac{1}{a}\rvert = 1, \arg(\frac{1}{a}) \in (\frac{4}{3}\pi, 2\pi)$

Table 3.4: Four-dimensional complex Lie algebras - appendix 2

L	$\mathcal{I}_6(L)$	Further specification
$\mathsf{s}_{4,3}(b^2,b)$, $b \notin \{\pm 1, -\frac{1}{2} - i\frac{\sqrt{3}}{2}\}$	$\{b, \frac{1}{b}\}$	$\lvert b\rvert = \lvert\frac{1}{b}\rvert = 1$, $\arg(b) \in (\pi, \frac{4}{3}\pi), \arg(\frac{1}{b}) \in (\frac{2}{3}\pi, \pi)$
$\mathsf{s}_{4,3}(a,a^2), \lvert a\rvert < 1$	$\{a, \frac{1}{a}\}$	$\lvert a\rvert < 1, \lvert\frac{1}{a}\rvert > 1$
$\mathsf{s}_{4,3}(a,a^2), \lvert a\rvert = 1$ $a \notin \{\pm 1, -\frac{1}{2} + i\frac{\sqrt{3}}{2}\}$	$\{a, \frac{1}{a}\}$	$\lvert a\rvert = \lvert\frac{1}{a}\rvert = 1$, $\arg(a) \in (0, \frac{2}{3}\pi), \arg(\frac{1}{a}) \in (\frac{4}{3}\pi, 2\pi)$

Table 3.5: Four-dimensional complex Lie algebras - appendix 3

L	$\mathcal{I}_7(L)$	Further specification
$\mathsf{s}_{4,3}(-1,b)$	$\{b, -b\}$	$\lvert b\rvert = \lvert -b\rvert < 1$, $\arg(b) \in \langle 0, \pi), \arg(-b) \in \langle \pi, 2\pi)$
$\mathsf{s}_{4,3}(a,-1)$, $a \notin \{1, i\}$	$\{a, -a\}$	$\lvert a\rvert = \lvert -a\rvert = 1$, $\arg(a) \in (0, \frac{2}{3}\pi), \arg(-a) \in (\pi, \frac{5}{3}\pi)$
$\mathsf{s}_{4,3}(a,-a)$, $\lvert a\rvert < 1$	$\{\frac{1}{a}, -\frac{1}{a}\}$	$\lvert\frac{1}{a}\rvert = \lvert -\frac{1}{a}\rvert > 1$, $\arg(\frac{1}{a}) \in (\pi, 2\pi) \cup \{0\}, \arg(-\frac{1}{a}) \in (0, \pi\rangle$
$\mathsf{s}_{4,3}(a,-a), a \neq 1$ $\lvert a\rvert = 1, a \neq 1$	$\{\frac{1}{a}, -\frac{1}{a}\}$	$\lvert\frac{1}{a}\rvert = \lvert -\frac{1}{a}\rvert = 1$, $\arg(\frac{1}{a}) \in (\frac{5}{3}\pi, 2\pi), \arg(-\frac{1}{a}) \in (\frac{2}{3}\pi, \pi)$

Table 3.6: Four-dimensional complex Lie algebras - appendix 4

$L = \mathsf{s}_{4,3}(a,b)$	$\mathcal{I}_8(L)$	Further specification
$\lvert b\rvert < \lvert a\rvert < 1, b \neq a^2$	$\{a, b, \frac{b}{a}\}$	$\lvert b\rvert < \min\{\lvert a\rvert, \lvert\frac{b}{a}\rvert\}$; $b + a \in \Phi_L^{(-1)}(13), b + \frac{b}{a} \notin \Phi_L^{(-1)}(13)$
$\lvert b\rvert = \lvert a\rvert < 1, a \neq \pm b$	$\{a, b\}$	$\arg(a) \leq \arg(b)$
$\lvert b\rvert < \lvert a\rvert = 1, a \neq \pm 1$	$\{b, \frac{b}{a}\}$	$\arg(b \cdot \frac{a}{b}) \in (0, \pi), \arg(\frac{1}{b} \cdot \frac{b}{a}) \in (\pi, 2\pi)$
$\lvert b\rvert = \lvert a\rvert = 1, a \notin \{1, b^2\}$ $b \notin \{\pm 1, \pm a, a^2\}$	$\{a, \frac{1}{a}, b, \frac{1}{b}, \frac{a}{b}, \frac{b}{a}\}$	appendix - Table 3.8

Table 3.7: Four-dimensional complex Lie algebras - appendix 5

$L = \mathsf{s}_{4,3}(a,b), \lvert b\rvert = \lvert a\rvert = 1,$ $a \notin \{1, b^2\}, b \notin \{\pm 1, \pm a, a^2\}$	Further specification
$\lvert a\rvert < \lvert\frac{b}{a}\rvert < \min\{\arg(z) \mid z = \frac{1}{a}, b, \frac{1}{b}, \frac{a}{b}\}$	$\frac{1}{a} + \frac{b}{a} \in \Phi_L^{(-1)}(13)$
$\lvert a\rvert < \left\lvert\frac{1}{b}\right\rvert < \min\{\arg(z) \mid z = \frac{1}{a}, b, \frac{a}{b}, \frac{b}{a}\}$	$\frac{1}{a} + \frac{1}{b} \notin \Phi_L^{(-1)}(13)$; $\arg(\frac{1}{b}) \geq \frac{2}{3}\pi$
$\lvert\frac{b}{a}\rvert < \lvert a\rvert < \min\{\arg(z) \mid z = \frac{1}{a}, b, \frac{1}{b}, \frac{a}{b}\}$	$\frac{a}{b} + a \notin \Phi_L^{(-1)}(13)$; $\arg(a) < \frac{2}{3}\pi$

Table 3.8: Four-dimensional complex Lie algebras - appendix 6

L	$\mathcal{I}_9(L)$	$\mathcal{I}_{10}(L)$
$\mathsf{s}_{4,8}(a), \lvert a\rvert = 1, a \neq 1$	$\{a\}$	$-$
$\mathsf{s}_{4,8}(a), \lvert a\rvert < 1, a \neq \frac{1}{2}$	$\emptyset$	$\{a\}$

Table 3.9: Four-dimensional complex Lie algebras - appendix 7

3.2.2 Lie Algebras over $\mathbb{R}$

Dimension one - only one possibility: $\mathfrak{n}_{1,1}$.

Dimension two - only one possibility: $\mathfrak{s}_{2,1}$.

Dimension three - the following invariants are used for identification:

$$
\begin{aligned}
\mathcal{I}_1(L) &:= \dim(L'),\\
\mathcal{I}_2(L) &:= \text{number of singularities of } \Psi_{L_{\mathbb{C}}},\\
\mathcal{I}_3(L) &:= \Psi_{L_{\mathbb{C}}}(1),\\
\mathcal{I}_4(L) &:= \Psi_{L_{\mathbb{C}}}^{(-1)}(4) \cap B_1,\\
\mathcal{I}_5(L) &:= \Psi_{L_{\mathbb{C}}}^{(-1)}(4) \cap \mathbb{C}^+/\{1\},\\
\mathcal{I}_6(L) &:= \text{signature of the Killing form on } L.
\end{aligned}
$$

L	$\mathcal{I}_1(L)$	$\mathcal{I}_2(L)$	$\mathcal{I}_3(L)$	$\mathcal{I}_4(L)$	$\mathcal{I}_5(L)$	$I_6(L)$
$\mathfrak{n}_{3,1}$	1	–	–	–	–	–
$\mathfrak{s}_{3,2}$	2	1	4	–	–	–
$\mathfrak{s}_{3,1}(1)$	2	1	6	–	–	–
$\mathfrak{s}_{3,1}(-1)$	2	2	–	–	–	$(1,0,2)$
$\mathfrak{s}_{3,3}(0)$	2	2	–	–	–	$(2,1,0)$
$\mathfrak{s}_{3,1}(\alpha), \lvert\alpha\rvert \neq 1$	2	3	–	$\{\alpha\}$	–	–
$\mathfrak{s}_{3,3}(\alpha), \alpha \neq 0$	2	3	–	$\emptyset$	$\{\frac{\alpha^2-1+i2\alpha}{\alpha^2+1}\}$	–
$\mathfrak{sl}(2,\mathbb{R})$	3	–	–	–	–	$(2,1,0)$
$\mathfrak{so}(3,\mathbb{R})$	3	–	–	–	–	$(0,3,0)$

Table 3.10: Three-dimensional real Lie algebras

Dimension four - the following invariants are used for identification:

$$
\begin{aligned}
\mathcal{I}_1(L) &:= \dim(Z_3(L)),\\
\mathcal{I}_2(L) &:= \left(\dim(L^{(1)}), \dim(L^{(2)})\right),\\
\mathcal{I}_3(L) &:= \text{number of singularities of } \Psi_{L_{\mathbb{C}}},\\
\mathcal{I}_4(L) &:= \Psi_{L_{\mathbb{C}}}(1),\\
\mathcal{I}_5(L) &:= \text{signature of the Killing form on } \operatorname{Der}(L),\\
\mathcal{I}_6(L) &:= \Phi_{L_{\mathbb{C}}}^{(-1)}(15),\\
\mathcal{I}_7(L) &:= \Psi_{L_{\mathbb{C}}}^{(-1)}(6)/\{1\},\\
\mathcal{I}_8(L) &:= \Phi_{L_{\mathbb{C}}}^{(-1)}(13) + 1,\\
\mathcal{I}_9(L) &:= \Psi_{L_{\mathbb{C}}}^{(-1)}(5) \cap B_1,\\
\mathcal{I}_{10}(L) &:= \Psi_{L_{\mathbb{C}}}^{(-1)}(5)/S_1,
\end{aligned}
$$

$$\mathcal{I}_{11}(L) := \Psi^{(-1)}_{L_{\mathbb{C}}}(4) \cap B_1,$$
$$\mathcal{I}_{12}(L) := \Psi^{(-1)}_{L_{\mathbb{C}}}(4) \cap S_1.$$

L	$\mathcal{I}_1(L)$	$\mathcal{I}_2(L)$	$\mathcal{I}_3(L)$	$\mathcal{I}_4(L)$	$\mathcal{I}_5(L)$	Appendix
$\mathsf{s}_{4,11}$	0	$(2,0)$	–	5	–	–
$\mathsf{s}_{4,12}$	0	$(2,0)$	–	4	–	–
$\mathsf{s}_{4,2}$	0	$(3,0)$	1	6	–	–
$\mathsf{s}_{4,4}(1)$	0	$(3,0)$	1	8	–	–
$\mathsf{s}_{4,3}(1,1)$	0	$(3,0)$	1	12	–	–
$\mathsf{s}_{4,4}(-1)$	0	$(3,0)$	2	5	–	–
$\mathsf{s}_{4,3}(1,-1)$	0	$(3,0)$	2	8	–	–
$\mathsf{s}_{4,4}(\alpha), \alpha \neq \pm 1$	0	$(3,0)$	3	6	–	Table 3.12
$\mathsf{s}_{4,5}(\frac{2}{\sqrt{3}}, -\frac{1}{\sqrt{3}})$	0	$(3,0)$	3	6	–	
$\mathsf{s}_{4,3}(1,\beta), \beta \neq \pm 1$	0	$(3,0)$	3	8	–	Table 3.13
$\mathsf{s}_{4,3}(\alpha,\alpha), \alpha \neq 1$	0	$(3,0)$	3	8	–	
$\mathsf{s}_{4,5}(1,0)$	0	$(3,0)$	4	–	–	–
$\mathsf{s}_{4,3}(\alpha,\alpha^2), \alpha \neq \pm 1$	0	$(3,0)$	5	–	–	Table 3.14
$\mathsf{s}_{4,5}(\sqrt{1+\beta^2},\beta), \beta \neq -\frac{1}{\sqrt{3}}$	0	$(3,0)$	5	–	–	
$\mathsf{s}_{4,3}(-1,\beta)$	0	$(3,0)$	6	–	–	Table 3.15
$\mathsf{s}_{4,3}(\alpha,-\alpha), \alpha \neq 1$	0	$(3,0)$	6	–	–	
$\mathsf{s}_{4,5}(\alpha,0), \alpha \neq 1$	0	$(3,0)$	6	–	–	
$\mathsf{s}_{4,3}(\alpha,\beta), \alpha \notin \{\pm 1, \pm\beta\}, \beta \notin \{\pm 1, \alpha^2\}$	0	$(3,0)$	7	–	$(3,0,3)$	Table 3.16
$\mathsf{s}_{4,5}(\alpha,\beta), \alpha \neq \sqrt{1+\beta^2}, \beta \neq 0$	0	$(3,0)$	7	–	$(2,1,3)$	Table 3.17
$\mathsf{s}_{4,10}$	0	$(3,1)$	2	5	–	–
$\mathsf{s}_{4,8}(1)$	0	$(3,1)$	2	7	–	–
$\mathsf{s}_{4,8}(\frac{1}{2})$	0	$(3,1)$	3	–	–	–
$\mathsf{s}_{4,8}(\alpha), \alpha \notin \{\frac{1}{2}, 1\}$	0	$(3,1)$	4	–	$(2,0,3)$	Table 3.18
$\mathsf{s}_{4,9}(\alpha)$	0	$(3,1)$	4	–	$(1,1,3)$	Table 3.19
$\mathsf{s}_{4,6}$	1	–	–	–	$(2,0,3)$	–
$\mathsf{s}_{4,7}$	1	–	–	–	$(1,1,3)$	–
$\mathsf{s}_{4,1}$	2	–	–	–	–	–
$\mathsf{n}_{4,1}$	4	–	–	–	–	–

Table 3.11: Four-dimensional real Lie algebras

L	Further specification
$\mathfrak{s}_{4,4}(\alpha), \alpha \neq \pm 1$	$7 \notin \Psi_{L_{\mathbb{C}}}(\mathbb{C})$; $\Psi_{L_{\mathbb{C}}}^{(-1)}(5) = \{\alpha, \frac{1}{\alpha}\}$, $\{\alpha + 1, \frac{1}{\alpha} + 1\} \cap \Phi_{L_{\mathbb{C}}}^{(-1)}(13) - 1 = \{\alpha\}$
$\mathfrak{s}_{4,5}(\frac{2}{\sqrt{3}}, -\frac{1}{\sqrt{3}})$	$7 \in \Psi_{L_{\mathbb{C}}}(\mathbb{C})$

Table 3.12: Four-dimensional real Lie algebras - appendix 1

L	$\mathcal{I}_6(L)$	Further spec.
$\mathfrak{s}_{4,3}(1, \beta), \beta \neq \pm 1$	$\{\beta\}$	$\lvert\beta\rvert < 1$
$\mathfrak{s}_{4,3}(\alpha, \alpha), \alpha \neq 1$	$\{\frac{1}{\alpha}\}$	$\lvert\frac{1}{\alpha}\rvert > 1$

Table 3.13: Four-dimensional real Lie algebras - appendix 2

L	$\mathcal{I}_7(L)$	Further specification
$\mathfrak{s}_{4,3}(\alpha, \alpha^2), \alpha \neq \pm 1$	$\{\alpha, \frac{1}{\alpha}\}$	$\lvert\alpha\rvert < 1, \lvert\frac{1}{\alpha}\rvert > 1$
$\mathfrak{s}_{4,5}(\sqrt{1+\beta^2}, \beta), \beta \neq -\frac{1}{\sqrt{3}}$	$\{\frac{\beta+i}{\sqrt{\beta^2+1}}, \frac{\beta-i}{\sqrt{\beta^2+1}}\}$	$\lvert\frac{\beta+i}{\sqrt{\beta^2+1}}\rvert = \lvert\frac{\beta-i}{\sqrt{\beta^2+1}}\rvert = 1$

Table 3.14: Four-dimensional real Lie algebras - appendix 3

L	$\mathcal{I}_8(L)$	Further specification
$\mathfrak{s}_{4,3}(-1, \beta)$	$\{\beta, -\beta\}$	$\pm\beta \in \mathbb{R}; \lvert\beta\rvert = \lvert-\beta\rvert < 1; \beta > 0, -\beta < 0$
$\mathfrak{s}_{4,3}(\alpha, -\alpha), \alpha \neq 1$	$\{\frac{1}{\alpha}, -\frac{1}{\alpha}\}$	$\pm\frac{1}{\alpha} \in \mathbb{R}; \lvert\frac{1}{\alpha}\rvert = \lvert-\frac{1}{\alpha}\rvert > 1; \frac{1}{\alpha} > 0, -\frac{1}{\alpha} < 0$
$\mathfrak{s}_{4,5}(\alpha, 0), \alpha \neq 1$	$\{i\alpha, -i\alpha\}$	$\pm i\alpha \notin \mathbb{R}; \mathrm{Im}(i\alpha) > 0, \mathrm{Im}(-i\alpha) < 0$

Table 3.15: Four-dimensional real Lie algebras - appendix 4

L	$\mathcal{I}_9(L)$	Further specification
$\mathfrak{s}_{4,3}(\alpha, \beta), \alpha \notin \{\pm 1, \pm\beta\}$, $\beta \notin \{\pm 1, \alpha^2\}$	$\{\alpha, \beta, \frac{\beta}{\alpha}\}$	$\lvert\beta\rvert < \min\{\lvert\alpha\rvert, \lvert\frac{\beta}{\alpha}\rvert\}$; $\beta + \alpha \in \Phi_{L_{\mathbb{C}}}^{(-1)}(13), \beta + \frac{\beta}{\alpha} \notin \Phi_{L_{\mathbb{C}}}^{(-1)}(13)$

Table 3.16: Four-dimensional real Lie algebras - appendix 5

L	$\mathcal{I}_{10}(L)$	Further specification
$\mathfrak{s}_{4,5}(\alpha, \beta)$, $\alpha \neq \sqrt{1+\beta^2}$, $\beta \neq 0$	$\left\{\frac{\beta+i}{\alpha}, \frac{\beta-i}{\alpha}, \frac{(\beta+i)\alpha}{\beta^2+1}, \frac{(\beta-i)\alpha}{\beta^2+1}\right\}$	$\overline{\frac{\beta+i}{\alpha}} = \frac{\beta-i}{\alpha}, \overline{\frac{(\beta+i)\alpha}{\beta^2+1}} = \frac{(\beta-i)\alpha}{\beta^2+1}$; $\frac{\beta+i}{\alpha} + \frac{\beta-i}{\alpha} \in \Phi_{L_{\mathbb{C}}}^{(-1)}(13)$, $\frac{(\beta+i)\alpha}{\beta^2+1} + \frac{(\beta-i)\alpha}{\beta^2+1} \notin \Phi_{L_{\mathbb{C}}}^{(-1)}(13)$

Table 3.17: Four-dimensional real Lie algebras - appendix 6

L	$\mathcal{I}_{11}(L)$
$\mathfrak{s}_{4,8}(\alpha), \alpha \notin \{\frac{1}{2}, 1\}$	$\{\alpha\}$

Table 3.18: Four-dimensional real Lie algebras - appendix 7

L	$\mathcal{I}_{12}(L)$	Further specification
$\mathfrak{s}_{4,9}(\alpha)$	$\{\frac{\alpha^2-1+i2\alpha}{\alpha^2+1}, \frac{\alpha^2-1-i2\alpha}{\alpha^2+1}\}$	$\mathrm{Re}(\frac{\alpha^2-1+i2\alpha}{\alpha^2+1}) = \mathrm{Re}(\frac{\alpha^2-1-i2\alpha}{\alpha^2+1}) = \frac{\alpha^2-1}{\alpha^2+1}$

Table 3.19: Four-dimensional real Lie algebras - appendix 8

3.3 LIEIDENTIFICATOR

As we have already found and computed the complete sets of invariants for all items in the list of indecomposable Lie algebras up to dimension four, we are now able to identify an arbitrary such a Lie algebra by computing the same invariants on it and by comparing with the values of the invariants computed on the items of the list. Naturally, we may use the same algorithms as for computing invariants for Lie algebras in the list, i.e. the ones described at the beginning of Section 3.2.

To automatize this process, we created a MAPLE procedure where we used all discussed algorithms and we considered all eventualities that may occur. This resulted into a programme identifying any indecomposable Lie algebra of dimension at most four. Besides, we made use of procedure Decompose, contained in subpackage LieAlgebras for decomposing a Lie algebra into the direct sum of its indecomposable ideals. Furthermore, a controlling php script for our MAPLE algorithm was written in order to put the procedure on-line.

All in all, the outcome of our work is a simple internet application, which we named LIEIDENTIFICATOR, that automatically identifies any finite-dimensional complex or real Lie algebra that is directly composed of at most four-dimensional indecomposable ideals. LIEIDENTIFICATOR is available on-line at

`http://kmlinux.fjfi.cvut.cz/~kotrbja2/LieIdentificator/`.

Conclusion

In our work we have focused on the automatic identification of Lie algebras over $\mathbb{R}$ or over $\mathbb{C}$, having the dimension four or less. For this purpose, we have made use of invariants.

First, we have introduced the basics of Lie algebras theory needed for establishing various invariants. Second, according to [13], we have listed all indecomposable Lie algebras up to dimension four. Finally, using methods described in [13], we have found complete sets of invariants for all Lie algebras in the list and we have used computed invariants for identification of any other such a Lie algebra among those in the list.

The classification process have been automatized by implementing in MAPLE 18 computer algebra system. Furthermore, this implementation have been put on-line as a simple internet application LIEIDENTIFICATOR, that automatically identify an arbitrary complex or real Lie algebra directly composed of indecomposable Lie algebras of dimensions at most four.

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