Czech Technical University in Prague
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Diploma Thesis

Construction of Representations of Lie Algebras and Lie Fields

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Prohlášení

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Abstract

Irreducible unitary representations of the Poincaré group $\mathcal{P}_4$ were classified in 1939 by E. P. Wigner. His result was later on broadened to much wider class of Lie groups by G. W. Mackey.

In the present thesis an alternative method for construction of irreducible unitary representations is suggested and illustrated on the Poincaré groups $\mathcal{P}_2$, $\mathcal{P}_3$ and $\mathcal{P}_4$. Our technique is motivated by the famous Gelfand-Kirillov conjecture, namely we make use of the relationship between the fields of fractions corresponding to Weyl algebras and universal enveloping algebras, respectively. Connection to Mackey theory is also discussed in each case in order to show that both methods lead to the same results.

Keywords: Lie field, Poincaré group, unitary representation, Gelfand-Kirillov conjecture

Abstrakt

Ireducibilní unitární reprezentace Poincarého grupy $\mathcal{P}_4$ byly klasifikovány již v roce 1939 ve slavné práci E. P. Wignera. Wignerova metoda bylo o několik let později zobecněna pro širokou třídu Lieových grup zásluhou G. W. Mackeyho.

V předkládané diplomové práci je představen postup konstrukce ireducibilních unitárních reprezentací, který je alternativou k výše uvedené metodě. Naše práce byla motivována takzvanou Gelfand-Kirillovovu domněnkou, jež dává do souvislosti tělesa obalových algeber s tělesy vhodných rozšíření algeber Weylových. S využitím této korespondence jsme schopni sestrojit kompletní množinu ireducibilních unitárních reprezentací pro Poincarého grupy $\mathcal{P}_2$, $\mathcal{P}_3$ a $\mathcal{P}_4$. Naše výsledky jsou v souladu s Mackeyho teorií, jak se ukazuje přímou konfrontací obou možných postupů.

Klíčová slova: Lieovské těleso, Poincarého grupa, unitární reprezentace,
Gelfand-Kirillovova domněnka

Contents

List of Notations

$\mathbb{1}$	identity operator
$\operatorname{Aut} V$	group of automorphisms of a vector space V
$\mathcal{B}(\mathcal{H})$	associative algebra of bounded linear operators on a Hilbert space $\mathcal{H}$
$\mathbb{C}$	field of complex numbers
$C_0^\infty(\mathcal{A})$	vector space of smooth, compactly supported functions $f : \mathcal{A} \subset \mathbb{R}^n \to \mathbb{C}$
char	characteristic of a field
det	determinant of a matrix
$\operatorname{diag}(\alpha_1 \ldots \alpha_n)$	diagonal $n \times n$ matrix with entries $\alpha_1, \ldots, \alpha_n$
dim	dimension of a vector space
$\operatorname{Dom} f$	domain of a map f
$\operatorname{End} V$	group of endomorphisms of a vector space V
$f^{(-1)}$	inverse image under a map f
f^{-1}	inverse mapping to a map f
$f\|_S$	restriction of a map f to a subset $S \subset \operatorname{Dom} f$
$\mathbb{F}[S]$	ring of polynomials over a set S with coefficients from a field $\mathbb{F}$
$\mathfrak{gl}(n, \mathbb{F})$	Lie algebra of $n \times n$ matrices over a field $\mathbb{F}$
$\mathrm{GL}(n, \mathbb{F})$	Lie group of invertible $n \times n$ matrices over a field $\mathbb{F}$
i	imaginary unit
$K \backslash S$	set of right cosets of a group K by a subgroup $S \subset K$
$L^2(\mathcal{A}, \mathrm{d}\mu; \mathcal{H})$	vector space of measurable functions $f : \mathcal{A} \to \mathcal{H}$, where $\mathcal{H}$ is a Hilbert space, with $\int_{\mathcal{A}} \|f\|_{\mathcal{H}}^2 \, \mathrm{d}\mu < +\infty$; in particular $L^2(\mathcal{A}, \mathrm{d}\mu; \mathbb{C}) \equiv L^2(\mathcal{A}, \mathrm{d}\mu)$
$\mathcal{L}(\mathcal{H})$	set of densely defined linear operators on a Hilbert space $\mathcal{H}$
max	the maximal element of a set
$\mathbb{N}$	set of natural numbers (without zero)
$\mathbb{N}_0$	set of natural numbers with zero
$\mathbb{R}$	field of real numbers
$\mathbb{R}^+$	set of positive real numbers
$\mathbb{R}^-$	set of negative real numbers
$R^\times$	set of non-zero elements of a ring R; in particular $\mathbb{R}^\times \equiv \mathbb{R}^+ \cup \mathbb{R}^-$
$\mathbb{R}^n$	vector space of n-tuples of real numbers
rank_R	rank of a matrix over a ring R
$\mathfrak{sl}(n, \mathbb{F})$	Lie algebra of $n \times n$ matrices with zero trace over a field $\mathbb{F}$
$\operatorname{Span}_{\mathbb{F}}$	linear span over a field $\mathbb{F}$
supp	support of a function

T^*	adjoint operator to $T \in \mathcal{L}(\mathcal{H})$; also an image under an involution
$\mathcal{U}(\mathcal{H})$	set of unitary linear operators on a Hilbert space $\mathcal{H}$
δ_{jk}	Kronecker delta
σ	spectrum of an operator
Λ^T	transpose of a matrix Λ
$[\ ,\ ]$	Lie bracket in a Lie algebra; commutator in an associative algebra
$\cdot$	group or scalar multiplication; the dot itself is often omitted
$\bullet$	inner product
$\circ$	composition of mappings
$\times$	Cartesian product
$\ltimes$	semidirect product of groups
$\oplus$	direct sum of vector spaces
$\otimes$	tensor product; $V^{\otimes n} \equiv V \otimes \cdots \otimes V$ (n times)
$\emptyset$	empty set

By *Hilbert space* we always mean a separable complex Hilbert space.
Further, any *representation* is always assumed to be faithful, if not otherwise stated.

Introduction

Irreducible unitary representations of the ten-dimensional Poincaré group (sometimes also referred to as the inhomogeneous Lorentz group) are of fundamental importance in relativistic quantum mechanics and consequently in quantum field theory. Evidence of this fact that has been well-known since the beginning of the quantum theory is outlined by the following simple observation.

First, the probability of transition between two quantum states must be invariant of the choice of the Lorentz frame of reference. Thus, suppose φ, ψ describe two states with respect to a Lorentz frame l and φ', ψ' describe the same states in another Lorentz frame l' that was obtained from l by transformation corresponding to an element g of the Poincaré group $\mathcal{P}_4$, i.e. the group of symmetries of the four dimensional Minkowski spacetime. Then by Wigner theorem $\varphi' = U(g)\varphi$ and $\psi' = U(g)\psi$, where $U(g)$ is a unitary operator on the Hilbert space of wave functions (cf. [44]).[1]

Second, the wave functions $U(g_1)U(g_2)\varphi$ and $U(g_1g_2)\varphi$ must obviously describe the same (normalized) state, i.e. $U(g_1)U(g_2)\varphi = \alpha\, U(g_1g_2)\varphi$ for some $\alpha \in \mathbb{C}$, $|\alpha| = 1$. It could be shown that such α is independent of the function φ and furthermore, that we may without loss of generality assume $\alpha = \pm 1$ (cf. [45], §5). Therefore $g \mapsto U(g)$ is in fact the so-called *two-valued* (or *projective*) unitary representation of the group $\mathcal{P}_4$.[2]. Under certain circumstances it may be further assumed $\alpha = 1$ and thus U is a unitary representation of $\mathcal{P}_4$.[3]

It was proven by Eugene P. Wigner that all such representations, in spite of being infinite-dimensional, are completely reducible, i.e. it is sufficient to consider entirely irreducible unitary representations of $\mathcal{P}_4$. He himself classified these representations in his famous paper [45].

Wigner's method was later generalized by George W. Mackey in [28] and [29] into much broader concept of the so-called *induced unitary representations* of Lie groups. In particular, Mackey theory also applies to the lower-dimensional analogues $\mathcal{P}_2$ and $\mathcal{P}_3$ of the Poincaré group. These are the groups of symmetries of the Minkowski spacetime in two and three dimensions, respectively.

In the present thesis we suggest an alternative method for construction of irreducible unitary representations. Use of the method is illustrated on the Poincaré groups $\mathcal{P}_2$, $\mathcal{P}_3$ and $\mathcal{P}_4$, but in principle it may be applied to much wider class of Lie groups.

Our technique is motivated by the famous *Gelfand-Kirillov conjecture* discussing the relationship between certain class of Lie algebras and suitably extended Weyl algebras. According to the conjecture, the elementary structure of universal enveloping algebras is closer to Weyl algebras then it may seem at first sight. To discover this consanguinity, one has to, however, go beyond the borders of associative algebras, to the so-called *fields of fractions* where "division" is taken into consideration. It turns out that under certain circumstances, the relationship, despite taking place on level of the respective fields of

[1] Notice that $U(g)$ could be also anti-unitary but let us for simplicity assume it is linear, hence unitary.
[2] Strictly speaking, $U(g)$ is *not* a representations in terms of contemporary conventions unless $\alpha \equiv 1$.
[3] Such an assumption corresponds to restricting ourselves to the states of "integer spin" (cf. [43]), §2.7).

fractions, can be used for inducing representations of Lie algebras and of Lie groups, consequently.

Briefly speaking, we proceed as follows. First, given a Lie group G with Lie algebra $\mathfrak{g}$, we consider the fields of fractions $\mathfrak{D}$ and $\mathfrak{D}(\mathfrak{g})$ corresponding to one of the extended Weyl algebras and to the enveloping algebra of $\mathfrak{g}$, respectively, and we find an involution-preserving isomorphism $\Psi : \mathfrak{D}(\mathfrak{g}) \to \mathfrak{D}' \subset \mathfrak{D}$. Then the composition of Ψ with a convenient representation of $\mathfrak{D}'$, induced by a representation of the underlying (extended) Weyl algebra, produces a skew-symmetric representation of the Lie algebra $\mathfrak{g} \subset \mathfrak{D}(\mathfrak{g})$. Second, the representation is "integrated" into a unitary representation of the Lie group G. This is done in several succesive steps, namely unitary one-parameter subgroups are constructed and then it is shown that their products form, or generate a representation of G. Finally, irreducibility and mutual non-equivalence of the constructed unitary representations are inspected.

The thesis itself is organized as follows.

In the first chapter necessary theoretical preliminaries are recalled. First, definitions and basic properties of fields of fractions, universal enveloping algebras and Weyl algebras are introduced. At that stage, the Gelfand-Kirillov conjecture is also investigated in some detail and the relationship between representations of Lie algebras and Lie groups is discussed. Second, the notion of Poincaré Lie groups and their Lie algebras is established. Finally, Mackey theory of induced representations is briefly explained and a bit more attention is paid to its application on the Poincaré groups.

In the second chapter the suggested technique of construction of unitary irreducible representations is demonstrated on the three-dimensional Poincaré group $\mathcal{P}_2$. This consists of finding an isomorphism between the field of fraction of the Lie algebra $\mathfrak{p}_2$ and field of fraction of an extended Weyl algebra, using the isomorphism for inducing skew-symmetric representations of $\mathfrak{p}_2$ and finally, of integrating these representations into irreducible unitary representations of $\mathcal{P}_2$. Afterwards, the complete family of such representations is constructed due to Mackey theory and then both methods are compared and proved to lead to the same results.

The third chapter is in fact a repetition of the second chapter for the six-dimensional Poincaré group $\mathcal{P}_3$. Although the discussion is more complicated and hence so are the involved computations, also in this case we are able to construct all irreducible unitary representations of the Lie group explicitly and to prove that our technique is completely equivalent to Mackey's approach.

Finally, the fourth chapter is devoted to application of our method to the Poincaré group $\mathcal{P}_4$. In contrary to the preceding chapters, the discussion on this case is not completely rigorous and does not go so much into detail. Neither explicit forms of the constructed representations are stated explicitly. This is, however, due to the heterogeneous structure of the set of representations in this case. Nevertheless, it is manifest that our method can again substitute Mackey theory and the complete set of irreducible unitary representations of the Lie group $\mathcal{P}_4$ can be independently constructed in the suggested way.

Chapter 1

Preliminaries

1.1 Lie Fields

1.1.1 Fields of Fractions

At the very beginning, we shall introduce the so-called *fields of fractions*. They can be associated to rings that fulfil certain additional conditions (cf. [12], [15], [27]).

Given a *ring*, i.e. a non-empty set R equipped with two binary operations, *addition* $+$ and *multiplication* $\cdot$, and containing *zero* $0 \in R$ and *unit* $1 \in R$ such that

(a) $(R, +, 0)$ is an abelian group,

(b) $(R, \cdot, 1)$ is a monoid, and

(c) $a \cdot (b + c) = a \cdot b + a \cdot c$ and $(a + b) \cdot c = a \cdot c + b \cdot c$ for any $a, b, c \in R$,

the following question arises. Under which circumstances, the ring can be extended into a "bigger" structure, where "division" is allowed? In other words, one would like the ring to be embedded in a *skew field*.

Remark 1.1. Note at this stage, that a *skew field* is defined to be a non-trivial ring (i.e. where $0 \neq 1$) in which each non-zero element has its inverse (with respect to multiplication). A skew field with commutative multiplication is then called a *field*. Although the notation is used in most of the standard literature (cf. e.g. [12], [16] or [27]), it could be easily misleading; one has to bear in mind that, in general, a skew field is *not* a field. Further, several examples of skew fields, usually called — fields (without "skew") but not being commutative in general, will be described.

Let R be a non-trivial ring. To go back to the question put above, observe at first the following fact. In a skew field K, uniqueness of the inverse implies that $(ab)^{-1}$, where $a, b \in K^{\times}$, must equal to $b^{-1}a^{-1}$. Therefore, if $ab = 0$ for some $a, b \in R^{\times}$, R could not be embedded in a field K since the product $b^{-1}a^{-1}$ would not be well-defined: regardless $b^{-1}a^{-1}$ was non-zero or not, the following contradiction would be reached:

$$0 \neq 1 = b^{-1}b = b^{-1}1b = b^{-1}a^{-1}ab = b^{-1}a^{-1}0 = 0.$$

To conclude, one has to insist on $ab \neq 0$ for any $a, b \in R^{\times}$. Fulfilling such a condition, the ring is called an *integral domain*. We shall see below that in case of commutative rings, this necessary condition is also sufficient.

Thus, assume R to be a (non-commutative, in general) integral domain. We define the following relation on $R^{\times} \times R$:

$$(a, b) \sim (a', b') \text{ if there exist } t, s \in R \text{ such that } tb = sb', ta = sa' \text{ and } sa' \in R^{\times}. \tag{1.1}$$

Notice that the last condition implies t and s must be from $R^{\times}$ in fact.

Let us discuss when $\sim$ can be an equivalence relation. First, one takes $t = s = 1$ to show reflexivity. Second, symmetry is proven by just interchanging s and t. Finally, to prove transitivity, suppose $(a, b) \sim (a', b')$ and $(a', b') \sim (a'', b'')$, i.e. there are $s, t, u, v \in R^\times$ such that $ta = sa'$, $tb = sb'$, $ua' = va''$ and $ub' = vb''$. We need to "connect" a, b with a'', b'', respectively, or to eliminate a' and b', in other words. Hence we need existence $x, y \in R$ such that

$$xsb' = yub', \quad xsa' = yua' \quad \text{and} \quad yua' \in R^\times.$$

From this reason the following additional condition is imposed on R:

$$Rz \cap R^\times w \neq \emptyset \quad \text{for any } z \in R^\times, w \in R. \tag{1.2}$$

This is a special case of the so-called *left Ore condition* on multiplicative subsets (i.e those containing 1 and being closed under multiplication) of a ring (cf. [13], p. 351). Taking the Ore condition and the fact that $s, u \in R^\times$ into account, there must be $x, y \in R^\times$ such that $xs = yu \in R^\times$. Therefore,

$$(xt)b = (yv)b'', \quad (xt)a = (yv)a'' \quad \text{and} \quad (yv)a'' \in R^\times$$

are the desired relations meaning $(a, b) \sim (a'', b'')$.

Remark 1.2. Notice that $(a, b) \sim (ta, tb)$ for any $a, t \in R^\times$ and $b \in R$.

Proposition 1.1. *If $(a, b) \sim (a', b')$ and $ta = sa'$ for some $t, s \in R^\times$, then $tb = sb'$.*

Proof. There are $t', s' \in R^\times$ with $t'b = s'b'$ and $t'a = s'a' \in R^\times$. Then there also exist $x, y \in R^\times$ satisfying $xt' = yt$. For them we have $ysa' = yta = xt'a = xs'a'$, implying $(ys - xs')a' = 0$ and consequently $ys = xs'$ since R is an integral domain. Therefore $ytb = xt'b = xs'b' = ysb'$ and finally $tb = sb'$. $\square$

Let $\frac{b}{a}$ denote the class of equivalence containing (a, b) and let $\mathfrak{D}(R)$ be the set of such classes. $\mathfrak{D}(R)$ can be equipped with addition and multiplication defined by

$$\frac{b_1}{a_1} + \frac{b_2}{a_2} := \frac{s_1 b_1 + s_2 b_2}{s_1 a_1}, \tag{1.3}$$

$$\frac{b_1}{a_1} \cdot \frac{b_2}{a_2} := \frac{t_2 b_2}{t_1 a_1}, \tag{1.4}$$

where $s_1, s_2, t_1 \in R^\times$ and $t_2 \in R$ obey relations $s_1 a_1 = s_2 a_2$ and $t_1 b_1 = t_2 a_2$. Existence of such elements clearly follows from (1.2) in each case except $b_1 = b_2 = 0$. But then t_1, t_2 could be chosen even arbitrarily. Nevertheless, one has to verify that the operations are independent of particular choice of $s_1, s_2, t_2 \in R^\times$ and $t_1 \in R$ as well as of representatives of equivalence classes.

Thus, take $\frac{b_j}{a_j} = \frac{b'_j}{a'_j} \in \mathfrak{D}(R)$, $j = 1, 2$. We have $s_1 a_1 = s_2 a_2$, $s'_1 a'_1 = s'_2 a'_2$, $t_1 b_1 = t_2 a_2$ and $t'_1 b'_1 = t'_2 a'_2$ for appropriate $s_1, s_2, s'_1, s'_2, t_1, t'_1 \in R^\times$ and $t_2, t'_2 \in R$. First, there are $x, x' \in R^\times$ such that $xs_1 a_1 = x's'_1 a'_1$. Then $xs_2 a_2 = x's'_2 a'_2$ and also, according to Proposition 1.1, $xs_1 b_1 = x's'_1 b'_1$ and $xs_2 b_2 = x's'_2 b'_2$. Therefore

$$\frac{b_1}{a_1} + \frac{b_2}{a_2} = \frac{s_1 b_1 + s_2 b_2}{s_1 a_1} = \frac{xs_1 b_1 + xs_2 b_2}{xs_1 a_1} = \frac{x's'_1 b'_1 + x's'_2 b'_2}{x's'_1 a'_1} = \frac{s'_1 b'_1 + s'_2 b'_2}{s'_1 a'_1} = \frac{b'_1}{a'_1} + \frac{b'_2}{a'_2}.$$

Second, there exist $y, y' \in R^\times$ with $yt_1 a_1 = y't'_1 a'_1$. Then $yt_2 a_2 = yt_1 b_1 = y't'_1 b'_1 = y't'_2 a'_2$, hence $yt_2 b_2 = y't'_2 b'_2$ and similarly

$$\frac{b_1}{a_1} \cdot \frac{b_2}{a_2} = \frac{t_2 b_2}{t_1 a_1} = \frac{yt_2 b_2}{yt_1 a_1} = \frac{y't'_2 b'_2}{y't'_1 a'_1} = \frac{b'_1}{a'_1} \cdot \frac{b'_2}{a'_2}.$$

Lemma 1.2. *The set $\mathfrak{D}(R)$ equipped with addition and multiplication defined by* (1.3) *and* (1.4)*, respectively, forms a skew field.*

Proof. First observe that (1.3) is symmetric in $1 \leftrightarrow 2$, hence the addition is commutative. Second, putting $b_2 = 0$ in (1.3), one has

$$\frac{b_1}{a_1} + \frac{0}{a_2} = \frac{s_1 b_1}{s_1 a_1} = \frac{b_1}{a_1},$$

hence $\frac{0}{1}$ plays the role of *zero* in $\mathfrak{D}(R)$. Similarly, for $a_2 = b_2 = 1$, (1.4) takes form

$$\frac{b_1}{a_1} \cdot \frac{1}{1} = \frac{t_2 a_2}{t_1 a_1} = \frac{t_1 b_1}{t_1 a_1} = \frac{b_1}{a_1},$$

thus $\frac{1}{1} \in \mathfrak{D}(R)$ is the *unit*. Concerning existence of opposite and inverse elements,

$$\frac{b_1}{a_1} + \frac{-b_1}{a_1} = \frac{s_1 b_1 - s_2 b_1}{s_1 a_1} = \frac{s_1 b_1 - s_1 b_1}{s_1 a_1} = \frac{0}{1},$$

because s_1 can be obviously taken equal to s_2 in this case, and

$$\frac{b_1}{a_1} \cdot \frac{a_1}{b_1} = \frac{t_2 a_1}{t_1 a_1} = \frac{t_1 a_1}{t_1 a_1} = \frac{1}{1},$$

provided $b_1 \neq 0$, or equivalently $\frac{b_1}{a_1} \neq \frac{0}{1}$, whence one can take $t_1 = t_2$.

Further, to show associativity of addition, we have

$$\left(\frac{b_1}{a_1} + \frac{b_2}{a_2}\right) + \frac{b_3}{a_3} = \frac{s_1 b_1 + s_2 b_2}{s_1 a_1} = \frac{s'_1 s_1 b_1 + s'_1 s_2 b_2 + s'_2 b_3}{s'_1 s_1 a_1} \tag{1.5}$$

for $s_1 a_1 = s_2 a_2$ and $s'_1 s_1 a_1 = s'_2 a_3$, and

$$\frac{b_1}{a_1} + \left(\frac{b_2}{a_2} + \frac{b_3}{a_3}\right) = \frac{\tilde{s}_1 b_2 + \tilde{s}_2 b_3}{\tilde{s}_1 a_2} = \frac{\tilde{s}'_1 b_1 + \tilde{s}'_2 \tilde{s}_1 b_2 + \tilde{s}'_2 \tilde{s}_2 b_3}{\tilde{s}'_1 a_1} \tag{1.6}$$

for $\tilde{s}_1 a_2 = \tilde{s}_2 a_3$ and $\tilde{s}'_1 a_1 = \tilde{s}'_2 \tilde{s}_1 a_2$. As before, there are $x, y \in R^\times$ with $x s'_1 s_1 = y \tilde{s}'_1$. But then $x s'_1 s_2 a_2 = x s'_1 s_1 a_1 = y \tilde{s}'_1 a_1 = y \tilde{s}'_2 \tilde{s}_1 a_2$ and hence $x s'_1 s_2 = y \tilde{s}'_2 \tilde{s}_1$. Analogously, $x s'_2 a_3 = x s'_1 s_1 a_1 = y \tilde{s}'_1 a_1 = y \tilde{s}'_2 \tilde{s}_1 a_2 = y \tilde{s}'_2 \tilde{s}_2 a_3$ and thus $x s'_2 = y \tilde{s}'_2 \tilde{s}_2$. Now it is necessary that (1.5) equals (1.6). Associativity of multiplication is analogous to prove:

$$\left(\frac{b_1}{a_1} \cdot \frac{b_2}{a_2}\right) \cdot \frac{b_3}{a_3} = \frac{t_2 b_2}{t_1 a_1} \cdot \frac{b_3}{a_3} = \frac{t'_2 b_3}{t'_1 t_1 a_1},$$

where $t_1 b_1 = t_2 a_2$ and $t'_1 t_2 b_2 = t'_2 a_3$, equals to

$$\frac{b_1}{a_1} \cdot \left(\frac{b_2}{a_2} \cdot \frac{b_3}{a_3}\right) = \frac{b_1}{a_1} \cdot \frac{\tilde{t}_2 b_3}{\tilde{t}_1 a_2} = \frac{\tilde{t}'_2 \tilde{t}_2 b_3}{\tilde{t}'_1 a_1},$$

where $\tilde{t}_1 b_2 = \tilde{t}_2 a_3$ and $\tilde{t}'_1 b_1 = \tilde{t}'_2 \tilde{t}_1 a_2$. This is because there are $x, y \in R^\times$ with $x t'_1 t_1 = y \tilde{t}'_1$, $x t'_1 t_2 a_2 = x t'_1 t_1 b_1 = y \tilde{t}'_1 b_1 = y \tilde{t}'_2 \tilde{t}_1 a_2$ and $x t'_1 t_2 = y \tilde{t}'_2 \tilde{t}_1$. Consequently, $x t'_2 a_3 = x t'_1 t_2 b_2 = y \tilde{t}'_2 \tilde{t}_1 b_2 = y \tilde{t}'_2 \tilde{t}_2 a_3$ and finally, $x t'_2 = y \tilde{t}'_2 \tilde{t}_2$.

It only remains to verify distributivity. First, concerning left distributivity,

$$\frac{b_1}{a_1} \cdot \left(\frac{b_2}{a_2} + \frac{b_3}{a_3}\right) = \frac{b_1}{a_1} \cdot \frac{s_1 b_2 + s_2 b_3}{s_1 a_2} = \frac{t_2 s_1 b_2 + t_2 s_2 b_3}{t_1 a_1},$$

with $s_1 a_2 = s_2 a_3$ and $t_1 b_1 = t_2 s_1 a_2$, equals

$$\frac{b_1}{a_1} \cdot \frac{b_2}{a_2} + \frac{b_1}{a_1} \cdot \frac{b_3}{a_3} = \frac{t'_2 b_2}{t'_1 a_1} + \frac{\tilde{t}_2 b_3}{\tilde{t}_1 a_1} = \frac{s'_1 t'_2 b_2 + s'_2 \tilde{t}_2 b_3}{s'_1 t'_1 a_1}$$

$t'_1 b_1 = t'_2 a_2$, $\tilde{t}_1 b_1 = \tilde{t}_2 a_3$, $s'_1 t'_1 a_1 = s'_2 \tilde{t}_1 a_1$, since the last relation implies $s'_1 t'_1 = s'_2 \tilde{t}_1$, and there are $x, y \in R^\times$ with $x t_1 = y s'_1 t'_1$; then $x t_2 s_1 a_2 = x t_1 b_1 = y s'_1 t'_1 b_1 = y s'_1 t'_2 a_2$ forces $x t_2 s_1 = y s'_1 t'_2$ and similarly $x t_2 s_2 a_3 = x t_2 s_1 a_2 = y s'_1 t'_2 a_2 = y s'_1 t'_1 b_1 = y s'_2 \tilde{t}_1 b_1 = y s'_2 \tilde{t}_2 a_3$ gives $x t_2 s_2 = y s'_2 \tilde{t}_2$. Second, the proof of right distributivity is completely analogous. □

Finally, let us discuss how R can be identified in the skew field $\mathfrak{D}(R)$. We define the mapping $\lambda : R \to \mathfrak{D}(R) : a \mapsto \frac{a}{1}$. For any $a, b \in R$ we obviously have

$$\lambda(a) + \lambda(b) = \frac{a}{1} + \frac{b}{1} = \frac{a+b}{1} = \lambda(a+b), \tag{1.7}$$

$$\lambda(a) \cdot \lambda(b) = \frac{a}{1} \cdot \frac{b}{1} = \frac{ab}{1} = \lambda(ab), \tag{1.8}$$

putting $a_1 = a_2 = s_1 = s_2 = t_1 = 1$, $b_1 = t_2 = a$ and $b_2 = b$ in (1.3) and (1.4). Further $\lambda(a) = 0$ implies $(1, a) \sim (1, 0)$ and hence $a = 0$. Therefore λ is an embedding of R in $\mathfrak{D}(R)$. Altogether, we have proved the following theorem.

Theorem 1.3. *An integral domain R satisfying the left Ore condition* (1.2) *is naturally embedded in the skew field $\mathfrak{D}(R)$ consisting of classes of equivalence* (1.1).

Definition 1.4. The skew-field $\mathfrak{D}(R)$ is called the *field of fractions* of R.

Since the mapping λ defined above is an embedding, the following convention will be used: for any $a \in R^\times$ and $b \in R$ we identify $\frac{b}{1} \equiv b$, $\frac{1}{a} \equiv a^{-1}$ and $\frac{b}{a} \equiv a^{-1}b$.

Further, for any $a_1, a_2 \in R^\times$ we have $\frac{1}{a_1} \cdot \frac{1}{a_2} = \frac{1}{a_2 a_1}$, for we could choose $t_1 = a_2$ and $t_2 = 1$ in (1.4). Consequently, the following rule holds:

$$(a_2 a_1)^{-1} = a_1^{-1} a_2^{-1}. \tag{1.9}$$

Remark 1.3. For a commutative ring R, the condition (1.2) is satisfied trivially.

1.1.2 Localizations

The concept of fields of fractions can be further broadened, at least in two directions. First, under certain assumptions, "fractions" can be defined and make sense even for a ring R not being an integral domain. As it was discussed above, one might abandon the requirement of the resulting set constituting a skew field. Second, it is possible to restrict the set of "denominators" from $R^\times$. Below we shall see a situation when this structure is convenient to work with.

Considering the described generalizations, we are getting from fields of fractions to the so-called *localizations* (cf. [13] or [23], but also [30] and [31] for illustration of importance of localizations to physics). The following stronger version of Theorem 1.3 holds (cf. [13], p. 350):

Theorem 1.5. *Let R be a ring and let $S \subset R$ be its multiplicative subset such that for any $a \in R$ and $s \in S$ one has*
(a) $Sa \cap Rs \neq \emptyset$, and
(b) if $sa = 0$ then there is $t \in S$ with $at = 0$.
Then the set $\mathfrak{D}_S(R)$ of classes of the following equivalence on $S \times R$:

$$(a, b) \sim (a', b') \text{ if there exist } t, s \in R \text{ such that } tb = sb', ta = sa' \text{ and } sa' \in S, \tag{1.10}$$

forms a ring and the mapping $\lambda : R \to \mathfrak{D}_S(R)$ sending $a \in R$ to the class containing $(1, a)$ is an S-inverting (i.e. mapping elements of S to invertible elements of $\mathfrak{D}_S(R)$) ring homomorphism.

Of course, one would have to check that the relation (1.10) is an equivalence indeed. Notice that the ring operations in $\mathfrak{D}_S(R)$ are defined in exactly the same way as in the case of $\mathfrak{D}(R) \equiv \mathfrak{D}_{R^\times}(R)$, i.e. by (1.3) and (1.4), only with $R^\times$ replaced by S.

Definition 1.6. The ring $\mathfrak{D}_S(R)$ is called a *localization* of R in S.

1.1.3 Universal Enveloping Algebras

Let us now skip from a general-algebra introduction to basic theory of enveloping algebras, in order to introduce the first example of a field of fractions (cf. [15] and [22]). Possibility of using the example for construction of representations of Lie algebras is also discussed in this section.

Let $\mathfrak{g}$ be a Lie algebra over a (commutative) field $\mathbb{F}$. It is sufficient for us to assume for simplicity that $\dim \mathfrak{g} < +\infty$ and $\operatorname{char} \mathbb{F} = 0$.

Recall that the *universal enveloping algebra* $\mathfrak{U}(\mathfrak{g})$ of $\mathfrak{g}$ is defined to be the quotient $\mathfrak{T}(\mathfrak{g})/\mathfrak{I}$, where $\mathfrak{T}(\mathfrak{g}) := \bigoplus_{i=0}^{+\infty} \mathfrak{g}^{\otimes i}$ is the *tensor algebra* of $\mathfrak{g}$, equipped with the tensor multiplication, and $\mathfrak{I} \subset \mathfrak{T}(\mathfrak{g})$ is the ideal generated by $\{x \otimes y - y \otimes x - [x,y] | x, y \in \mathfrak{g}\}$. To clarify the notation, we put $\mathfrak{g}^{\otimes 0} \equiv \mathbb{F}$ and $\mathfrak{g}^{\otimes 1} \equiv \mathfrak{g}$, and similarly $x^{\otimes 0} \equiv x^0 \equiv 1$.

Clearly, $\mathfrak{U}(\mathfrak{g})$ is an associative unital algebra over $\mathbb{F}$. As usual, we shall omit the tensor-product sign while working within $\mathfrak{U}(\mathfrak{g})$. Furthermore, since the canonical projection $\pi : \mathfrak{g} \to \mathfrak{U}(\mathfrak{g}) : x \mapsto x + \mathfrak{I}$ is injective and satisfies

$$\pi[x,y] = \pi(x)\pi(y) - \pi(y)\pi(x) \equiv [x,y] + \mathfrak{I} \tag{1.11}$$

for any $x, y \in \mathfrak{g}$, (cf. e.g. [15]), $\mathfrak{g}$ can be identified in $\mathfrak{U}(\mathfrak{g})$ with $\pi(\mathfrak{g})$. This fact allows us to denote an element of $\mathfrak{U}(\mathfrak{g})$ simply by x instead of $x + \mathfrak{I}$.

There is an important and famous theorem constituting a basis of $\mathfrak{U}(\mathfrak{g})$. For the complete proof see [15] or [22]. The theorem is referred as "PBW theorem" and similarly the basis is usually called the *PBW basis* of $\mathfrak{U}(\mathfrak{g})$.

Theorem 1.7 (Poincaré-Birkhoff-Witt). *Let $\{x_1, \dots, x_n\}$ be a basis of $\mathfrak{g}$. Then*

$$\left\{ x_1^{k_1} \cdots x_n^{k_n} \middle| k_1, \dots, k_n \in \mathbb{N}_0 \right\}$$

is a basis for $\mathfrak{U}(\mathfrak{g})$.

The *centre* of a universal enveloping algebra $\mathfrak{U}(\mathfrak{g})$ is defined as (cf. [22], p. 128)

$$\mathfrak{Z}(\mathfrak{g}) := \{c \in \mathfrak{U}(\mathfrak{g}) | cx = xc \text{ for any } x \in \mathfrak{U}(\mathfrak{g})\}. \tag{1.12}$$

A non-trivial element, i.e. not a multiple of the identity, of $\mathfrak{Z}(\mathfrak{g})$ is called a *Casimir operator* (or *Casimir element*). It turns out that there is always a finite number of functionally independent Casimir elements. Namely, it was shown in [5] that for an n-dimensional Lie algebra $\mathfrak{g}$ with a basis $\{x_1, \dots, x_n\}$ there exist precisely

$$\operatorname{index} \mathfrak{g} \equiv n - \operatorname{rank}_{\mathfrak{S}(\mathfrak{g})} \mathbf{S}(\mathfrak{g}) \in \mathbb{N}_0 \tag{1.13}$$

independent Casimir elements and any other is then expressed as a polynomial in them. Here $\mathfrak{S}(\mathfrak{g}) := \mathbb{F}[x_1, \dots, x_n]$ and $\mathbf{S}(\mathfrak{g})$ is the matrix over $\mathfrak{S}(\mathfrak{g})$ with entries

$$\mathbf{S}(\mathfrak{g})_{j,k} := [x_j, x_k], \quad 1 \le j, k \le n. \tag{1.14}$$

Remark 1.4. To be precise, the *index* of a Lie algebra is defined in a different way then expressed in (1.13) and afterwards, the relation (1.13) is proven (cf. [15], p. 47 and 64, respectively). Note that the original definition is independent of a particular choice of basis for $\mathfrak{g}$ and hence so is (1.13).

It can be proven (cf. [15], p. 122; or [4], p. 269) and it is crucial for our work, that any universal enveloping algebra (regarded as a ring) satisfies the conditions of Theorem 1.3 and thus possesses the field of fractions. This fact justifies the following definition.

Definition 1.8. The skew field $\mathfrak{D}(\mathfrak{U}(\mathfrak{g})) \equiv \mathfrak{D}(\mathfrak{g})$ is called the *Lie field* of $\mathfrak{g}$.

Notice that $\mathfrak{D}(\mathfrak{g})$ can be also regarded as an associative algebra over $\mathbb{F}$, with scalar multiplication inherited from $\mathfrak{U}(\mathfrak{g})$. Hence an (algebra) homomorphism to another associative algebra may be taken into consideration. In principle, the case could occur that two non-isomorphic Lie algebras (over the same field $\mathbb{F}$) possess mutually isomorphic Lie fields. We shall see immediately that, under certain circumstances, this could be used for construction of representations.

Let $\mathfrak{g}_1$ and $\mathfrak{g}_2$ be such Lie algebras and let $\Psi : \mathfrak{D}(\mathfrak{g}_1) \to \mathfrak{D}(\mathfrak{g}_2)$ be an isomorphism. Take a representation Φ_2 of the Lie algebra $\mathfrak{g}_2$ on a Hilbert space $\mathcal{H}$, i.e. an injective homomorphism $\Phi_2 : \mathfrak{g}_2 \to \mathcal{L}(\mathcal{H})$ such that all operators from $\Phi_2(\mathfrak{g}_2)$ share a common dense invariant domain (cf. [4], p. 31). The representation can be uniquely extended to a representation of the whole $\mathfrak{U}(\mathfrak{g}_2)$ (cf. [15], p. 70). Let Φ_2 denote the extension as well and suppose that it can be further extended to a certain localization $\mathfrak{D}_S(\mathfrak{U}(\mathfrak{g}_2)) \subset \mathfrak{D}(\mathfrak{g}_2)$ containing $\Psi(\mathfrak{g}_1)$. In other words, we need operators from $\Phi_2(S)$, $S \subset \mathfrak{U}(\mathfrak{g})$, to have well-defined inverses. Then the restriction of

$$\Phi_1 := \Phi_2 \circ \Psi \tag{1.15}$$

to $\mathfrak{g}_1$ is obviously a (faithful) representation of the Lie algebra $\mathfrak{g}_1$ on $\mathcal{H}$.

Remark 1.5. In fact, we do not have to strictly insist on Ψ being an isomorphism. It is enough to have an algebra homomorphism $\Psi : \mathfrak{g}_1 \subset \mathfrak{U}(\mathfrak{g}_1) \to \mathfrak{D}(\mathfrak{g}_2)$ and it is reasonable to require injectivity of Ψ in order to preserve faithfulness of the representation. Then, however, Ψ extends uniquely to $\mathfrak{U}(\mathfrak{g}_1)$ (cf. [15], p. 70) and further to the whole $\mathfrak{D}(\mathfrak{g}_1)$ because $\Psi(x) = 0$ only if $x = 0$ (cf. [15], p. 119). Therefore we eventually leave only the requirement of surjectivity.

We shall see below why it is reasonable for us to consider representations that send elements of a Lie algebra to skew-symmetric operators. On this account, let us explain how the involution on a Lie field is defined.

First, it is natural to put $x^* := -x$ for any $x \in \mathfrak{g}$, and than require a representation to be *involutive* in order to fulfil the skew-symmetry condition. Second, there is an assertion (cf. [15], p. 73) that such defined involution extends uniquely to $\mathfrak{U}(\mathfrak{g})$. This is done in an obvious way, following the rule $(ab)^* := b^* a^*$ for $a, b \in \mathfrak{U}(\mathfrak{g})$. Finally, for $a, b \in \mathfrak{U}(\mathfrak{g})$, $a \neq 0$, we define (cf. [7], p. 5)

$$\left(a^{-1}b\right)^* := b^* \left(a^*\right)^{-1}. \tag{1.16}$$

To check (1.16) is well-defined, take $\frac{b_1}{a_1} = \frac{b_2}{a_2} \in \mathfrak{D}(\mathfrak{g})$ and choose $s, t \in \mathfrak{U}(\mathfrak{g})$ such that $sa_1 = ta_2$ and $sb_1 = tb_2$. Then, according to (1.9),

$$\begin{aligned}
\left(a_1^{-1}b_1\right)^* &= b_1^* \left(a_1^*\right)^{-1} = b_1^* s^* \left(s^*\right)^{-1} \left(a_1^*\right)^{-1} = \left(sb_1\right)^* \left(a_1^* s^*\right)^{-1} = \left(sb_1\right)^* \left(\left(sa_1\right)^*\right)^{-1} \\
&= \left(tb_2\right)^* \left(\left(ta_2\right)^*\right)^{-1} \\
&= \left(a_2^{-1}b_2\right)^*.
\end{aligned}$$

In order to preserve skew-symmetry of the resulting representations, also the Lie field isomorphism Ψ considered above is desired to be involutive.

Remark 1.6. Strictly speaking, the operation $T \mapsto T^*$ is *not*, in general, an involution on $\mathcal{L}(\mathcal{H})$. Clarify that for a representation $\Phi : \mathcal{A} \to \mathcal{L}(\mathcal{H})$ of an (associative unital) $*$-algebra $\mathcal{A}$, to be *involutive* means to fulfil the condition $\Phi(x^*) \subset \Phi(x)^*$, $x \in \mathcal{A}$. To further avoid an ambiguity, remark that despite this fact, the adjoint operator is for simplicity denoted in the same way as the image under an involution.

At the end of this section, we mention an obvious but useful rule for computation within a Lie field. If $[a, b] \equiv ab - ba = c$ for $a, b, c \in \mathfrak{D}(\mathfrak{g})$, then $b^{-1}a - ab^{-1} = b^{-1}cb^{-1}$, and hence

$$\left[a, b^{-1}\right] = -b^{-1}[a, b]b^{-1}. \tag{1.17}$$

1.1.4 Weyl Algebras and Their Extensions

The notion of well-known Weyl algebras and their central extensions shall be recalled now (cf. e.g. [14], [15], [18]). The reason for us to do so is that they provide another possible starting point for the construction described in the previous section.

Let $\mathbb{F}$ be a field with char $\mathbb{F} = 0$ and let $m \in \mathbb{N}_0$. The *Weyl algebra* over $\mathbb{F}$ is defined to be the unital associative $\mathbb{F}$-algebra $\mathfrak{W}_m(\mathbb{F})$ generated by $p_1, \ldots, p_m, q_1, \ldots, q_m$ subject to the following relations:

$$p_j q_k - q_k p_j = \delta_{jk}, \quad p_j p_k - p_k p_j = q_j q_k - q_k q_j = 0, \quad 1 \le j, k \le m. \tag{1.18}$$

Let further r be a non-negative integer. We define the *extended Weyl algebra* $\mathfrak{W}_{m,r}(\mathbb{F})$ to be the Weyl algebra extended by r commuting elements $\theta_1, \ldots, \theta_r$, i.e.

$$\mathfrak{W}_{m,r}(\mathbb{F}) := \mathfrak{W}_m(\mathbb{F}) \otimes \mathbb{F}[\theta_1, \ldots, \theta_r]. \tag{1.19}$$

In particular, $\mathfrak{W}_{m,0}(\mathbb{F}) \equiv \mathfrak{W}_m(\mathbb{F})$. For completeness, note that $\mathfrak{W}_{0,0}(\mathbb{F}) \equiv \mathfrak{W}_0(\mathbb{F}) \equiv \mathbb{F}$.

Remark 1.7. In the language of the previous section where universal enveloping algebras were introduced, we may equivalently define $\mathfrak{W}_{m,r}(\mathbb{F})$ to be $\mathfrak{T}(W_{m,r}(\mathbb{F}))/\mathfrak{I}$, where $W_{m,r}(\mathbb{F}) := \mathrm{Span}_{\mathbb{F}}\{p_1, \ldots, p_m, q_1, \ldots, q_m, \theta_1, \ldots, \theta_r\}$ is the $\mathbb{F}$-vector space of formal sums of the respective elements, equipped with the tensor product, and $\mathfrak{I}$ is the ideal of $\mathfrak{T}(W_{m,r}(\mathbb{F}))$ generated by

$$\left\{ p_j q_k - q_k p_j - \delta_{jk}, p_j \theta_l - \theta_l p_j, q_k \theta_l - \theta_l q_k, \theta_l \theta_s - \theta_s \theta_l \middle| 1 \le j, k \le m, 1 \le l, s \le r \right\}.$$

Despite not "enveloping" any Lie algebra, Weyl algebras share certain important properties with universal enveloping algebras. First, there is an analogue of Theorem 1.7 introducing a basis in $\mathfrak{W}_{m,r}(\mathbb{F})$.

Theorem 1.9. *Suppose $m, r \in \mathbb{N}_0$ and $\mathbb{F} \in \{\mathbb{R}, \mathbb{C}\}$. The set*

$$\left\{ \theta_1^{j_1} \cdots \theta_r^{j_r} q_1^{k_1} \cdots q_m^{k_m} p_1^{l_1} \cdots p_m^{l_m} \middle| j_1, \ldots, j_r, k_1, \ldots, k_m, l_1, \ldots, l_m \in \mathbb{N}_0 \right\}$$

is a basis for $\mathfrak{W}_{m,r}(\mathbb{F})$.

Second, also Weyl algebras have the important property of possessing fields of fractions. We denote $\mathfrak{D}(\mathfrak{W}_m(\mathbb{F})) \equiv \mathfrak{D}_m(\mathbb{F})$ and $\mathfrak{D}(\mathfrak{W}_{m,r}(\mathbb{F})) \equiv \mathfrak{D}_{m,r}(\mathbb{F})$.

Remark 1.8. Although both these assertions can be proved directly (cf. [4], [15], [18]), it is certainly interesting to mention that the respective proofs can be obtained from a much broader concept of the so-called *G-algebras* (cf. [19], §1.9). Apropos of universal enveloping algebras, the same applies to them as well.

An involution on $\mathfrak{W}_{m,r}(\mathbb{F})$ can be defined as follows:

$$p_j^* := p_j, \quad q_j^* := -q_j, \quad \theta_k^* := \theta_k, \quad 1 \le j \le m, 1 \le k \le r, \tag{1.20}$$

and naturally $(ab)^* := b^* a^*$, $a, b \in \mathfrak{W}_{m,r}(\mathbb{F})$. It is readily seen that the defining relations (1.18) are preserved for this choice. The mapping extends to the involution on $\mathfrak{D}_{m,r}(\mathbb{F})$ according to (1.16).

Remark 1.9. Notice that the choice of involution on $\mathfrak{D}_{m,r}(\mathbb{F})$ is far from unique. Namely, each of the following involutions is obviously admissible:

$$p_j^* := \varepsilon_j p_j, \quad q_j^* := -\varepsilon_j q_j, \quad \theta_k^* := \theta_k, \quad \varepsilon_j = \pm 1, 1 \le j \le m, 1 \le k \le r. \tag{1.21}$$

See e.g. [17] for an example of the, in some sense "opposite", involution to ours. Later we shall see that the involution (1.20) is a convenient one for us to work with.

As before, existence of the fields of fractions $\mathfrak{D}_{m,r}(\mathbb{F})$ can be used for inducing a representation of a Lie algebra (over $\mathbb{F}$). In fact, we may repeat the discussion from the previous section just with the enveloping algebra $\mathfrak{U}(\mathfrak{g}_2)$ replaced by $\mathfrak{W}_{m,r}(\mathbb{F})$. Let us skip the discussion directly to skew-symmetric representations.

Thus, let $\mathfrak{g}$ be a Lie algebra over $\mathbb{F}$. If there are $m, r \in \mathbb{N}_0$ such that a $*$-isomorphism $\Psi : \mathfrak{D}(\mathfrak{g}) \to \mathfrak{D}_{m,r}(\mathbb{F})$ exists, then the involutive representations of $\mathfrak{W}_{m,r}(\mathbb{F})$ (on a Hilbert space $\mathcal{H}$) that can be extended either to the whole $\mathfrak{D}_{m,r}(\mathbb{F})$, or at least to a certain localization $\mathfrak{D}_S(\mathfrak{W}_{m,r}(\mathbb{F})) \subset \mathfrak{D}_{m,r}(\mathbb{F})$ containing $\Psi(\mathfrak{g})$, induce involutive (i.e. skew-symmetric) representations of $\mathfrak{g}$ on $\mathcal{H}$.

This is the crucial point of our work. Namely, following the pattern we just sketched, we shall construct skew-symmetric representations of the (real) Poincaré algebras (cf. §1.2 below). In the construction, the following involutive representations of the *real* Weyl algebras $\mathfrak{W}_{m,r}(\mathbb{R})$ are involved: we choose $\mathcal{H}_m := L^2(\mathbb{R}^\times \times \mathbb{R}^{m-1}, \mathrm{d}^m x)$ on which we define the family $\Phi_{c_1,\dots,c_r} : \mathfrak{W}_{m,r}(\mathbb{R}) \to \mathcal{L}(\mathcal{H}_m)$ of representations

$$\Phi_{c_1,\dots,c_r}(p_j)\psi(x) := -i\partial_{x_j}\psi(x), \tag{1.22}$$

$$\Phi_{c_1,\dots,c_r}(q_j)\psi(x) := ix_j\psi(x), \tag{1.23}$$

$$\Phi_{c_1,\dots,c_r}(\theta_k)\psi(x) := c_k\psi(x), \tag{1.24}$$

where $c_k \in \mathbb{R}$, $\partial_{x_j} \equiv \frac{\partial}{\partial x_j}$, $1 \le j \le m$, $1 \le k \le r$, and $x = (x_1, \dots, x_m) \in \mathbb{R}^\times \times \mathbb{R}^{m-1}$. Notice that the definition is well-posed since all the operators (1.22) - (1.24) are well-defined on $C_0^\infty(\mathbb{R}^\times \times \mathbb{R}^{m-1})$ which is obviously invariant and also dense in $\mathcal{H}_m$ (cf. e.g. [42], p. 10), and $[-i\partial_{x_j}, ix_k\mathbb{1}] = \delta_{jk}\mathbb{1}$, $1 \le j,k \le m$. Furthermore, the representations are involutive because $(\partial_{x_j})^* \supset -\partial_{x_j}$, $(x_j\mathbb{1})^* \supset x_j\mathbb{1}$ and $(c_k\mathbb{1})^* \supset c_k\mathbb{1}$, for $1 \le j \le m$ and $1 \le k \le r$.

It is, however, far from clear whether it is possible to extend $\Phi_{c_1,\dots,c_r}$ to the whole $\mathfrak{D}_{m,r}(\mathbb{R})$ or not; do e.g. the inverse operators to $\Phi_{c_1,\dots,c_r}(p_j)$, $1 \le j \le m$ exist? In fact we will not need them either. It will became apparent below that enough for us is to extend the representations to the localization $\mathfrak{D}'_{m,r}(\mathbb{R}) := \mathfrak{D}_{\mathfrak{Q}}(\mathfrak{W}_{m,r}(\mathbb{R}))$, where $\mathfrak{Q}$ is the subalgebra of $\mathfrak{W}_{m,r}(\mathbb{R})$ generated by $q_1 \in \mathfrak{W}_{m,r}(\mathbb{R})$. This is done via

$$\Phi_{c_1,\dots,c_r}\left(q_1^{-1}\right) := -\frac{i}{x_1}\mathbb{1}. \tag{1.25}$$

Remark 1.10. Notice that $\mathcal{H}_m = \mathcal{H}_m^+ \oplus \mathcal{H}_m^-$, where

$$\mathcal{H}_m^\pm = \left\{\psi \in \mathcal{H}_m | \psi(x_1, \dots, x_m) = 0 \text{ for almost any } x_1 \in \mathbb{R}^\pm\right\} \equiv L^2(\mathbb{R}^\pm \times \mathbb{R}^{m-1}, \mathrm{d}^m x).$$

Further, note that the inner product of $\phi, \psi \in \mathcal{H}_m$ is

$$(\phi, \psi) = \int_{\mathbb{R}^\times \times \mathbb{R}^{m-1}} \overline{\phi(x)}\, \psi(x)\, \mathrm{d}^m x = \int_{\mathbb{R}^m} \overline{\phi(x)}\, \psi(x)\, \mathrm{d}^m x.$$

1.1.5 Gelfand-Kirillov Conjecture

The question is, however, whether the technique described above could be used for a given Lie algebra $\mathfrak{g}$ or nor, i.e. whether a $*$-isomorphism from $\mathfrak{D}(\mathfrak{g})$ onto some $\mathfrak{D}_{m,r}(\mathbb{F})$ exists. At least a glimpse of an answer is provided by the *Gelfand-Kirillov conjecture*.

Let us recall the notion of the so-called *algebraic Lie algebras* at this stage (cf. [11]). First, a subgroup of the group $\mathrm{Aut}\, V$ of automorphisms of an $\mathbb{F}$-vector space V is called *algebraic* if there is a *defining* set $D \subset \mathbb{F}[\mathrm{End}\, V]$ such that

$$G = \{\eta \in \mathrm{Aut}\, V | \pi(\eta) = 0 \text{ for all } \pi \in D\}.$$

In other words, G is given as a set of solutions of a system of polynomial equations. Then a *Lie algebra* is said to be *algebraic* if it is isomorphic to a Lie algebra of an algebraic Lie group. An alternative definition can be found in [10].

A lot of known finite-dimensional Lie algebra arises in this way; e.g.
(a) $\mathfrak{gl}(n, \mathbb{F})$,
(b) nilpotent Lie subalgebras of $\mathfrak{gl}(n, \mathbb{F})$,
(c) $\mathfrak{sl}(n, \mathbb{F})$, for $\mathbb{F}$ being algebraically closed and of characteristic zero,
(d) semisimple Lie algebras over $\mathbb{F}$, provided $\operatorname{char}\mathbb{F} = 0$,
are algebraic (cf. [11]). To see that not every Lie algebra has this property, consider the following counterexample: the (solvable) complex Lie algebra generated by x_1, x_2, x_3, x_4 due to $[x_1, x_2] = x_2 + x_3$, $[x_1, x_3] = x_3$ and $[x_1, x_4] = -2x_4$, is not the Lie algebra of any algebraic Lie group (cf. [36], p. 16).

In 1966, I. M. Gelfand and A. A. Kirillov stated their famous *Hypothèse fondamentale* (cf. [18]). In its original version, it read as follows:

Conjecture 1.10 (Gelfand-Kirillov)**.** *Let $\mathbb{F}$ be an algebraically closed field of characteristic zero. For any finite-dimensional algebraic Lie algebra $\mathfrak{g}$ over $\mathbb{F}$, there exist $m, r \in \mathbb{N}_0$ such that $\mathfrak{D}(\mathfrak{g}) \cong \mathfrak{D}_{m,r}(\mathbb{F})$.*

Remark 1.11. It is well-know (cf. e.g. [35]) that if $\mathfrak{D}(\mathfrak{g}) \cong \mathfrak{D}_{m,r}(\mathbb{F})$, then necessarily

$$r = \operatorname{index}\mathfrak{g} \quad \text{and} \quad m = \frac{1}{2}(\dim\mathfrak{g} - \operatorname{index}\mathfrak{g}). \tag{1.26}$$

The conjecture was verified for nilpotent Lie algebras, for $\mathfrak{sl}(n, \mathbb{F})$ and $\mathfrak{gl}(n, \mathbb{F})$ in [18], by authors themselves. A later on, in 1973, it was confirmed for solvable Lie algebras, independently in [9], [24] and [32]. In 1979 validity of the conjecture was further extended for certain semidirect products of simple Lie algebras with their standard modules (cf. [33]). In 1996, however, J. Alev, A. Ooms and M. Van den Bergh construed in [1] a series of counterexamples, starting with a Lie algebra of dimension nine, that finally disproved the original assertion. Four years later, the same trinity proved that their nine-dimensional example is in fact the simplest one and that the Gelfand-Kirillov conjecture holds true for all Lie algebras up to dimension eight (cf. [2]). Finally, in 2010 A. Premet showed in [38] that the conjecture fails for simple Lie algebras of type B_n, for $n \geq 3$, D_n, for $n \geq 4$, E_6, E_7, E_8 and F_4 (cf. [26], ch. 3, for explanation of the "types").

In spite of the great achievement, the validity of Gelfand-Kirillov conjecture in the general case remains an open problem. Furthermore, it is far from clear whether and how the results could be reframed for a ground field not being algebraically closed, though the conjecture makes perfect sense for such Lie algebras (cf. [11]). In particular, one would be of course interested in the case $\mathbb{F} = \mathbb{R}$. Apparently, if $\mathfrak{D}(\mathfrak{g}) \cong \mathfrak{D}_{m,r}(\mathbb{R})$ for a real Lie algebra $\mathfrak{g}$, then also $\mathfrak{D}(\mathfrak{g}_{\mathbb{C}}) \cong \mathfrak{D}_{m,r}(\mathbb{C})$, where $\mathfrak{g}_{\mathbb{C}}$ is its complexification. This means that the conjecture fails for any real form of a complex Lie algebra for which the conjecture was disproved. Furthermore, also for a real Lie algebra, the only field $\mathfrak{D}_{m,r}(\mathbb{R})$ potentially isomorphic to the respective Lie field is specified by (1.26).

Remark 1.12. Notice that the problem of Gelfand and Kirillov does not deal with the involutive property of the respective isomorphisms at all. It only provides us, in the cases where it holds, with a necessary condition for the existence of a $*$-isomorphism. On the other hand, we know for sure that it is a waste of time to seek for a $*$-isomorphism in cases where the conjecture is contradicted.

1.1.6 Representations of Lie Groups

We end the first part of the opening chapter discussing correspondence between (skew-symmetric) representations of Lie algebras and (unitary) representations of Lie groups.

Let G be a connected Lie group and let $\mathfrak{g}$ be its Lie algebra. Enough for us is to consider only Lie groups that do not cover any other group but itself. In other words, we shall assume that $G \cong \widetilde{G}/\widetilde{N}$, where $\widetilde{G}$ is the universal covering group of G and $\widetilde{N}$ is the maximal discrete normal subgroup of $\widetilde{G}$.

Any representation Φ of $\mathfrak{g}$ can be, in principle, uniquely integrated into a representation of G (cf. [41], p. 71). Locally, this is realized as in the case of Lie algebras and Lie groups themselves, i.e. by the well-known exponential mapping (cf. [41], sec. 2.10). Since we aim at unitary Lie group representations in particular and since unitarity of $e^{\Phi(x)}$, $x \in \mathfrak{g}$, obviously corresponds to skew-symmetry of $\Phi(x)$, $x \in \mathfrak{g}$, we are occupied entirely by skew-symmetric representations of Lie algebras (cf. [4], p. 322).

Regarding the question of globality, there are several powerful criteria to decide whether a skew-symmetric representation of $\mathfrak{g}$ is integrable into a unitary representation of the whole G, such as theory of the so-called *analytic vectors* (cf. [4], §11.4) or properties of the so-called *Nelson operator* (cf. [4], §11.5). Nevertheless, none of the tools is suitable for us. Instead, we are able to substitute their role by simple algebraic computations. To be precise, we shall proceed as follows:

(a) First, given a skew-symmetric representation Φ of a Lie algebra $\mathfrak{g}$ on a Hilbert space $\mathcal{H}$, we choose a basis $\{x_1, \dots, x_n\}$ for $\mathfrak{g}$ and evaluate $\Phi(x_j)$, $1 \le j \le n$.
(b) Second, for each $1 \le j \le n$ we compute one-parameter subgroup

$$U^{(j)}(t) = \exp\Big\{it\left[-i\widetilde{\Phi}(x_j)\right]\Big\} \equiv \exp\{t\,\Phi(x_j)\} \tag{1.27}$$

whose generator $-i\widetilde{\Phi}(x_j) := \frac{1}{i}\frac{\mathrm{d}}{\mathrm{d}t}U^{(j)}(t)\big|_{t=0}$ is self-adjoint extension of the (symmetric) operator $-i\Phi(x_j)$ to domain $\Big\{\psi \in \mathcal{H}\Big|\lim_{t\to 0}\left[\frac{1}{t}\left(U^{(j)}(t) - \mathbb{1}\right)\psi\right] \text{ exists}\Big\}$ (cf. [6] §5.9).[1]
This consists of
– guessing or computing such additive one-parameter set $U^{(j)}(t)$ of operators that its derivative in $t = 0$ formally agrees with $\Phi(x_i)$;
– verifying that $U^{(j)}(t)$ is a strongly continuous one-parameter subgroup of unitary operators on $\mathcal{H}$;
– verifying that $-i\Phi(x_j) \subset -i\widetilde{\Phi}(x_j)$, i.e. that $\lim_{t\to 0}\left[\frac{1}{t}\left(U^{(j)}(t) - \mathbb{1}\right)\psi\right]$ exists for any $\psi \in \operatorname{Dom}\Phi(x_j)$.
(c) Third, for an appropriate permutation π of $\{1, \dots, n\}$ we define

$$U(t_1, \dots, t_n) := U^{(\pi(1))}(t_{\pi(1)}) \cdots U^{(\pi(n))}(t_{\pi(n)}). \tag{1.28}$$

It is clear that the mapping $(t_1, \dots, t_n) \in \mathbb{R}^n \mapsto U(t_1, \dots, t_n)$ is strongly continuous.
(d) Fourth, we show that for certain neighbourhood $\mathbb{T}_n$ of $0 \in \mathbb{R}^n$ there are continuous functions $f_j : \mathbb{T}_n \times \mathbb{T}_n \to \mathbb{R}$, $1 \le j \le n$, such that

$$U(t)U(t') = U\big(f_1(t,t'), \dots, f_n(t,t')\big) \tag{1.29}$$

for any $t \equiv (t_1, \dots, t_n)$ and $t' \equiv (t'_1, \dots, t'_n)$ from $\mathbb{T}_n$.

Then there certainly exits a neighbourhood $\widetilde{\mathbb{T}}_n \subset \mathbb{T}_n$, $0 \in \widetilde{\mathbb{T}}_n$, and continuous functions $\widetilde{f}_j : \widetilde{\mathbb{T}}_n \to \mathbb{R}$, $1 \le j \le n$, such that

$$U(t)^{-1} = U^{(\pi(n))}(-t_{\pi(n)}) \cdots U^{(\pi(1))}(-t_{\pi(1)}) = U\Big(\widetilde{f}_1(t), \dots, \widetilde{f}_n(t)\Big) \tag{1.30}$$

[1] Notice that we only identify $U^{(j)}(t) \equiv \exp\Big\{t\,\Phi(x_j)\Big\}$ in order to "label" the one-parameter subgroups.

for any $t \equiv (t_1, \dots, t_n) \in \widetilde{\mathbb{T}}_n$. This follows from continuity of functions f_j, $1 \le j \le n$, by $n-1$ repetitions of the rule (1.29). Therefore the group, denote it by $\mathcal{G}$, generated by $U(t)$, $t \in \widetilde{\mathbb{T}}_n$, is a connected Lie group; it is locally homeomorphic to $\mathbb{R}^n$ with continuous multiplication and inversion (cf. also [41], p. 88).

It is obvious from the construction that the Lie algebra of $\mathcal{G}$ is isomorphic to $\mathfrak{g}$, therefore the Lie groups G and $\mathcal{G}$ are locally isomorphic (cf. [41], p. 73). But this means nothing less than $\mathcal{G} \cong \widetilde{G}/N$, where N is a subgroup of $\widetilde{N}$, with $\widetilde{N}$ and $\widetilde{G}$ being as above (cf. [4], p. 90). Thus, if $N = \widetilde{N}$, then $\mathcal{G} \cong G$ and hence $\mathcal{G}$ is a (faithful, unitary) representation of G. The question whether N is a proper subgroup of $\widetilde{N}$, or $\widetilde{N}$ itself shall be discussed for each $\mathcal{G}$ separately.

Remark 1.13. The family of real parameters t_j, $1 \le j \le n$, is an example of the so-called *canonical coordinate system of the second kind* in $\mathcal{G}$ (cf. [41], p. 89). Although such a chart may be global in some cases, in general it provides a homeomorphism to $\mathbb{R}^n$ only for a certain small neighbourhood of the identity. Outside of the neighbourhood, there could exist a group element that cannot be described in terms of the coordinates.

1.2 Poincaré Groups and Algebras

Let us now introduce the class of the co-called Poincaré Lie groups a their, Poincaré, Lie algebras (cf. [4], p. 431, and [3]). The first three non-trivial representatives of the class provide us with examples on which our method of construction of representations based on the Lie-field technique will be illustrated.

1.2.1 Poincaré Groups

Assume $n \in \mathbb{N}$, $n \ge 2$. *Minkowski space* M^n is the real linear space $\mathbb{R}^n$ equipped with the following inner product, $x = (x_0, \dots, x_{n-1}), y = (y_0, \dots, y_{n-1}) \in M^n$:

$$x \bullet y := x_0 y_0 - \sum_{j=1}^{n-1} x_j y_j = \sum_{\mu,\nu=0}^{n-1} \eta_{\mu\nu} x_\mu y_\nu, \tag{1.31}$$

where $\eta = (\eta_{\mu\nu})_{\mu,\nu=0}^{n-1} = \mathrm{diag}(1, -1, \dots, -1) \in \mathbb{R}^{n,n}$.

The *Poincaré group* $\mathcal{P}_n$ is defined to be the group of transformations in Minkowski space M^n that preserve the inner product. Such a transformation $x \mapsto x'$ is of the form

$$x'_\mu = \sum_{\nu,\sigma=0}^{n-1} \Lambda_{\mu\nu} x_\nu + a_\mu, \tag{1.32}$$

$0 \le \mu \le n-1$, where

$$\Lambda = (\Lambda_{\mu\nu})_{\mu,\nu=0}^{n-1} \in \mathrm{SO}_0(1, n-1) = \left\{ \Lambda \in \mathbb{R}^{n,n} \middle| \Lambda^T \eta \Lambda = \eta, \det \Lambda = 1, \Lambda_{00} \ge 1 \right\},$$

and

$$a = \begin{pmatrix} a_0 \\ \vdots \\ a_{n-1} \end{pmatrix} \in \mathrm{T}^n = \{ a | a \in \mathbb{R}^n \}.$$

$\mathrm{SO}_0(1, n-1)$ is the pseudo-orthogonal group of *rotations* and T^n is the additive group of *translations*. The group multiplication in $\mathcal{P}_n$ corresponds to the composition of the transformations. It is easily seen from (1.32) that transformations (Λ, a) and (Λ', a') are composed as

$$(\Lambda', a') \circ (\Lambda, a) = (\Lambda'\Lambda, \Lambda' a + a'), \tag{1.33}$$

hence $\mathcal{P}_n$ is in fact the semidirect product $\mathrm{SO}_0(1, n-1) \ltimes \mathrm{T}^n$, where the determining left action of $\mathrm{SO}_0(1, n-1)$ on T^n is nothing else but the natural representation.

It is convenient to realize that such a semidirect product can be also viewed as the subgroup of $\mathrm{GL}(n+1, \mathbb{R})$ consisting of matrices of the form

$$\begin{pmatrix} & & & a_0 \\ & \Lambda & & \vdots \\ & & & a_{n-1} \\ 0 & \cdots & 0 & 1 \end{pmatrix}, \quad \Lambda \in \mathrm{SO}_0(1, n-1), \begin{pmatrix} a_0 \\ \vdots \\ a_{n-1} \end{pmatrix} \in \mathrm{T}^n.$$

As a subgroup of $\mathrm{GL}(n+1, \mathbb{R})$, $\mathcal{P}_n$ is obviously a (real) Lie group.

Remark 1.14. To be precise, the general transformation (1.32) allows Λ to be from

$$\mathrm{O}(1, n-1) = \left\{ \Lambda \in \mathbb{R}^{n,n} \middle| \Lambda^T \eta \Lambda = \eta \right\}. \tag{1.34}$$

In general, this group has, and hence so would have the group $\mathcal{P}_n$, more than one mutually disconnected connected components (cf. e.g. [4], p. 513, for discussion on the well-know case $n = 4$). Since we shall construct representations of $\mathcal{P}_n$ by integrating representations of its Lie algebra, we restrict ourselves to the connected component of $\mathrm{O}(1, n-1)$ that contains the identity, i.e. to $\mathrm{SO}_0(1, n-1)$. Under this assumption, the Poincaré group $\mathcal{P}_n$ is connected.

1.2.2 Poincaré Algebras

The Lie algebra $\mathfrak{p}_n$ of $\mathcal{P}_n$ is a real $\frac{n}{2}(n+1)$-dimensional Lie algebra spanned by P_μ, $0 \le \mu \le n-1$, and $L_{\mu\nu}$, $0 \le \mu < \nu \le n-1$, subject to the following commutation relations ($0 \le \mu, \nu, \sigma, \rho \le n-1$):

$$\begin{aligned} [L_{\mu\nu}, L_{\sigma\rho}] &= -\eta_{\mu\sigma} L_{\nu\rho} + \eta_{\mu\rho} L_{\nu\sigma} - \eta_{\nu\rho} L_{\mu\sigma} + \eta_{\nu\sigma} L_{\mu\rho}, \\ [L_{\mu\nu}, P_\rho] &= -\eta_{\mu\rho} P_\nu + \eta_{\nu\rho} P_\mu, \\ [P_\rho, P_\sigma] &= 0, \end{aligned} \tag{1.35}$$

where we put $L_{00} := 0$ and $L_{\nu\mu} := -L_{\mu\nu}$ for $\nu > \mu$. Notice that P_ρ, $0 \le \rho \le n-1$, generate the translation group T^n while $L_{\mu\nu}$, $0 \le \mu < \nu \le n-1$, are generators for the group $\mathrm{SO}_0(1, n-1)$ of rotations.

1.2.3 Coordinates in $\mathcal{P}_n$

With respect to the realization of the Lie group $\mathcal{P}_n$ as a matrix group, the Lie algebra $\mathfrak{p}_n$ can be regarded as a subalgebra of $\mathfrak{gl}(n+1, \mathbb{R})$. One can easily check that we may send $L_{\mu\nu} \mapsto \mathbf{L}_{\mu\nu}$, $0 \le \mu < \nu \le n-1$, and $P_\rho \mapsto \mathbf{P}_\rho$, $0 \le \rho \le n-1$, where

$$(\mathbf{L}_{\mu\nu})_{\alpha\beta} = \delta_{\mu\alpha} \eta_{\nu\beta} - \delta_{\nu\beta} \eta_{\mu\alpha} \quad \text{and} \quad (\mathbf{P}_\rho)_{\alpha\beta} = \delta_{\rho\alpha} \delta_{n\beta}, \tag{1.36}$$

$0 \le \alpha, \beta \le n$. From this realization, the Lie group $\mathcal{P}_n$ can be, at least locally, reconstructed in terms of the canonical coordinates of the second kind (cf. Remark 1.13). Namely, a neighbourhood of the identity in $\mathcal{P}_n$ can be written as

$$\left\{ g(t_1, \ldots, t_N) \equiv \exp\left(t_{\pi(1)} \mathbf{A}_{\pi(1)}\right) \cdots \exp\left(t_{\pi(N)} \mathbf{A}_{\pi(N)}\right) \middle| (t_1, \ldots, t_N) \in \mathbb{T} \right\},$$

where $N := \frac{n}{2}(n+1)$, $\mathbb{T} \subset \mathbb{R}^N$ is a neighbourhood of zero, π is a permutation of $\{1, \ldots, N\}$ and $\{\mathbf{A}_1, \ldots, \mathbf{A}_N\}$ is a basis of the matrix Lie algebra $\mathfrak{p}_n \subset \mathfrak{gl}(n+1, \mathbb{R})$, i.e. of $\mathrm{Span}_{\mathbb{R}} \left\{ \mathbf{L}_{\mu\nu}, \mathbf{P}_\rho \middle| 0 \le \mu < \nu \le n-1, 0 \le \rho \le n-1 \right\}$. Notice that "exp" stands for the matrix exponential now (cf. e.g. [25], p. 76). Again the coordinates may or may not be extendable to the whole $\mathfrak{p}_n$.

The following convention shall be adopted: we put $\mathbf{A}_{\pi(j)} := \mathbf{P}_{j-1}$ for $1 \leq j \leq n$, and $\mathbf{A}_{\pi(j)} \in \mathrm{Span}_{\mathbb{R}}\{\mathbf{L}_{\mu\nu} | 0 \leq \mu < \nu \leq n-1\}$ for $n < j \leq N$. It is clear from the form of matrices (1.36) that then, for $(t_1, \dots, t_N) \in \mathbb{T}$,

$$g(t_1, \dots, t_N) = \begin{pmatrix} & & & t_{\pi(1)} \\ & \Lambda(t_{\pi(n+1)}, \dots, t_{\pi(N)}) & & \vdots \\ & & & t_{\pi(n)} \\ 0 & \cdots & 0 & 1 \end{pmatrix},$$

where $\Lambda(t_{\pi(n+1)}, \dots, t_{\pi(N)}) \in \mathrm{SO}_0(1, n-1)$. Hence $g(t_1, \dots, t_N)$ may be (and will be) regarded also as an ordered pair $\Big(\Lambda(t_{\pi(n+1)}, \dots, t_{\pi(N)}), a(t_{\pi(1)}, \dots, t_{\pi(n)})\Big)$, with

$$a(t_{\pi(1)}, \dots, t_{\pi(n)}) = \begin{pmatrix} t_{\pi(1)} \\ \vdots \\ t_{\pi(n)} \end{pmatrix} \in \mathrm{T}^n. \tag{1.37}$$

Remark 1.15. Notice that there is a serious reason to distinguish elements of an abstract Lie algebra and its matrix realization. Namely, we shall also work with universal enveloping algebras and since they are (for non-trivial Lie algebras) infinite-dimensional, their elements cannot be any more faithfully represented by matrices. To illustrate this, in $\mathfrak{p}_2$, for instance, we have

$$\mathbf{P}_0 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \text{thus} \quad \mathbf{P}_0^2 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

But, in contrast, $P_0^2 \in \mathfrak{U}(\mathfrak{p}_2)$ is a non-zero element.

1.3 Mackey Theory

In this section we shall introduce the "standard" framework for construction of irreducible unitary representations of the Lie groups $\mathcal{P}_n$, within which the representations are induced by representations of certain "smaller" groups. The method was first used for $\mathcal{P}_4$ by E. P. Wigner in his famous paper [45] and a decade later, it was generalized by G. W. Mackey, by far not only to $\mathcal{P}_n$. For further details on the theory behind as well as for derivation of the results we use, we refer the reader to [4], where the original Mackey's papers [28] and [29] are cited and his results are presented clearly.

1.3.1 Induced Unitary Representations

The first important result due to George W. Mackey is the so-called *Mackey decomposition theorem* (cf. [4], p. 70). It says that for each locally compact separable topological group S with a closed subgroup K there is a Borel set $H \subset S$ such that each $\Lambda \in S$ uniquely decomposes as

$$\Lambda = k_\Lambda h_\Lambda, \quad k_\Lambda \in K \text{ and } h_\Lambda \in H. \tag{1.38}$$

Second, let μ be a quasi-invariant measure on $K\backslash S$, that means the measures $\mu(x)$ and $\mu_\Lambda(x) \equiv \mu(x\Lambda)$ are equivalent (i.e. having the same sets of measure zero) for each $\Lambda \in S$. Here $(x, \Lambda) \in K\backslash S \times S \mapsto x\Lambda$ is the natural right action of the group S on the set $K\backslash S = \{K\Lambda | \Lambda \in S\}$. Note that the homogeneous space $K\backslash S$ always admits such a measure (cf. [4], p. 130). Then there exists a real function ρ on $K\backslash S \times S$ such that

$$\mathrm{d}\mu(x\Lambda) = \rho(x, \Lambda)\mathrm{d}\mu(x) \tag{1.39}$$

for all $x \in K\backslash S$ and $\Lambda \in S$. The function ρ is the so-called *Radon-Nikodym derivative* of the measure μ and it is unique to within a set of measure zero (cf. also [40]).

Now consider a unitary representation W of K in a Hilbert space $\mathcal{H}$. Then the following formula (cf. [4], eq. (15) on p. 479) defines a unitary representation U_W of S on $L^2(K\backslash S, \mathrm{d}\mu; \mathcal{H})$:

$$U_W(\Lambda)\psi(x) = \rho(x,\Lambda)^{\frac{1}{2}}\, W(k_{h_{\Lambda_x}\Lambda})\psi(x\Lambda), \tag{1.40}$$

valid for any $\Lambda \in S$ and $\psi \in L^2(K\backslash S, \mathrm{d}\mu; \mathcal{H})$, thus for μ-almost every $x = K\Lambda_x \in K\backslash S$. Note that we still keep the notation of the Mackey decomposition (1.38) and that $k_{h_{\Lambda_x}\Lambda}$ does not depend on the particular choice of $\Lambda_x \in S$. More precisely,

Proposition 1.11. *If $K\Lambda = K\tilde{\Lambda}$, then $h_\Lambda = h_{\tilde{\Lambda}}$.*

Proof. Suppose $h_\Lambda \neq h_{\tilde{\Lambda}}$. Then we have $Kh_\Lambda = K\Lambda = K\tilde{\Lambda} = Kh_{\tilde{\Lambda}}$ and there is $k \in K$ with $kh_\Lambda = h_{\tilde{\Lambda}}$. But this contradicts uniqueness of the Mackey decomposition. □

1.3.2 Irreducible Unitary Representations of Semidirect Products

The concept of induced representations is most powerful when S is a factor of the so-called *regular* semidirect product $G = S \ltimes N$ of separable locally compact groups, with N being abelian. See [4] for the precise definition as well as for the proof of regularity in the case we are interested in, i.e. the Poincaré groups $\mathcal{P}_n \equiv \mathrm{SO}_0(1, n-1) \ltimes \mathrm{T}^n$. Being the case, it turns out that *every* irreducible unitary representation of the product arises in this way.

The natural left action of S on N arising from the definition of the semidirect product translates onto the right action of S on the dual group $\hat{N}$, setting

$$(\chi\Lambda)(a) := (\chi \circ \Lambda)(a) \tag{1.41}$$

for any $\chi \in \hat{N}$, $a \in N$ and $\Lambda \in S$. Then $\hat{N}$ decomposes into a (disjoint) union of orbits under this action and each orbit $\mathcal{O}_\xi$, with an origin $\xi \in \hat{N}$, is homeomorphic to $S_\xi\backslash S$, where S_ξ is the (closed) stabilizer of ξ (cf. [21], p. 121).

We can therefore put $K := S_\xi$ in the previous paragraph, identify $\mathcal{O}_\xi$ with the factor group $K\backslash S \equiv S_\xi\backslash S$ and use (1.40) to induce representations of S from those of S_ξ. Furthermore, since N is abelian, the extension of such an induced representation to the whole G differs only by a scalar factor, namely the action of $\chi \in \mathcal{O}_\xi$ on a represented abelian element $a \in N$ (cf. [4], p. 507). Thus, for chosen orbit $\mathcal{O}_\xi$ and representation W of S_ξ on $\mathcal{H}$, any pair $(\Lambda, a) \in G$ is represented as

$$U_{\mathcal{O}_\xi, W}(\Lambda, a)\psi(\chi) = \chi(a)\, U_W(\Lambda)\psi(\chi), \tag{1.42}$$

with $\psi \in L^2(\mathcal{O}_\xi, \mathrm{d}\mu; \mathcal{H})$, μ is a quasi-invariant measure on $\mathcal{O}_\xi$, and U_W given by (1.40).

In order to determine the element $k_{h_{\Lambda_\chi}\Lambda} \equiv k_{(\chi,\Lambda)}$ in (1.40), we have to specify the subset $H \subset S$. This is equivalent to choosing a mapping $h : \mathcal{O}_\xi \to S$ fulfilling $\chi = \xi h(\chi)$ for any $\chi \in \mathcal{O}_\xi$, and setting $H := h(\mathcal{O}_\xi)$. Then $h_\Lambda = h(\xi\Lambda)$ and $k_{(\chi,\Lambda)}$ is the (unique) solution of $h(\chi)\Lambda = k_{(\chi,\Lambda)}h(\chi\Lambda)$.

Altogether, for each orbit $\mathcal{O}_\xi$ and unitary irreducible representation W of S_ξ on $\mathcal{H}$ we have the following unitary irreducible representation of $S \ltimes N$ on $L^2(\mathcal{O}_\xi, \mathrm{d}\mu; \mathcal{H})$:

$$U_{\mathcal{O}_\xi, W}(\Lambda, a)\psi(\chi) = \sqrt{\rho(\chi,\Lambda)}\, \chi(a)\, W(k_{(\chi,\Lambda)})\, \psi(\chi\Lambda). \tag{1.43}$$

The most important aspects of the construction are its completeness and uniqueness. Namely all mutually non-equivalent irreducible unitary representations of $S \ltimes N$ are in one-to-one correspondence with all pairs $(W, \mathcal{O}_\xi)$ of orbits $\mathcal{O}_\xi$ and mutually non-equivalent irreducible unitary representations W of S_ξ (cf. [4], p. 508, 509).

1.3.3 Irreducible Unitary Representations of $\mathcal{P}_n$

In the case we are interested in, the formula (1.43) can be further specified (cf. §1.2.1). Namely for $G = \mathcal{P}_n$, $n \geq 2$, we have $S = \mathrm{SO}_0(1, n-1)$, $N = \mathrm{T}^n$,

$$\hat{N} = \hat{\mathrm{T}}^n = \left\{ \chi = \begin{pmatrix} \chi_0 \\ \vdots \\ \chi_{n-1} \end{pmatrix} \middle| \chi_k \in \mathbb{R}, 0 \leq k \leq n-1 \right\} \tag{1.44}$$

and

$$\chi(a) = \exp\left\{ i \left(\chi_0 a_0 - \sum_{j=1}^{n-1} \chi_j a_j \right) \right\} \equiv \exp\{i\chi \bullet a\}, \quad \chi \in \hat{\mathrm{T}}^n, a \in \mathrm{T}^n. \tag{1.45}$$

Further, since the action Λa of $\mathrm{SO}_0(1, n-1)$ on T^n is the standard matrix multiplication, the (right) action on $\hat{\mathrm{T}}^n$ is represented by inverse-matrix multiplication $\Lambda^{-1}\chi$. All in all, we transformed (1.43) into the following form:

$$U_{\mathcal{O}_\xi, W}(\Lambda, a)\psi(\chi) = \sqrt{\rho(\chi, \Lambda)}\ \exp\{i\chi \bullet a\}\ W(k_{(\chi,\Lambda)})\ \psi(\Lambda^{-1}\chi). \tag{1.46}$$

Regarding the orbits of the (right) action of $\mathrm{SO}_0(1, n-1)$ on $\hat{\mathrm{T}}^n$, we will distinguish two cases. First, for $n = 4$, the classification of orbits was first given by Wigner in [45]. The generalization of his result for $n \geq 3$ is straightforward (cf. [3]) and we present it in Table 1.1. We denote $e_i \in \hat{\mathrm{T}}^n$ fulfilling $(e_i)_j = \delta_{ij}$.

Type	Orbit	Stabilized point	Stabilizer
0	$\xi = 0$	origin	$\mathrm{SO}_0(1, n-1)$
$\mathrm{I}^{\pm}$	$\xi \bullet \xi = 0$, $\pm\xi_0 > 0$	$\pm(e_0 + e_1)$	E_{n-2}
$\mathrm{II}^{\pm}_{\lvert m \rvert}$	$\xi \bullet \xi = \lvert m \rvert^2 > 0$, $\pm\xi_0 > 0$	$\pm\lvert m \rvert e_0$	$\mathrm{SO}(n-1, \mathbb{R})$
$\mathrm{III}_{\lvert m \rvert}$	$\xi \bullet \xi = -\lvert m \rvert^2 < 0$	$\lvert m \rvert e_1$	$\mathrm{SO}_0(1, n-2)$

Table 1.1: Orbits of the right action of $\mathrm{SO}_0(1, n-1)$ on $\hat{\mathrm{T}}^n$, $n \geq 3$; $|m| \in \mathbb{R}^+$

Here, for $n \in \mathbb{N}$,

$$\mathrm{SO}(n, \mathbb{R}) := \left\{ \Lambda \in \mathbb{R}^{n,n} \middle| \Lambda^T \Lambda = \mathbb{1}, \det \Lambda = 1 \right\} \subset \mathrm{GL}(n, \mathbb{R})$$

is the *special orthogonal group* (cf. [20], p. 5) and

$$\mathrm{E}_n := \mathrm{SO}(n, \mathbb{R}) \ltimes \mathrm{T}^n \subset \mathrm{GL}(n+1, \mathbb{R})$$

with the natural semidirect product is the *Euclidean group* (cf. [4], p. 431).

Second, somewhat special is the case $n = 2$ which cannot be contained in the table above. Roughly speaking, since $\hat{\mathrm{T}}^2$ is a "plane" and the rotation around e_0 is not available in this case, one cannot connect (by action of any element from $\mathrm{SO}_0(1,1)$) the ray standing for the axis of the first quadrant in the $\xi_1\xi_0$-plane with the axis of the second quadrant. Similarly, the respective rays in the half-plane $\xi_0 < 0$ cannot be connected either. Hence there are four distinct orbits of the type I now. Analogically, there are two distinct orbits of the type III for each $|m| \in \mathbb{R}^+$.

Our considerations, that will become evident later when the action of $\mathrm{SO}_0(1,1)$ will be stated explicitly, are summarized in Table 1.2. Notice that all the stabilizers are trivial (containing entirely the identity) except the one corresponding to the orbit $\xi = 0$ which is, in contrary, equal to the whole $\mathrm{SO}_0(1,1)$.

Type	Orbit	Stabilized point
0	$\xi = 0$	origin
$\mathrm{I}^{\pm}_{\varepsilon}$	$\xi \bullet \xi = 0, \pm\xi_0 > 0, \pm\varepsilon\xi_1 > 0$	$\pm(e_0 + \varepsilon e_1)$
$\mathrm{II}^{\pm}_{\|m\|}$	$\xi \bullet \xi = \|m\|^2 > 0, \pm\xi_0 > 0$	$\pm\|m\|e_0$
$\mathrm{III}^{\pm}_{\|m\|}$	$\xi \bullet \xi = -\|m\|^2 < 0, \pm\xi_1 > 0$	$\pm\|m\|e_1$

Table 1.2: Orbits of the right action of $\mathrm{SO}_0(1,1)$ on $\hat{\mathrm{T}}^2$; $|m| \in \mathbb{R}^+$, $\varepsilon = \pm 1$

Remark 1.16. Note that we shall be only interested in representations corresponding to non-trivial orbits. For orbits of type 0 we have $\exp(i\chi \bullet a) = 1$ and hence the resulting unitary operator (1.46) is independent of $a \in \mathrm{T}^n$. This means that such representations are not faithful.

Chapter 2

Representations of $\mathcal{P}_2$

The first Lie group to deal with is $\mathcal{P}_2 = \mathrm{SO}_0(1,1) \ltimes \mathrm{T}^2$. The notation from §1.2 is used for $n = 2$. In order to introduce the second-kind coordinates in $\mathcal{P}_2$, we compute

$$\exp(t_1 \mathbf{L}_{01}) = \exp \begin{pmatrix} 0 & -t_1 & 0 \\ -t_1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} \cosh t_1 & -\sinh t_1 & 0 \\ -\sinh t_1 & \cosh t_1 & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

$$\exp(t_2 \mathbf{P}_0) = \exp \begin{pmatrix} 0 & 0 & t_2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 & t_2 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

$$\exp(t_3 \mathbf{P}_1) = \exp \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & t_3 \\ 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & t_3 \\ 0 & 0 & 1 \end{pmatrix}.$$

It it not difficult to show that the coordinates

$$g : t \equiv (t_1, t_2, t_3) \mapsto g(t) \equiv e^{t_2 \mathbf{P}_0}\, e^{t_3 \mathbf{P}_1}\, e^{t_1 \mathbf{L}_{01}} = \begin{pmatrix} \cosh t_1 & -\sinh t_1 & t_2 \\ -\sinh t_1 & \cosh t_1 & t_3 \\ 0 & 0 & 1 \end{pmatrix} \tag{2.1}$$

are global in this case (see the Appendix). Therefore

$$\mathcal{P}_2 = \{ g(t_1, t_2, t_3) \equiv (\Lambda(t_1), a(t_2, t_3)) | t_1, t_2, t_3 \in \mathbb{R} \},$$

where

$$\Lambda(t_1) \equiv \begin{pmatrix} \cosh t_1 & -\sinh t_1 \\ -\sinh t_1 & \cosh t_1 \end{pmatrix} \in \mathrm{SO}_0(1,1) \quad \text{and} \quad a(t_2, t_3) \equiv \begin{pmatrix} t_2 \\ t_3 \end{pmatrix} \in \mathrm{T}^2.$$

From (2.1), group multiplication in terms of the coordinates can be easily uncovered. Namely for any $t \equiv (t_1, t_2, t_3), t' \equiv (t'_1, t'_2, t'_3) \in \mathbb{R}^3$ we have

$$g(t) \cdot g(t') = g(t_1 + t'_1, t_2 + t'_2 \cosh t_1 - t'_3 \sinh t_1, t_3 + t'_3 \cosh t_1 - t'_2 \sinh t_1). \tag{2.2}$$

2.1 Lie Field Technique

Now we shall make use of the method introduced in §1.1.4 to induce skew-symmetric representations of the Poincaré algebra $\mathfrak{p}_2$ from well-known representations of certain extended Weyl algebra. Further, integrating the representations by virtue of section §1.1.6, the complete family of irreducible unitary representations of the Lie group $\mathcal{P}_2$ shall be constructed in an unified way.

In agreement with the convention established in §1.2, the abstract Lie algebra $\mathfrak{p}_2$ is a three-dimensional real Lie algebra, generated by P_0, P_1 and L_{01} subject to

$$[L_{01}, P_0] = -P_1, \quad [L_{01}, P_1] = -P_0, \quad [P_0, P_1] = 0. \tag{2.3}$$

With respect to (1.13), in this case we have

$$\operatorname{index}\mathfrak{p}_2 = 3 - \operatorname{rank}_{\mathfrak{S}(\mathfrak{p}_2)} \begin{pmatrix} 0 & -P_1 & -P_0 \\ P_1 & 0 & 0 \\ P_0 & 0 & 0 \end{pmatrix} = 1,$$

therefore $\mathfrak{Z}(\mathfrak{p}_2)$ is generated by the only Casimir operator, namely (cf. [34], p. 226)

$$M^2 := P_1^2 - P_0^2. \tag{2.4}$$

Further, $\frac{1}{2}(3-1) = 1$ and hence it is reasonable to search for a connection to $\mathfrak{D}_{1,1}(\mathbb{R})$.

2.1.1 Isomorphism of $\mathfrak{D}(\mathfrak{p}_2)$ and $\mathfrak{D}_{1,1}(\mathbb{R})$

Let ε be either 1, or -1. From (2.3) we have

$$[\varepsilon L_{01}, P_0 - \varepsilon P_1] = \varepsilon(-P_1 + \varepsilon P_0) = P_0 - \varepsilon P_1,$$

hence

$$\begin{aligned} 1 &= (P_0 - \varepsilon P_1)^{-1}[\varepsilon L_{01}, P_0 - \varepsilon P_1] \\ &= (P_0 - \varepsilon P_1)^{-1}\varepsilon L_{01}(P_0 - \varepsilon P_1) - \varepsilon L_{01} \\ &= (P_0 - \varepsilon P_1)^{-1}\varepsilon L_{01}(P_0 - \varepsilon P_1) - (P_0 - \varepsilon P_1)(P_0 - \varepsilon P_1)^{-1}\varepsilon L_{01} \\ &= \left[(P_0 - \varepsilon P_1)^{-1}\varepsilon L_{01}, P_0 - \varepsilon P_1\right] \end{aligned}$$

and similarly

$$\begin{aligned} 1 &= [\varepsilon L_{01}, P_0 - \varepsilon P_1](P_0 - \varepsilon P_1)^{-1} \\ &= \varepsilon L_{01} - (P_0 - \varepsilon P_1)\varepsilon L_{01}(P_0 - \varepsilon P_1)^{-1} \\ &= \varepsilon L_{01}(P_0 - \varepsilon P_1)^{-1}(P_0 - \varepsilon P_1) - (P_0 - \varepsilon P_1)\varepsilon L_{01}(P_0 - \varepsilon P_1)^{-1} \\ &= \left[\varepsilon L_{01}(P_0 - \varepsilon P_1)^{-1}, P_0 - \varepsilon P_1\right]. \end{aligned}$$

Putting these relations together, we may also write

$$1 = \left[\frac{1}{2}\left((P_0 - \varepsilon P_1)^{-1}\varepsilon L_{01} + \varepsilon L_{01}(P_0 - \varepsilon P_1)^{-1}\right), P_0 - \varepsilon P_1\right]. \tag{2.5}$$

Making use of (1.17),

$$\begin{aligned} \varepsilon L_{01}(P_0 - \varepsilon P_1)^{-1} &= \left[\varepsilon L_{01}, (P_0 - \varepsilon P_1)^{-1}\right] + (P_0 - \varepsilon P_1)^{-1}\varepsilon L_{01} \\ &= -(P_0 - \varepsilon P_1)^{-1}(P_0 - \varepsilon P_1)(P_0 - \varepsilon P_1)^{-1} + (P_0 - \varepsilon P_1)^{-1}\varepsilon L_{01} \\ &= (P_0 - \varepsilon P_1)^{-1}(\varepsilon L_{01} - 1) \end{aligned}$$

and we finally rewrite (2.5) in the form

$$1 = \left[(P_0 - \varepsilon P_1)^{-1}\left(\varepsilon L_{01} - \frac{1}{2}\right), P_0 - \varepsilon P_1\right]. \tag{2.6}$$

Thus for $\hat{p}_\varepsilon, \hat{q}_\varepsilon \in \mathfrak{D}(\mathfrak{p}_2)$ defined by

$$\hat{p}_\varepsilon := (P_0 - \varepsilon P_1)^{-1}\left(\varepsilon L_{01} - \frac{1}{2}\right) = \frac{1}{2}\left[(P_0 - \varepsilon P_1)^{-1}\varepsilon L_{01} + \varepsilon L_{01}(P_0 - \varepsilon P_1)^{-1}\right], \tag{2.7}$$

$$\hat{q}_\varepsilon := P_0 - \varepsilon P_1, \tag{2.8}$$

we have $[\hat{p}_\varepsilon, \hat{q}_\varepsilon] = 1$. Moreover

$$\begin{aligned}
\hat{p}_\varepsilon^* &= \frac{1}{2}\left[(P_0 - \varepsilon P_1)^{-1}\varepsilon L_{01} + \varepsilon L_{01}(P_0 - \varepsilon P_1)^{-1}\right]^* \\
&= \frac{1}{2}\left(\varepsilon L_{01}^*\left[P_0^* - \varepsilon P_1^*\right)^{-1} + \left(P_0^* - \varepsilon P_1^*\right)^{-1}\varepsilon L_{01}^*\right] \\
&= \frac{1}{2}\left(\varepsilon L_{01}\left[P_0 - \varepsilon P_1\right)^{-1} + \left(P_0 - \varepsilon P_1\right)^{-1}\varepsilon L_{01}\right] \\
&= \hat{p}_\varepsilon
\end{aligned}$$

and

$$\hat{q}_\varepsilon^* = (P_0 - \varepsilon P_1)^* = -P_0 + \varepsilon P_1 = -\hat{q}_\varepsilon.$$

Since $[P_1, P_0] = 0$, for the Casimir operator we have

$$M^2 = -(P_0 - \varepsilon P_1)(P_0 + \varepsilon P_1). \tag{2.9}$$

Now the relations (2.7), (2.8) and (2.9) can be easily inverted as follows:

$$L_{01} = \varepsilon\left(\hat{q}_\varepsilon \hat{p}_\varepsilon + \frac{1}{2}\right), \tag{2.10}$$

$$P_0 = \frac{1}{2}\left(\hat{q}_\varepsilon - \hat{q}_\varepsilon^{-1}M^2\right), \tag{2.11}$$

$$P_1 = -\frac{\varepsilon}{2}\left(\hat{q}_\varepsilon + \hat{q}_\varepsilon^{-1}M^2\right). \tag{2.12}$$

Let us define the following linear mapping $\Psi_\varepsilon : \mathfrak{p}_2 \to \mathfrak{D}_{1,1}(\mathbb{R})$:

$$\Psi_\varepsilon(L_{01}) := \varepsilon\left(qp + \frac{1}{2}\right), \tag{2.13}$$

$$\Psi_\varepsilon(P_0) := \frac{1}{2}\left(q - q^{-1}\theta\right), \tag{2.14}$$

$$\Psi_\varepsilon(P_1) := -\frac{\varepsilon}{2}\left(q + q^{-1}\theta\right). \tag{2.15}$$

Clearly $\Psi_\varepsilon[x, y] = [\Psi_\varepsilon(x), \Psi_\varepsilon(y)]$ for any $x, y \in \mathfrak{p}_2$ because p, q and θ satisfy the same commutation relations as $\hat{p}_\varepsilon$, $\hat{q}_\varepsilon$ and M^2, respectively. Therefore Ψ_ε extends uniquely to a homomorphism $\Psi_\varepsilon : \mathfrak{U}(\mathfrak{p}_2) \to \mathfrak{D}_{1,1}(\mathbb{R})$ with $\Psi_\varepsilon(1) = 1$.

Proposition 2.1. *In $\mathfrak{W}_1(\mathbb{R})$ we have $p^n q = qp^n + np^{n-1}$ for all $n \in \mathbb{N}$.*

Proof. By induction. For $n = 1$ we have nothing else but the identity $pq - qp = 1$. For the inductive step, suppose $p^n q = qp^n + np^{n-1}$. Then

$$p^{n+1}q = pp^n q = p\left(qp^n + np^{n-1}\right) = p^{n+1} + [p, q]p^n + np^n = p^{n+1} + (n+1)p^n. \quad \square$$

Proposition 2.2. *In $\mathfrak{W}_1(\mathbb{R})$ we have $(qp)^n = q^n p^n + f_n(q, p)$, where $f_n \in \mathfrak{W}_1(\mathbb{R})$ contains p at most to the power of $n - 1$, for any $n \in \mathbb{N}$.*

Proof. We use induction again. For $n = 1$ the relation holds trivially with $f_1(q, p) = 0$. For the inductive step, we make use of the previous proposition to write

$$\begin{aligned}
(qp)^{n+1} &= (qp)^n qp = [q^n p^n + f_n(q, p)]\, qp = q^n\left(qp^n + np^{n-1}\right)p + f_n(q, p)qp \\
&= q^{n+1}p^{n+1} + f_{n+1}(q, p),
\end{aligned}$$

where $f_{n+1}(q, p) := nq^n p^n + f_n(q, p)qp \in \mathfrak{W}_1(\mathbb{R})$ contains at most p^n. $\square$

Lemma 2.3. *For $\Psi_\varepsilon : \mathfrak{U}(\mathfrak{p}_2) \to \mathfrak{D}_{1,1}(\mathbb{R})$ and $x \in \mathfrak{U}(\mathfrak{p}_2)$ one has $\Psi_\varepsilon(x) = 0$ only if $x = 0$.*

Proof. Due to PBW theorem we have $x = \sum_{j,k,l=0}^{N} \alpha_{j,k,l} P_0^j P_1^k L_{01}^l$ for some $N \in \mathbb{N}$ and $\alpha_{j,k,l} \in \mathbb{R}$, $0 \leq j,k,l \leq N$. Hence

$$\begin{aligned}
0 &= \Psi_\varepsilon(x) \\
&= \sum_{j,k,l=0}^{N} \alpha_{i,j,k} \Psi_\varepsilon(P_0)^j \Psi_\varepsilon(P_1)^k \Psi_\varepsilon(L_{01})^l \\
&= \sum_{l=0}^{N} \left[\sum_{j,k=0}^{N} \tilde{\alpha}_{j,k,l} \left(q - q^{-1}\theta\right)^j \left(q + q^{-1}\theta\right)^k \right] \left(qp + \frac{1}{2}\right)^l \\
&= f(q, q^{-1}, \theta, p) + \left[\sum_{j,k=0}^{N} \tilde{\alpha}_{j,k,N} \left(q - q^{-1}\theta\right)^j \left(q + q^{-1}\theta\right)^k \right] q^N p^N,
\end{aligned}$$

where $\tilde{\alpha}_{j,k,l} := \alpha_{j,k,l}(-1)^j \frac{\varepsilon^{k+l}}{2^{j+k}}$ and $f(q, q^{-1}, \theta, p) \in \mathfrak{D}_{1,1}(\mathbb{R})$ contains at most p^{N-1}. By custom of Theorem 1.9, we consequently have

$$\sum_{j,k=0}^{N} \tilde{\alpha}_{j,k,N} \left(q - q^{-1}\theta\right)^j \left(q + q^{-1}\theta\right)^k = 0.$$

But now we can write

$$\Psi_\varepsilon(x) = \sum_{l=0}^{N-1} \left[\sum_{j,k=0}^{N} \tilde{\alpha}_{j,k,l} \left(q - q^{-1}\theta\right)^j \left(q + q^{-1}\theta\right)^k \right] \left(qp + \frac{1}{2}\right)^l,$$

and hence, repeating the procedure N-times, we uncover that

$$\sum_{j,k=0}^{N} \tilde{\alpha}_{j,k,l} \left(q - q^{-1}\theta\right)^j \left(q + q^{-1}\theta\right)^k = 0 \tag{2.16}$$

holds for any $0 \leq l \leq N$.

Since all q, q^{-1} and θ all commute, (2.16) implies (cf. [7]) that, for an arbitrary l,

$$\sum_{j,k=0}^{N} \tilde{\alpha}_{j,k,l} \left(x - \frac{y}{x}\right)^j \left(x + \frac{y}{x}\right)^k = 0 \tag{2.17}$$

for $(x, y) \in \mathbb{R}^\times \times \mathbb{R}$. Because the Jacobian of mapping $u := \left(x - \frac{y}{x}\right)$, $v := \left(x + \frac{y}{x}\right)$ is

$$\det \begin{pmatrix} 1 + \frac{y}{x^2} & -\frac{1}{x} \\ 1 - \frac{y}{x^2} & \frac{1}{x} \end{pmatrix} = \frac{1}{x} + \frac{y}{x^3} + \frac{1}{x} - \frac{y}{x^3} = \frac{2}{x},$$

the mapping is regular on $\mathbb{R}^\times \times \mathbb{R}$, thus the polynomial $\sum_{j,k=0}^{N} \tilde{\alpha}_{j,k,l} u^j v^k = 0$ on an open subset of $\mathbb{R}^2$. Consequently, it is the zero polynomial, with $\tilde{\alpha}_{j,k,l} = 0$, and therefore finally $\alpha_{j,k,l} = 0$, for any $0 \leq j,k,l \leq N$. $\square$

An immediate consequence of the previous lemma is that $\Psi_\varepsilon : \mathfrak{U}(\mathfrak{p}_2) \to \mathfrak{D}_{1,1}(\mathbb{R})$ is injective and it can be extended uniquely to an (injective) homomorphism $\Psi_\varepsilon : \mathfrak{D}(\mathfrak{p}_2) \to \mathfrak{D}_{1,1}(\mathbb{R})$ such that $\Psi_\varepsilon(x^{-1}) = \Psi_\varepsilon(x)^{-1}$, $x \in \mathfrak{U}(\mathfrak{p}_2)$. Furthermore, it is easily seen from the above computations that the extended mapping is surjective - it is enough to realize

$$\Psi_\varepsilon^{(-1)}(p) = \hat{p}_\varepsilon, \quad \Psi_\varepsilon^{(-1)}(q) = \hat{q}_\varepsilon, \quad \Psi_\varepsilon^{(-1)}(\theta) = M^2.$$

Finally, since $\hat{p}_\varepsilon^* = \hat{p}_\varepsilon$, $\hat{q}_\varepsilon^* = -\hat{q}_\varepsilon$ and $(M^2)^* = (P_1^*)^2 - (P_0^*)^2 = P_1^2 - P_0^2 = M^2$, Ψ_ε is moreover involutive. All in all, the following theorem has been proven.

Theorem 2.4. *The mapping $\Psi_\varepsilon : \mathfrak{D}(\mathfrak{p}_2) \to \mathfrak{D}_{1,1}(\mathbb{R})$ is a $*$-isomorphism.*

2.1.2 Skew-symmetric Representations of $\mathfrak{p}_2$

Now, because $\Psi_\varepsilon(\mathfrak{p}_2) \subset \mathfrak{D}'_{1,1}(\mathbb{R})$, as seen from (2.13) - (2.15), we may apply the technique introduced in §1.1.4 in order to obtain skew-symmetric representations of $\mathfrak{p}_2$ from those of $\mathfrak{W}_{1,1}(\mathbb{R})$. In this case, we make use of the following family Φ_{m^2}, $m^2 \in \mathbb{R}$, of representations of $\mathfrak{W}_{1,1}(\mathbb{R})$ on $\mathcal{H}_1 \equiv L^2(\mathbb{R}^\times, \mathrm{d}x)$:

$$\Phi_{m^2}(p)\psi(x) = -i\partial_x\psi(x), \tag{2.18}$$

$$\Phi_{m^2}(q)\psi(x) = ix\psi(x), \tag{2.19}$$

$$\Phi_{m^2}(\theta)\psi(x) = m^2\psi(x). \tag{2.20}$$

Now the restriction of Ψ_ε to $\mathfrak{p}_2$, composed with Φ_{m^2}, provides us with the following family $\Omega_{m^2,\varepsilon}$, $m^2 \in \mathbb{R}$, $\varepsilon = \pm 1$, of skew-symmetric representations of $\mathfrak{p}_2$ on $\mathcal{H}_1$:

$$\Omega_{m^2,\varepsilon}(L_{01})\psi(x) = \varepsilon\left(x\partial_x + \frac{1}{2}\right)\psi(x), \tag{2.21}$$

$$\Omega_{m^2,\varepsilon}(P_0)\psi(x) = \frac{i}{2}\left(x + \frac{m^2}{x}\right)\psi(x), \tag{2.22}$$

$$\Omega_{m^2,\varepsilon}(P_1)\psi(x) = -\frac{i\varepsilon}{2}\left(x - \frac{m^2}{x}\right)\psi(x). \tag{2.23}$$

Remark 2.1. Recall that the domain of all operators considered here is assumed to be $C_0^\infty(\mathbb{R}^\times)$, as discussed in §1.1.4.

2.1.3 Irreducible Unitary Representations of $\mathcal{P}_2$

One-parameter Subgroups

The operators (2.21) - (2.23) can be easily integrated into one-parameter subgroups of unitary operators on $\mathcal{H}$. For any $t \in \mathbb{R}$ we define

$$U^{(1)}_{m^2,\varepsilon}(t)\psi(x) \equiv \exp\left\{t\,\Omega_{m^2,\varepsilon}(L_{01})\right\}\psi(x) = e^{\frac{\varepsilon t}{2}}\psi\left(e^{\varepsilon t}x\right), \tag{2.24}$$

$$U^{(2)}_{m^2,\varepsilon}(t)\psi(x) \equiv \exp\left\{t\,\Omega_{m^2,\varepsilon}(P_0)\right\}\psi(x) = e^{\frac{it}{2}\left(x+\frac{m^2}{x}\right)}\psi(x), \tag{2.25}$$

$$U^{(3)}_{m^2,\varepsilon}(t)\psi(x) \equiv \exp\left\{t\,\Omega_{m^2,\varepsilon}(P_1)\right\}\psi(x) = e^{-\frac{\varepsilon it}{2}\left(x-\frac{m^2}{x}\right)}\psi(x). \tag{2.26}$$

Nevertheless, one has to verify that the definition is well-posed, as explained in §1.1.6.

Lemma 2.5. *Let $\left\{U(t) \mid t \in \mathbb{R}\right\}$ be a set of unitary operators on $\mathcal{H}_m$, $m \in \mathbb{N}$, such that $U(t+s) = U(t)U(s)$ for any $t, s \in \mathbb{R}$.*
(a) If there are continuous functions

$$\alpha : \mathbb{R}^\times \times \mathbb{R}^{m-1} \times \mathbb{R} \to \mathbb{C},$$

$$X_1 : \mathbb{R}^\times \times \mathbb{R}^{m-1} \times \mathbb{R} \to \mathbb{R}^\times,$$

$$X_j : \mathbb{R}^\times \times \mathbb{R}^{m-1} \times \mathbb{R} \to \mathbb{R}, \quad 2 \le j \le m,$$

such that

$$U(t)\psi(x_1, \ldots, x_m) = \alpha(x_1, \ldots, x_m; t)\,\psi(X_1(x_1, \ldots, x_m; t), \ldots, X_m(x_1, \ldots, x_m; t)) \tag{2.27}$$

for any $\psi \in \mathcal{H}_m$, then the mapping $t \mapsto U(t)$ is strongly continuous (on $\mathbb{R}$).
(b) If the functions α and X_j, $1 \le j \le m$, are moreover differentiable, then

$$\lim_{t\to 0}\left[\frac{1}{t}\left(U^{(j)}(t) - \mathbb{1}\right)\psi\right]$$

exists for any $\psi \in C_0^\infty(\mathbb{R}^\times \times \mathbb{R}^{m-1})$.

Proof. To prove (a), one has to show the following holds for any $t_0 \in \mathbb{R}$ and $\psi \in \mathcal{H}_m$:

$$\lim_{t\to t_0} \|U(t)\psi - U(t_0)\psi\| = 0. \tag{2.28}$$

Since $U(t)$ is unitary and thus $\|U(t)\| = 1$, $t \in \mathbb{R}$, we have

$$\|U(t)\psi - U(t_0)\psi\| = \|U(t_0)\,[U(t-t_0)\psi - U(0)\psi]\| \leq \|U(t-t_0)\psi - \psi\|$$

and thus we may assume, without loss of generality, $t_0 = 0$. Further, it is sufficient to prove (2.28) for any ψ from a dense subset of $\mathcal{H}_m$.

Thus, take any $\psi \in C_0^\infty(\mathbb{R}^\times \times \mathbb{R}^{m-1})$ and $\delta > 0$. For any $x \in \mathbb{R}^\times \times \mathbb{R}^{m-1}$ and $t \in \langle\delta,\delta\rangle$ we have

$$\left|\overline{\psi(x)}\,U(t)\psi(x)\right| \leq \max_{\operatorname{supp}\psi\times\langle\delta,\delta\rangle} |\alpha(x;t)| \cdot \|\psi\|_\infty \cdot |\psi(x)| \equiv \hat{\psi}(x).$$

Clearly $\hat{\psi} \in \mathcal{H}_m$ and we may therefore use Lebesgue theorem in order to show

$$\begin{aligned}
\lim_{t\to 0}\,(\psi, U(t)\psi) &= \lim_{t\to 0}\int_{\mathbb{R}^m} \overline{\psi(x)}\,U(t)\psi(x)\,\mathrm{d}x \\
&= \int_{\mathbb{R}^m}\left[\lim_{t\to 0}\overline{\psi(x)}\,U(t)\psi(x)\right]\mathrm{d}x \\
&= \|\psi\|^2.
\end{aligned}$$

But then

$$\lim_{t\to 0}\|U(t)\psi - \psi\|^2 = \lim_{t\to 0}\left[\|U(t)\psi\|^2 - (\psi, U(t)\psi) - (U(t)\psi,\psi) + \|\psi\|^2\right] = 0.$$

Part (b) is trivial. □

Proposition 2.6. *For any $j = 1,2,3$, $m^2 \in \mathbb{R}$ and $\varepsilon = \pm 1$, $U^{(j)}_{m^2,\varepsilon}(t)$ are one-parameter subgroups of unitary operators on $\mathcal{H}_1$.*

Proof. Take arbitrary $m^2 \in \mathbb{R}$ and $\varepsilon = \pm 1$.

(a) First, $U^{(1)}_{m^2,\varepsilon}(t)\psi \in \mathcal{H}_1$ for any $t \in \mathbb{R}$ and $\psi \in \mathcal{H}_1$ because

$$\left\|U^{(1)}_{m^2,\varepsilon}(t)\psi\right\|^2 = \int_{\mathbb{R}}\left|e^{\frac{\varepsilon t}{2}}\psi(e^{\varepsilon t}x)\right|^2\mathrm{d}x = \int_{\mathbb{R}} e^{\varepsilon t}\,|\psi(y)|^2\,e^{-\varepsilon t}\mathrm{d}y = \int_{\mathbb{R}}|\psi(y)|^2\,\mathrm{d}y = \|\psi\|^2.$$

Second, for any $t, s \in \mathbb{R}$ we obviously have $U^{(1)}_{m^2,\varepsilon}(t+s) = U^{(1)}_{m^2,\varepsilon}(t)\,U^{(1)}_{m^2,\varepsilon}(s)$, in particular $U^{(1)}_{m^2,\varepsilon}(t)^{-1} = U^{(1)}_{m^2,\varepsilon}(-t)$. Third, for any $\phi, \psi \in \mathcal{H}_1$ and $t \in \mathbb{R}$ we can write

$$\begin{aligned}
\left(\phi, U^{(1)}_{m^2,\varepsilon}(t)\psi\right) &= \int_{\mathbb{R}}\overline{\phi(x)}\,e^{\frac{\varepsilon t}{2}}\psi(e^{\varepsilon t}x)\,\mathrm{d}x = \int_{\mathbb{R}}\overline{\phi(e^{-\varepsilon t}y)}e^{\frac{\varepsilon t}{2}}\varepsilon^{-\varepsilon t}\psi\,(y)\,\mathrm{d}y \\
&= \int_{\mathbb{R}}\overline{e^{-\frac{\varepsilon t}{2}}\phi(e^{-\varepsilon t}y)}\psi(y)\mathrm{d}y = \left(U^{(1)}_{m^2,\varepsilon}(-t)\phi,\psi\right),
\end{aligned}$$

hence $U^{(1)}_{m^2,\varepsilon}(t)^* = U^{(1)}_{m^2,\varepsilon}(-t) = U^{(1)}_{m^2,\varepsilon}(t)^{-1}$. Finally, we can see that all the assumptions of Lemma 2.5 are fulfilled, and thus $U^{(1)}_{m^2,\varepsilon}(t)$ is strongly continuous in t.

(b) As above, $U^{(2)}_{m^2,\varepsilon}(t)\psi \in \mathcal{H}_1$ for any $t \in \mathbb{R}$ and $\psi \in \mathcal{H}_1$ since

$$\left\|U^{(2)}_{m^2,\varepsilon}(t)\psi\right\|^2 = \int_{\mathbb{R}}\left|e^{\frac{it}{2}\left(x+\frac{m^2}{x}\right)}\psi(x)\right|^2\mathrm{d}x = \int_{\mathbb{R}}|\psi(x)|^2\,\mathrm{d}x = \|\psi\|^2.$$

Also in this case is clear that $U^{(2)}_{m^2,\varepsilon}(t+s) = U^{(2)}_{m^2,\varepsilon}(t)\,U^{(2)}_{m^2,\varepsilon}(s)$, $t, s \in \mathbb{R}$, and

$$\begin{aligned}
\left(\phi, U^{(2)}_{m^2,\varepsilon}(t)\psi\right) &= \int_{\mathbb{R}}\overline{\phi(x)}\,e^{\frac{it}{2}\left(x+\frac{m^2}{x}\right)}\psi(x)\mathrm{d}x = \int_{\mathbb{R}}\overline{e^{-\frac{it}{2}\left(x+\frac{m^2}{x}\right)}\phi(x)}\,\psi(x)\mathrm{d}x \\
&= \left(U^{(2)}_{m^2,\varepsilon}(-t)\phi,\psi\right),
\end{aligned}$$

for any $\phi, \psi \in \mathcal{H}_1$ and $t \in \mathbb{R}$, therefore $U^{(2)}_{m^2,\varepsilon}(t)^* = U^{(2)}_{m^2,\varepsilon}(t)^{-1}$. Finally, strong continuity follows from Lemma 2.5 again.

(c) For $U^{(3)}_{m^2,\varepsilon}(t)$ the proof is completely analogous to the previous case. $\square$

Notice that, for any $m^2 \in \mathbb{R}$ and $\varepsilon = \pm 1$, assumptions of part (b) of Lemma 2.5 are fulfilled as well and hence the generators for one-parameter subgroups $U_{m^2,\varepsilon}$ extend the respective operators $\Omega_{m^2,\varepsilon}$. This finally justify labelling by "exp" in (2.24) - (2.26).

Unitary Representations

Take any $m^2 \in \mathbb{R}$ and $\varepsilon = \pm 1$. As a consequence of Proposition 2.6, the mapping

$$(t_1, t_2, t_3) \mapsto U_{m^2,\varepsilon}(t_1, t_2, t_3) \equiv U^{(2)}_{m^2,\varepsilon}(t_2) U^{(3)}_{m^2,\varepsilon}(t_3) U^{(1)}_{m^2,\varepsilon}(t_1)$$

maps from $\mathbb{R}^3$ to $\mathcal{U}(\mathcal{H}_1)$ and it is unitary and strongly continuous. Explicitly,

$$U_{m^2,\varepsilon}(t_1, t_2, t_3)\psi(x) = \exp\left\{\frac{\varepsilon t_1}{2} + \frac{it_2}{2}\left(x + \frac{m^2}{x}\right) - \frac{\varepsilon i t_3}{2}\left(x - \frac{m^2}{x}\right)\right\} \psi\left(e^{\varepsilon t_1} x\right). \tag{2.29}$$

We claim that, for each m^2 and ε, (2.29) defines a unitary representation $U_{m^2,\varepsilon}$ of $\mathcal{P}_2$ by

$$g(t_1, t_2, t_3) \equiv (\Lambda(t_1), a(t_2, t_3)) \in \mathcal{P}_2 \mapsto U_{m^2,\varepsilon}(t_1, t_2, t_3) \tag{2.30}$$

In order to confirm this assertion, it only remains to verify (2.30) is a homomorphism, i.e. the composition rule (2.2) is respected.

Proposition 2.7. *For any $t_1, t_2, t_3, t_1', t_2', t_3' \in \mathbb{R}$ we have*

$$\begin{aligned} &U_{m^2,\varepsilon}(t_1, t_2, t_3) U_{m^2,\varepsilon}(t_1', t_2', t_3') \\ &\quad = U_{m^2,\varepsilon}(t_1 + t_1', t_2 + t_2' \cosh t_1 - t_3' \sinh t_1, t_3 + t_3' \cosh t_1 - t_2' \sinh t_1) \end{aligned} \tag{2.31}$$

Proof. For any $\psi \in \mathcal{H}_1$ we have

$$\begin{aligned} &U_{m^2,\varepsilon}(t_1, t_2, t_3) U_{m^2,\varepsilon}(t_1', t_2', t_3')\psi(x) \\ &\quad = U_{m^2,\varepsilon}(t_1, t_2, t_3) e^{\frac{\varepsilon t_1'}{2} + \frac{it_2'}{2}\left(x + \frac{m^2}{x}\right) - \frac{\varepsilon i t_3'}{2}\left(x - \frac{m^2}{x}\right)} \psi\left(e^{\varepsilon t_1'} x\right) \\ &\quad = e^{\frac{\varepsilon(t_1 + t_1')}{2} + \frac{it_2}{2}\left(x + \frac{m^2}{x}\right) + \frac{it_2'}{2}\left(e^{\varepsilon t_1} x + \frac{m^2}{e^{\varepsilon t_1} x}\right) - \frac{\varepsilon i t_3}{2}\left(x - \frac{m^2}{x}\right) - \frac{\varepsilon i t_3'}{2}\left(e^{\varepsilon t_1} x - \frac{m^2}{e^{\varepsilon t_1} x}\right)} \psi\left(e^{\varepsilon(t_1 + t_1')} x\right) \\ &\quad = e^{\frac{\varepsilon(t_1 + t_1')}{2} + \frac{it_2}{2}\left(x + \frac{m^2}{x}\right) - \frac{\varepsilon i t_3}{2}\left(x - \frac{m^2}{x}\right) + \frac{i}{2}\left[t_2'\left(e^{\varepsilon t_1} x + \frac{m^2}{e^{\varepsilon t_1} x}\right) - \varepsilon t_3'\left(e^{\varepsilon t_1} x - \frac{m^2}{e^{\varepsilon t_1} x}\right)\right]} \psi\left(e^{\varepsilon(t_1 + t_1')} x\right), \end{aligned}$$

while

$$\begin{aligned} &U_{m^2,\varepsilon}(t_1 + t_1', t_2 + t_2' \cosh t_1 - t_3' \sinh t_1, t_3 + t_3' \cosh t_1 - t_2' \sinh t_1)\psi(x) \\ &\quad = e^{\frac{\varepsilon(t_1 + t_1')}{2} + \frac{i(t_2 + t_2' \cosh t_1 - t_3' \sinh t_1)}{2}\left(x + \frac{m^2}{x}\right) - \frac{\varepsilon i(t_3 + t_3' \cosh t_1 - t_2' \sinh t_1)}{2}\left(x - \frac{m^2}{x}\right)} \psi\left(e^{\varepsilon(t_1 + t_1')} x\right) \\ &\quad = e^{\frac{\varepsilon(t_1 + t_1')}{2} + \frac{it_2}{2}\left(x + \frac{m^2}{x}\right) - \frac{\varepsilon i t_3}{2}\left(x - \frac{m^2}{x}\right)} e^{\frac{i}{2}\left[(t_2' \cosh t_1 - t_3' \sinh t_1)\left(x + \frac{m^2}{x}\right) - \varepsilon(t_3' \cosh t_1 - t_2' \sinh t_1)\left(x - \frac{m^2}{x}\right)\right]} \\ &\qquad \times \psi\left(e^{\varepsilon(t_1 + t_1')} x\right). \end{aligned}$$

Now the proof is complete, since

$$\begin{aligned} &(t_2' \cosh t_1 - t_3' \sinh t_1)\left(x + \frac{m^2}{x}\right) - \varepsilon(t_3' \cosh t_1 - t_2' \sinh t_1)\left(x - \frac{m^2}{x}\right) \\ &\quad = (t_2' \cosh \varepsilon t_1 - \varepsilon t_3' \sinh \varepsilon t_1)\left(x + \frac{m^2}{x}\right) - (\varepsilon t_3' \cosh \varepsilon t_1 - t_2' \sinh \varepsilon t_1)\left(x - \frac{m^2}{x}\right) \end{aligned}$$

$$
\begin{aligned}
&= \frac{t_2'}{2}\left(e^{\varepsilon t_1}x + \frac{e^{\varepsilon t_1}m^2}{x} + \frac{x}{e^{\varepsilon t_1}} + \frac{m^2}{e^{\varepsilon t_1}x} + e^{\varepsilon t_1}x - \frac{e^{\varepsilon t_1}m^2}{x} - \frac{x}{e^{\varepsilon t_1}} + \frac{m^2}{e^{\varepsilon t_1}x}\right)\\
&\quad - \frac{\varepsilon t_3'}{2}\left(e^{\varepsilon t_1}x + \frac{e^{\varepsilon t_1}m^2}{x} - \frac{x}{e^{\varepsilon t_1}} - \frac{m^2}{e^{\varepsilon t_1}x} + e^{\varepsilon t_1}x - \frac{e^{\varepsilon t_1}m^2}{x} + \frac{x}{e^{\varepsilon t_1}} - \frac{m^2}{e^{\varepsilon t_1}x}\right)\\
&= t_2'\left(e^{\varepsilon t_1}x + \frac{m^2}{e^{\varepsilon t_1}x}\right) - \varepsilon t_3'\left(e^{\varepsilon t_1}x - \frac{m^2}{e^{\varepsilon t_1}x}\right). \qquad \square
\end{aligned}
$$

Remark 2.2. Since both groups $\mathrm{SO}_0(1,1)$ and T^2 are obviously simply connected, so is $\mathcal{P}_2 \equiv \mathrm{SO}_0(1,1) \ltimes \mathrm{T}^2$ (cf. [37], p. 224). Therefore the Lie groups $\{U_{m^2,\varepsilon}(t) | t \in \mathbb{R}^3\}$ are all isomorphic to $\mathcal{P}_2$ itself and no discussion as outlined at the end of §1.1.6 is needed in the case.

Irreducibility

Let us now discuss irreducibility of the representations. Take again any real m^2 and $\varepsilon = \pm 1$. It is clear directly from (2.29) that $U_{m^2,\varepsilon}$ is reducible; it possesses two invariant subspaces, namely $\mathcal{H}_1^+ \equiv L^2(\mathbb{R}^+, \mathrm{d}x)$ and $\mathcal{H}_1^- \equiv L^2(\mathbb{R}^-, \mathrm{d}x)$. Thus let us denote

$$
U^{\pm}_{m^2,\varepsilon}(t_1,t_2,t_3) := U_{m^2,\varepsilon}(t_1,t_2,t_3)\big|_{\mathcal{H}_1^{\pm}}. \tag{2.32}
$$

Notice please, that here as well as everywhere else, the signum of ε is independent of any other considered or explicitly stated sign.

It turns out that no "finer" invariant subspaces exist. In other words,

Proposition 2.8. *Each of the representations $U^{\pm}_{m^2,\varepsilon}$ is irreducible.*

Proof. We shall follow the Schur's lemma (cf. [4], p. 144). Let $m^2 \in \mathbb{R}$ and $\varepsilon = \pm 1$ be fixed. Consider $T \in \mathcal{B}(\mathcal{H}_1^{\pm})$ such that $TU^{\pm}_{m^2,\varepsilon}(t_1,t_2,t_3) = U^{\pm}_{m^2,\varepsilon}(t_1,t_2,t_3)T$, for all $t_1, t_2, t_3 \in \mathbb{R}$. Then the same rule must hold also for the restrictions of the generators for one-parameter subgroups to $\mathcal{H}_1^{\pm}$, i.e. for any $\Omega^{\pm}_{m^2,\varepsilon}(z) := \Omega_{m^2,\varepsilon}(z)\big|_{C_0^\infty(\mathbb{R}^{\pm})}$, $z \in \mathfrak{p}_2$.

In particular, for $\Omega^{\pm}_{m^2,\varepsilon}(P_0 - \varepsilon P_1)\psi(x) = ix\psi(x)$, $\psi \in C_0^\infty(\mathbb{R}^{\pm})$, the requirement of commutativity with T implies $T\psi(x) = \tau(x)\psi(x)$ for some bounded function $\tau\colon \mathbb{R}^{\pm} \to \mathbb{C}$ (cf. [6], p. 180 and 233). Commuting T further with $U^{\pm}_{m^2,\varepsilon}(t_1,0,0)$, one finds the condition $\tau(e^{t_1 x}) = \tau(x)$ has to be satisfied for almost any $x \in \mathbb{R}^{\pm}$. But this could not be fulfilled without $\tau(x)$ being constant. Thus, T is in fact a multiple of the identity. $\square$

Mutual Non-equivalence

It only remains to answer the question whether the constructed representations can be mutually equivalent. At this place we recall several rough but useful criteria for irreducible unitary group representations to be non-equivalent.

Lemma 2.9. *Let U_1, U_2 be irreducible unitary representations of a real Lie group G on Hilbert spaces $\mathcal{H}^{(1)}$ and $\mathcal{H}^{(2)}$, respectively. Let Φ_1 and Φ_2 denote the extensions of the respective induced representations of the Lie algebra $\mathfrak{g}$ of G to $\mathfrak{U}(\mathfrak{g})$. If U_1 and U_2 are equivalent, then*
(a) if $x \in \mathfrak{Z}(\mathfrak{g})$ is a quadratic Casimir element (i.e. $x \in \mathfrak{g}^{\otimes 2}$) and hence $\Phi_j(x) = \alpha_j \mathbb{1}$ for some $\alpha_j \in \mathbb{R}$, $j = 1,2$, then $\alpha_1 = \alpha_2$;
(b) for any $x \in \mathfrak{U}(\mathfrak{g})$, $\sigma[\Phi_1(x)] = \sigma[\Phi_2(x)]$.

Proof. Part (a) is obviously a special case of (b). Thus, consider $\alpha \in \mathbb{C}$ such that there is $\psi \in \mathcal{H}^{(1)}$ with $\psi \neq 0$ and $\Phi_1(x)\psi = \alpha\psi$. Since $U_1 \cong U_2$, there is an isometry $\mathcal{R}\colon \mathcal{H}^{(1)} \to \mathcal{H}^{(2)}$ with $\mathcal{R}U_1(g) = U_2(g)\mathcal{R}$ for any $g \in G$. But then also $\mathcal{R}\Phi_1(x) = \Phi_2(x)\mathcal{R}$ for any $x \in \mathfrak{U}(\mathfrak{g})$. Then $\Phi_2(x)\mathcal{R}\psi = \mathcal{R}\Phi_1(x)\psi = \alpha\mathcal{R}\psi$ and hence $\alpha \in \sigma[\Phi_2(x)]$. $\square$

Recall that the restrictions of the considered skew-symmetric Lie algebra representations corresponding to $U^{\pm}_{m^2,\varepsilon}$ are

$$\Omega^{\pm}_{m^2,\varepsilon}(x) := \Omega_{m^2,\varepsilon}(x)\big|_{C_0^\infty(\mathbb{R}^{\pm})}, \tag{2.33}$$

$x \in \mathfrak{p}_2$, $m^2 \in \mathbb{R}$, $\varepsilon = \pm 1$. Extension to $\mathfrak{U}(\mathfrak{p}_2)$ is straightforward. It is easily seen from part (a) of Lemma 2.9 that the representations $U^{\pm}_{m^2,\varepsilon}$ corresponding to distinct values of the parameter m^2 cannot be equivalent. Let us now fix m^2 and let us look at the four representations $U^{\pm}_{m^2,\varepsilon}$ in some detail.

Consider first the case $m^2 = 0$. For any $\psi \in C_0^\infty(\mathbb{R}^{\pm})$ we have

$$\Omega^{\pm}_{0,\varepsilon}(P_0)\psi(x) = \frac{ix}{2}\psi(x) \quad \text{and} \quad \Omega^{\pm}_{0,\varepsilon}(P_1)\psi(x) = -\frac{i\varepsilon x}{2}\psi(x)$$

According to [6], p. 102, none of operators $\Omega^{\pm}_{0,\varepsilon}(P_j)$, $j = 0, 1$, has empty spectrum. But $\sigma\left[\Omega^{+}_{0,\varepsilon}(P_0)\right] \subset i\mathbb{R}^+$ while $\sigma\left[\Omega^{-}_{0,\varepsilon}(P_0)\right] \subset i\mathbb{R}^-$, regardless what ε is. Similarly, $\sigma\left[\Omega^{\pm}_{0,+1}(P_1)\right] \subset i\mathbb{R}^{\mp}$ simultaneously with $\sigma\left[\Omega^{\pm}_{0,-1}(P_1)\right] \subset i\mathbb{R}^{\pm}$. Altogether this means that all the four "massless" representations are pairwise non-equivalent.

For $m^2 \neq 0$ fixed, the same argument can be used only partially. Now

$$\Omega^{\pm}_{m^2,\varepsilon}(P_0)\psi(x) = \frac{i}{2}\left(x + \frac{m^2}{x}\right)\psi(x) \quad \text{and} \quad \Omega^{\pm}_{m^2,\varepsilon}(P_1)\psi(x) = -\frac{i\varepsilon}{2}\left(x - \frac{m^2}{x}\right)\psi(x),$$

$\psi \in C_0^\infty(\mathbb{R}^{\pm})$. Although the spectra are non-empty again (cf. [6]), for $m^2 > 0$ only $\Omega^{\pm}_{m^2,\varepsilon}(P_0)$ could be useful since $\operatorname{sgn}\left(x - \frac{m^2}{x}\right)$ varies. Similarly, for $m^2 < 0$ only the other generator is available. Therefore, comparing the spectra of $\Omega^{\pm}_{m^2>0,\varepsilon}(P_0)$ and $\Omega^{\pm}_{m^2<0,\varepsilon}(P_1)$, respectively, we obtain $U^{+}_{m^2,\varepsilon} \ncong U^{-}_{m^2,\varepsilon}$, for any $m^2 \neq 0$, but we are not able to prove inequivalence for distinct ε. In fact,

Proposition 2.10. *If $m^2 > 0$, then $U^{\pm}_{m^2,+1} \cong U^{\pm}_{m^2,-1}$. If $m^2 < 0$, then $U^{\pm}_{m^2,+1} \cong U^{\mp}_{m^2,-1}$.*

Instead of searching the respective isometries to show the equivalences now, proof of the assertion as well as the isometry mappings will be given later, as an elegant consequence of comparison with "Mackey's" list of representations of $\mathcal{P}_2$. With respect to the proposition, we may denote, for $m^2 \neq 0$, $U^{\pm}_{m^2} := U^{\pm}_{m^2,+1}$.

Summary

To summarize, the following theorem holds.

Theorem 2.11. *The set $\left\{U^{\pm}_{m^2} \equiv U^{\pm}_{m^2,+1} \,\middle|\, m^2 \in \mathbb{R}^\times\right\} \cup \left\{U^{\pm}_{0,\varepsilon} \,\middle|\, \varepsilon = \pm 1\right\}$, where*

$$U^{\pm}_{m^2,\varepsilon}(t_1,t_2,t_3)\psi(x) = \exp\left\{\frac{\varepsilon t_1}{2} + \frac{it_2}{2}\left(x + \frac{m^2}{x}\right) - \frac{\varepsilon i t_3}{2}\left(x - \frac{m^2}{x}\right)\right\}\psi\left(e^{\varepsilon t_1}x\right), \tag{2.34}$$

where $m^2 \in \mathbb{R}$, $\varepsilon = \pm 1$ and $\psi \in L^2(\mathbb{R}^{\pm})$, is a family of pairwise non-equivalent irreducible unitary representations of the Lie group $\mathcal{P}_2$.

Above all, we shall see below, by comparison with the representations constructed within the frame of Mackey theory, that our construction exhausts the whole list of all irreducible unitary representations of the Lie group $\mathcal{P}_2$.

2.2 Mackey's Technique

In order to independently verify our results presented in the previous section, we shall construct the set of irreducible unitary representations of the Lie group $\mathcal{P}_2$ within the (standard) framework of Mackey theory. Namely we shall make use of the device introduced in §1.3.3, for $n = 2$.

The dual group to T^2 is

$$\hat{T}^2 = \left\{ \begin{pmatrix} \chi_0 \\ \chi_1 \end{pmatrix} \middle| \chi_0, \chi_1 \in \mathbb{R} \right\}. \tag{2.35}$$

Any (non-zero) orbit is expressed as

$$\mathcal{O}_\xi = \left\{ \Lambda^{-1}\xi \middle| \Lambda \in \mathrm{SO}_0(1,1) \right\} = \left\{ \chi(x) := \Lambda(x)^{-1}\xi = \Lambda(-x)\xi \middle| x \in \mathbb{R} \right\} \cong \mathbb{R}, \tag{2.36}$$

hence it inherits the Lebesgue measure, namely we put $\mu(\chi(x)) := x$. Then

$$\mu\left[\Lambda(t_1)^{-1}\chi(x)\right] = \mu\left[\Lambda(-t_1)\Lambda(-x)\xi\right] = \mu\left[\Lambda(-t_1 - x)\right] = \mu\left[\chi(x+t_1)\right] = x + t_1,$$

$x, t_1 \in \mathbb{R}$, and therefore, as $\mathrm{d}x = \mathrm{d}(x+t_1)$, we have $\rho \equiv 1$. Further, since $S_\xi = \{1\}$ in each case, the general formula (1.46) takes the following form:

$$\begin{aligned} U_{\mathcal{O}_\xi}(t_1, t_2, t_3)\psi(\chi(x)) &\equiv U_{\mathcal{O}_\xi}\left(\Lambda(t_1), a(t_2, t_3)\right)\psi(\chi(x)) \\ &= \exp\left\{i\chi(x)\cdot a(t_2,t_3)\right\}\psi\left(\Lambda(t_1)^{-1}\chi(x)\right) \\ &= \exp\left\{i(\chi_0(x)t_2 - \chi_1(x)t_3)\right\}\psi(\chi(x+t_1)), \end{aligned}$$

$\psi \in L^2(\mathcal{O}_\xi, \mathrm{d}\mu)$. However, since $\mathcal{O}_\xi \cong \mathbb{R}$ and $\mathrm{d}\mu(\chi(x)) = \mathrm{d}x$, we may identify $\psi(\chi(x)) \equiv \psi(x)$ in order to finally obtain, for any $\psi \in L^2(\mathbb{R}, \mathrm{d}x)$,

$$U_{\mathcal{O}_\xi}(t_1, t_2, t_3)\psi(x) = \exp\left\{i(\chi_0(x)t_2 - \chi_1(x)t_3)\right\}\psi(x+t_1). \tag{2.37}$$

2.2.1 Orbits of Type I

First, let us take an orbit of type $\mathrm{I}^\pm_\varepsilon$, $\varepsilon = \pm 1$.[1] In this case we have $\xi = \pm\begin{pmatrix} 1 \\ \varepsilon \end{pmatrix}$, hence

$$\chi(x) = \Lambda(-x)\xi = \pm\begin{pmatrix} \cosh x & \sinh x \\ \sinh x & \cosh x \end{pmatrix}\begin{pmatrix} 1 \\ \varepsilon \end{pmatrix} = \pm\begin{pmatrix} \cosh \varepsilon x + \sinh \varepsilon x \\ \varepsilon \sinh \varepsilon x + \varepsilon \cosh \varepsilon x \end{pmatrix} = \pm e^{\varepsilon x}\begin{pmatrix} 1 \\ \varepsilon \end{pmatrix}$$

and therefore the respective representation is of the form, $\psi \in L^2(\mathbb{R})$,

$$U^{\mathrm{I}}_{\varepsilon,\pm}(t_1, t_2, t_3)\psi(x) = \exp\left\{\pm i e^{\varepsilon x}(t_2 - \varepsilon t_3)\right\}\psi(x+t_1). \tag{2.38}$$

2.2.2 Orbits of Type II

Second, for an orbit of type $\mathrm{II}^\pm_{|m|}$, $|m| > 0$, we have $\xi = \pm\begin{pmatrix} |m| \\ 0 \end{pmatrix}$, then

$$\chi(x) = \pm\begin{pmatrix} \cosh x & \sinh x \\ \sinh x & \cosh x \end{pmatrix}\begin{pmatrix} |m| \\ 0 \end{pmatrix} = \pm|m|\begin{pmatrix} \cosh x \\ \sinh x \end{pmatrix}$$

and therefore the representation is, $\psi \in L^2(\mathbb{R})$,

$$U^{\mathrm{II}}_{|m|,\pm}(t_1, t_2, t_3)\psi(x) = \exp\left\{\pm i|m|\left(t_2\cosh x - t_3 \sinh x\right)\right\}\psi(x+t_1) \tag{2.39}$$

[1] Notice please, that the signum of ε is independent of the sign in the superscript of the orbit type.

2.2.3 Orbits of Type III

Finally, if an orbit is of type $\mathrm{III}^{\pm}_{|m|}$, $|m| > 0$, then $\xi = \pm \begin{pmatrix} 0 \\ |m| \end{pmatrix}$, $\chi(x) = \pm |m| \begin{pmatrix} \sinh x \\ \cosh x \end{pmatrix}$ and thus we obtain, $\psi \in L^2(\mathbb{R})$,

$$U^{\mathrm{III}}_{|m|,\pm}(t_1, t_2, t_3)\psi(x) = \exp\left\{\pm i\, |m|\, (t_2 \sinh x - t_3 \cosh x)\right\} \psi(x + t_1). \tag{2.40}$$

2.3 Comparison of Results

Now we shall show that our approach to construction of irreducible unitary representations of the Lie group $\mathcal{P}_2$ is completely equivalent to the Mackey's technique.

2.3.1 Spectra of Generators and Casimir Operators

To begin with, we shall determine how certain elements of the Lie algebra $\mathfrak{p}_2$ and its enveloping algebra $\mathfrak{U}(\mathfrak{p}_2)$ are represented within the representations on $L^2(\mathbb{R})$, denote them Θ, induced by the Lie group representations U constructed in the previous section. Then we may be able to compare, at least indirectly, the results obtained there with the representations constructed from Lie field correspondence. Namely, Lemma 2.9 comparing spectra of represented operators is used for this purpose.

First, for the representations of type I we have

$$\Theta^{\mathrm{I}}_{\varepsilon,\pm}(P_0) \equiv \frac{\mathrm{d}}{\mathrm{d}t_2} U^{\mathrm{I}}_{\varepsilon,\pm}(0, t_2, 0)\Big|_{t_2=0} = \left(\pm i e^{\varepsilon x}\right) \mathbb{1}, \tag{2.41}$$

$$\Theta^{\mathrm{I}}_{\varepsilon,\pm}(P_1) \equiv \frac{\mathrm{d}}{\mathrm{d}t_3} U^{\mathrm{I}}_{\varepsilon,\pm}(0, 0, t_3)\Big|_{t_3=0} = \left(\mp i \varepsilon e^{\varepsilon x}\right) \mathbb{1} \tag{2.42}$$

and hence

$$\Theta^{\mathrm{I}}_{\varepsilon,\pm}(M^2) = -\left[\Theta^{\mathrm{I}}_{\varepsilon,\pm}(P_0)\right]^2 + \left[\Theta^{\mathrm{I}}_{\varepsilon,\pm}(P_1)\right]^2 = \left(e^{2\varepsilon x}\right) \mathbb{1} - \left(e^{2\varepsilon x}\right) \mathbb{1} = 0. \tag{2.43}$$

Analogously, for the type II of representations we have

$$\Theta^{\mathrm{II}}_{|m|,\pm}(P_0) = \left(\pm i\, |m| \cosh x\right) \mathbb{1}, \tag{2.44}$$

$$\Theta^{\mathrm{II}}_{|m|,\pm}(P_1) = \left(\mp i\, |m| \sinh x\right) \mathbb{1}, \tag{2.45}$$

$$\Theta^{\mathrm{II}}_{|m|,\pm}(M^2) = |m|^2\, \mathbb{1}. \tag{2.46}$$

Finally, the representations of type III induce

$$\Theta^{\mathrm{III}}_{|m|,\pm}(P_0) = \left(\pm i\, |m| \sinh x\right) \mathbb{1}, \tag{2.47}$$

$$\Theta^{\mathrm{III}}_{|m|,\pm}(P_1) = \left(\mp i\, |m| \cosh x\right) \mathbb{1}, \tag{2.48}$$

$$\Theta^{\mathrm{III}}_{|m|,\pm}(M^2) = -\,|m|^2\, \mathbb{1}. \tag{2.49}$$

Now, comparing how the Casimir operator M^2 is represented by Θ and Ω, we have the following correspondences:

$$\left\{U^{\pm}_{0,\varepsilon} \,\middle|\, \varepsilon = \pm 1\right\} \longleftrightarrow \left\{U^{\mathrm{I}}_{\varepsilon,\pm} \,\middle|\, \varepsilon = \pm 1\right\}, \tag{2.50}$$

$$\left\{U^{\pm}_{m^2}\right\} \longleftrightarrow \left\{U^{\mathrm{II}}_{|m|,\pm}\right\}, 0 < m^2 = |m|^2, \tag{2.51}$$

$$\left\{U^{\pm}_{m^2}\right\} \longleftrightarrow \left\{U^{\mathrm{III}}_{|m|,\pm}\right\}, 0 > m^2 = -\,|m|^2. \tag{2.52}$$

By this notation we mean that just representations from the corresponding sets could be eventually equivalent. To uncover if and which representations are equivalent indeed, one has to inspect spectra of represented operators P_0 and P_1.

Thus, first we have $\sigma\left[\Theta^{\mathrm{I}}_{\varepsilon,\pm}(P_0)\right] \subset i\mathbb{R}^{\pm}$ and $\sigma\left[\Theta^{\mathrm{I}}_{\varepsilon,\pm}(P_1)\right] \subset i\varepsilon\mathbb{R}^{\mp}$, $\varepsilon = \pm 1$. Comparing with the discussion on spectra in 2.1.3, one can see that there is only one possibility of mutual correspondence within (2.50), namely, $\varepsilon = \pm 1$,

$$U^{\pm}_{0,\varepsilon} \longleftrightarrow U^{\mathrm{I}}_{\varepsilon,\pm}. \tag{2.53}$$

Similarly, for any $0 < m^2 = |m|^2$, we have $\sigma\left[\Theta^{\mathrm{II}}_{|m|,\pm}(P_0)\right] \subset i\mathbb{R}^{\pm}$ and hence

$$U^{\pm}_{m^2} \longleftrightarrow U^{\mathrm{II}}_{|m|,\pm}. \tag{2.54}$$

Finally, if $0 > m^2 = -|m|^2$, we have $\sigma\left[\Theta^{\mathrm{III}}_{|m|,\pm}(P_1)\right] \subset i\mathbb{R}^{\mp}$ and hence

$$U^{\pm}_{m^2} \longleftrightarrow U^{\mathrm{III}}_{|m|,\pm}. \tag{2.55}$$

Again, "$\longleftrightarrow$" means the respective representations could be possibly equivalent. If we, however, admit that the set of representations constructed due to the Mackey theory exhausts the entire list of irreducible unitary representations of $\mathcal{P}_2$, then there is no other eventuality but the corresponding representations are equivalent indeed. Nevertheless, below we shall confirm this assertion by introducing isometry transformations explicitly.

Remark 2.3. Comparing how the Casimir operator M^2 is represented within the representations constructed due to Lie fields and Mackey approaches, we can finally relate parameters m^2 and $|m|$. Recall that the parameters have been totally independent until now. Namely we can see that "$|m|^2 = \left|m^2\right|$". Notice, and it is not surprising, that in both expressions $|m|$ and m^2, the *mass* m is determined up so sign. An exception is the so-called *massless* case $m^2 = |m| = 0$, where $m = 0$. Notice for completeness that in the case $m^2 = |m|^2 > 0$ the mass $m = \pm |m|$ is *real*, while in the case $m^2 = -|m|^2 < 0$ the mass $m = \pm i\,|m|$ is *purely imaginary*.

2.3.2 Explicit Isometries

The final part of the second chapter is devoted to explicit demonstration of the equivalences derived above. For this purpose, let us define, for any $|m| > 0$ and $\varepsilon = \pm 1$, the following mappings $\mathcal{R}^{\pm}_{|m|,\varepsilon} : L^2(\mathbb{R}^{\pm}, \mathrm{d}x) \to L^2(\mathbb{R}, \mathrm{d}x)$:

$$\mathcal{R}^{\pm}_{|m|,\varepsilon}\psi(x) := \sqrt{|m|\, e^{\varepsilon x}}\,\psi(\pm\,|m|\, e^{\varepsilon x}). \tag{2.56}$$

Proposition 2.12. *Each $\mathcal{R}^{\pm}_{|m|,\varepsilon}$ is an isometry.*

Proof. For any $m > 0$, $\varepsilon = \pm 1$ and $\psi, \phi \in L^2(\mathbb{R}^{\pm}, \mathrm{d}x)$ we have

$$\begin{aligned}\left(\mathcal{R}^{\pm}_{|m|,\varepsilon}\phi, \mathcal{R}^{\pm}_{|m|,\varepsilon}\psi\right)_{L^2(\mathbb{R},\mathrm{d}x)} &= \int_{-\infty}^{+\infty} |m|\, e^{\varepsilon x}\,\overline{\phi(\pm\,|m|\, e^{\varepsilon x})}\,\psi(\pm\,|m|\, e^{\varepsilon x})\,\mathrm{d}x \\ &= \pm\int_0^{\pm\infty} \overline{\phi(y)}\,\psi(y)\,\mathrm{d}x \\ &= (\phi, \psi)_{L^2(\mathbb{R}^{\pm},\mathrm{d}x)}.\end{aligned}$$

□

Now we are ready to prove the concluding theorem:

Theorem 2.13. *With the above notation, for any $|m| > 0$ and $\varepsilon = \pm 1$ we have*

$$U^{\pm}_{0,\varepsilon} \cong U^{\mathrm{I}}_{\varepsilon,\pm}, \quad U^{\pm}_{|m|^2} \cong U^{\mathrm{II}}_{|m|,\pm} \quad \textit{and} \quad U^{\pm}_{-|m|^2} \cong U^{\mathrm{III}}_{|m|,\pm}. \tag{2.57}$$

Proof. Take arbitrary $|m|>0$, $\varepsilon=\pm1$, $t\equiv(t_1,t_2,t_3)\in\mathbb{R}^3$ and $\psi\in L^2(\mathbb{R}^\pm)$.
(a) First, on the one hand we have

$$\begin{aligned}\mathcal{R}^\pm_{2,\varepsilon}U^\pm_{0,\varepsilon}(t)\psi(x) &= \mathcal{R}^\pm_{2,\varepsilon}\exp\left\{\frac{\varepsilon t_1}{2}+\frac{it_2x}{2}-\frac{\varepsilon it_3x}{2}\right\}\psi\left(e^{\varepsilon t_1}x\right)\\ &= \sqrt{2}\exp\left\{\frac{\varepsilon(t_1+x)}{2}\pm ie^{\varepsilon x}(t_2-\varepsilon t_3)\right\}\psi\left(\pm2e^{\varepsilon(t_1+x)}\right),\end{aligned}$$

on the other hand,

$$\begin{aligned}U^{\mathrm{I}}_{\varepsilon,\pm}(t)\mathcal{R}^\pm_{2,\varepsilon}\psi(x) &= U^{\mathrm{I}}_{\varepsilon,\pm}(t)\sqrt{2e^{\varepsilon x}}\,\psi(\pm2e^{\varepsilon x})\\ &= \exp\left\{\pm ie^{\varepsilon x}(t_2-\varepsilon t_3)\right\}\sqrt{2e^{\varepsilon(x+t_1)}}\,\psi\left(\pm2e^{\varepsilon(x+t_1)}\right).\end{aligned}$$

Hence $U^\pm_{0,\varepsilon}\cong U^{\mathrm{I}}_{\varepsilon,\pm}$.
(b) Second,

$$\begin{aligned}\mathcal{R}^\pm_{|m|,\varepsilon}U^\pm_{|m|^2,\varepsilon}(t)\psi(x) &= \mathcal{R}^\pm_{|m|,\varepsilon}\,e^{\frac{\varepsilon t_1}{2}+\frac{it_2}{2}\left(x+\frac{|m|^2}{x}\right)-\frac{\varepsilon it_3}{2}\left(x-\frac{|m|^2}{x}\right)}\psi\left(e^{\varepsilon t_1}x\right)\\ &= \sqrt{|m|}\,e^{\frac{\varepsilon(t_1+x)}{2}\pm\frac{i|m|}{2}[t_2(e^{\varepsilon x}+e^{-\varepsilon x})-\varepsilon t_3(e^{\varepsilon x}-e^{-\varepsilon x})]}\psi\left(\pm|m|\,e^{\varepsilon(t_1+x)}\right),\\ &= \sqrt{|m|}\,e^{\frac{\varepsilon(t_1+x)}{2}\pm\frac{i|m|}{2}[t_2(e^{x}+e^{-x})-t_3(e^{x}-e^{-x})]}\psi\left(\pm|m|\,e^{\varepsilon(t_1+x)}\right),\\ &= \sqrt{|m|}\,e^{\frac{\varepsilon(t_1+x)}{2}\pm i|m|(t_2\cosh x-t_3\sinh x)}\psi\left(\pm|m|\,e^{\varepsilon(t_1+x)}\right),\end{aligned}$$

equals to

$$\begin{aligned}U^{\mathrm{II}}_{|m|,\pm}(t)\mathcal{R}^\pm_{|m|,\varepsilon}\psi(x) &= U^{\mathrm{II}}_{|m|,\pm}(t)\sqrt{|m|\,e^{\varepsilon x}}\,\psi(\pm|m|\,e^{\varepsilon x})\\ &= e^{\pm i|m|(t_2\cosh x-t_3\sinh x)}\sqrt{|m|\,e^{\varepsilon(x+t_1)}}\,\psi\left(\pm|m|\,e^{\varepsilon(x+t_1)}\right).\end{aligned}$$

Hence, $U^\pm_{|m|^2,\varepsilon}\cong U^{\mathrm{II}}_{|m|,\pm}$. In particular, $U^\pm_{|m|^2}\cong U^{\mathrm{II}}_{|m|,\pm}$.
(c) Finally,

$$\begin{aligned}\mathcal{R}^\pm_{|m|,\varepsilon}U^\pm_{-|m|^2,\varepsilon}(t)\psi(x) &= \mathcal{R}^\pm_{|m|,\varepsilon}\,e^{\frac{\varepsilon t_1}{2}+\frac{it_2}{2}\left(x-\frac{|m|^2}{x}\right)-\frac{\varepsilon it_3}{2}\left(x+\frac{|m|^2}{x}\right)}\psi\left(e^{\varepsilon t_1}x\right)\\ &= \sqrt{|m|}\,e^{\frac{\varepsilon(t_1+x)}{2}\pm\frac{i|m|}{2}[t_2(e^{\varepsilon x}-e^{-\varepsilon x})-\varepsilon t_3(e^{\varepsilon x}+e^{-\varepsilon x})]}\psi\left(\pm|m|\,e^{\varepsilon(t_1+x)}\right),\\ &= \sqrt{|m|}\,e^{\frac{\varepsilon(t_1+x)}{2}\pm i\varepsilon|m|(t_2\sinh x-t_3\cosh x)}\psi\left(\pm|m|\,e^{\varepsilon(t_1+x)}\right),\end{aligned}$$

is equal to

$$\begin{aligned}U^{\mathrm{III}}_{|m|,\pm\varepsilon}(t)\mathcal{R}^\pm_{|m|,\varepsilon}\psi(x) &= U^{\mathrm{III}}_{|m|,\pm\varepsilon}(t)\sqrt{|m|\,e^{\varepsilon x}}\,\psi(\pm|m|\,e^{\varepsilon x})\\ &= e^{\pm i\varepsilon|m|(t_2\sinh x-t_3\cosh x)}\sqrt{|m|\,e^{\varepsilon(x+t_1)}}\,\psi\left(\pm|m|\,e^{\varepsilon(x+t_1)}\right),\end{aligned}$$

where "$\pm\varepsilon$" stands for $\pm$ or $\mp$ if ε is $+1$ or -1 respectively. Therefore, $U^\pm_{-|m|^2,\varepsilon}\cong U^{\mathrm{III}}_{|m|,\pm\varepsilon}$ and again, this in particular means $U^\pm_{-|m|^2}\cong U^{\mathrm{III}}_{|m|,\pm}$. $\square$

Remark 2.4. It follows from parts (b) and (c), respectively, of the previous proof, that

$$U^\pm_{|m|^2,+1}\cong U^{\mathrm{II}}_{|m|,\pm}\cong U^\pm_{|m|^2,-1}\quad\text{and}\quad U^\pm_{-|m|^2,+1}\cong U^{\mathrm{III}}_{|m|,\pm}\cong U^\mp_{-|m|^2,-1}. \tag{2.58}$$

Since $\cong$ is an equivalence relation, this proves Proposition 2.10.

Chapter 3

Representations of $\mathcal{P}_3$

The other Lie group introduced in §1.2 is $\mathcal{P}_3 = \mathrm{SO}_0(1,2) \ltimes \mathrm{T}^3$. In this case the second-kind canonical coordinates are chosen as follows:

$$g : t \equiv (t_1, \ldots, t_6) \mapsto g(t) \equiv e^{t_2\mathbf{P}_0}\, e^{t_3\mathbf{P}_1}\, e^{t_4\mathbf{P}_2}\, e^{t_5(\mathbf{L}_{12}-\mathbf{L}_{02})}\, e^{t_1\mathbf{L}_{01}}\, e^{t_6(\mathbf{L}_{12}+\mathbf{L}_{02})}, \tag{3.1}$$

where $t \in \mathbb{R}^6$ and

$$\exp(t_1\mathbf{L}_{01}) = \exp\begin{pmatrix} 0 & -t_1 & 0 & 0 \\ -t_1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} \cosh t_1 & -\sinh t_1 & 0 & 0 \\ -\sinh t_1 & \cosh t_1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix},$$

$$\exp\{t_5(\mathbf{L}_{12}-\mathbf{L}_{02})\} = \exp\begin{pmatrix} 0 & 0 & t_5 & 0 \\ 0 & 0 & -t_5 & 0 \\ t_5 & t_5 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 1+\frac{t_5^2}{2} & \frac{t_5^2}{2} & t_5 & 0 \\ -\frac{t_5^2}{2} & 1-\frac{t_5^2}{2} & -t_5 & 0 \\ t_5 & t_5 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix},$$

$$\exp\{t_6(\mathbf{L}_{12}+\mathbf{L}_{02})\} = \exp\begin{pmatrix} 0 & 0 & -t_6 & 0 \\ 0 & 0 & -t_6 & 0 \\ -t_6 & t_5 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 1+\frac{t_6^2}{2} & -\frac{t_6^2}{2} & -t_6 & 0 \\ \frac{t_6^2}{2} & 1-\frac{t_6^2}{2} & -t_6 & 0 \\ -t_6 & t_6 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix},$$

$$\exp(t_2\mathbf{P}_0) = \exp\begin{pmatrix} 0 & 0 & 0 & t_2 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & t_2 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix},$$

$$\exp(t_3\mathbf{P}_1) = \exp\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & t_3 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & t_3 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix},$$

$$\exp(t_4\mathbf{P}_2) = \exp\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & t_4 \\ 0 & 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & t_4 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

Restricting to a sufficiently small neighbourhood of zero, the group multiplication rule can be translated into the language of the coordinates. Namely, at least for

$$t, t' \in \mathbb{T}_6 := \mathbb{R} \times \mathbb{R}^3 \times (-1,1) \times (-1,1)$$

we have

$$g(t) \cdot g(t') = g(t''), \tag{3.2}$$

where

$$\begin{aligned}
t_1'' &= t_1 + t_1' - 2\ln(1 - t_6 t_5'),\\
t_5'' &= \frac{t_5 + e^{t_1} t_5' - t_5 t_6 t_5'}{1 - t_6 t_5'},\\
t_6'' &= \frac{t_6' + e^{t_1'} t_6 - t_6 t_5' t_6'}{1 - t_6 t_5'},\\
t_2'' &= t_2 + t_5\{t_4' + t_6(t_3' - t_2')\} + \frac{e^{t_1}}{2}(t_2' - t_3')\\
&\quad + \frac{1}{2e^{t_1}}\{t_2' + t_3' + t_6^2(t_2' - t_3') + t_5^2(t_2' + t_3') + t_5^2 t_6^2(t_2' - t_3') - 2t_6 t_4'(1 + t_5^2)\},\\
t_3'' &= t_3 - t_5\{t_4' + t_6(t_3' - t_2')\} - \frac{e^{t_1}}{2}(t_2' - t_3')\\
&\quad + \frac{1}{2e^{t_1}}\{t_2' + t_3' + t_6^2(t_2' - t_3') - t_5^2(t_2' + t_3') - t_5^2 t_6^2(t_2' - t_3') - 2t_6 t_4'(1 - t_5^2)\},\\
t_4'' &= t_4 + t_4' + t_6(t_3' - t_2') + e^{-t_1} t_5\{t_2' + t_3' - 2t_6 t_4' + t_6^2(t_2' - t_3')\}.
\end{aligned}$$

The rule is not so easy to be uncovered by hand, nevertheless it can be readily obtained using e.g. MAPLE computer algebra system (CAS).

Notice the coordinates system g is not global in this case. To illustrate this fact, one can easily convince her- or himself by writing the product $g(t)$ in a single matrix form, that the matrix

$$R_0(\pi) \equiv \begin{pmatrix} 1 & 0 & 0 & 0\\ 0 & -1 & 0 & 0\\ 0 & 0 & -1 & 0\\ 0 & 0 & 0 & 1 \end{pmatrix}, \tag{3.3}$$

in spite of being from $\mathcal{P}_3$, is indescribable in terms of the coordinates. Notice that $R_0(\pi)$ represents rotation around the 0-th axis of the Minkowski space M^3 by angle π.

Remark 3.1. It may be suggested that our choice of coordinates is far from best and that global coordinates of $\mathcal{P}_3$ could possibly exists. However, we shall see further, that this coordinate system is by far the most convenient one for us and it simplifies our calculations rapidly. Remark that there is no special demand for global coordinates in the process of construction of representations for the Lie group.

3.1 Lie Field Technique

Now we shall repeat the procedure of §2.1, where a relation between fields of fractions was used with advantage, in order to construct skew-symmetric representations of the Poincaré algebra $\mathfrak{p}_3$ and consequently the complete family of irreducible unitary representations of the Lie group $\mathcal{P}_3$.

The Lie algebra $\mathfrak{p}_3$ is a six-dimensional real Lie algebra, generated by P_0, P_1, P_2, L_{01}, L_{02} and L_{12} subject to the following non-zero commutation relations:

$$\begin{aligned} [L_{01}, L_{02}] &= -L_{12}, & [L_{01}, L_{12}] &= -L_{02}, & [L_{02}, L_{12}] &= L_{01}, \\ [L_{01}, P_0] &= -P_1, & [L_{01}, P_1] &= -P_0, & [L_{02}, P_0] &= -P_2, \\ [L_{02}, P_2] &= -P_0, & [L_{12}, P_1] &= P_2, & [L_{12}, P_2] &= -P_1. \end{aligned} \tag{3.4}$$

The other commutation relations are trivial. Since

$$\operatorname{index} \mathfrak{p}_3 = 6 - \operatorname{rank}_{\mathfrak{S}(\mathfrak{p}_3)} \begin{pmatrix} 0 & -L_{12} & -L_{02} & -P_1 & -P_0 & 0 \\ L_{12} & 0 & L_{01} & -P_2 & 0 & -P_0 \\ L_{02} & -L_{01} & 0 & 0 & P_2 & -P_1 \\ P_1 & P_2 & 0 & 0 & 0 & 0 \\ P_0 & 0 & -P_2 & 0 & 0 & 0 \\ 0 & P_0 & P_1 & 0 & 0 & 0 \end{pmatrix} = 2,$$

there are two independent Casimir elements of $\mathfrak{Z}(\mathfrak{p}_3)$, namely (cf. [34], p. 297)

$$M^2 := P_2^2 + P_1^2 - P_0^2 \tag{3.5}$$

and

$$C := L_{02}P_1 - L_{01}P_2 - L_{12}P_0 = P_1L_{02} - P_2L_{01} - P_0L_{12}. \tag{3.6}$$

Further, as $\frac{1}{2}(6-2) = 2$, $\mathfrak{D}_{2,2}(\mathbb{R})$ is the candidate for isomorphic "partner" of $\mathfrak{D}(\mathfrak{p}_3)$.

3.1.1 Isomorphism of $\mathfrak{D}(\mathfrak{p}_3)$ and $\mathfrak{D}_{2,2}(\mathbb{R})$

Since L_{01}, P_0 and P_1 commute identically as in the case $\mathfrak{p}_2$, for $\hat{p}_1, \hat{q}_1 \in \mathfrak{D}(\mathfrak{p}_3)$ defined by

$$\hat{p}_1 := (P_0 - P_1)^{-1}\left(L_{01} - \frac{1}{2}\right) = \frac{1}{2}\left((P_0 - P_1)^{-1}L_{01} + L_{01}(P_0 - P_1)^{-1}\right), \tag{3.7}$$

$$\hat{q}_1 := P_0 - P_1, \tag{3.8}$$

we have $[\hat{p}_1, \hat{q}_1] = 1$ as well as $\hat{p}_1^* = \hat{p}_1$ and $\hat{q}_1^* = -\hat{q}_1$. Further, according to (3.4),

$$[L_{12} - L_{02}, P_2] = P_0 - P_1,$$

therefore

$$\begin{aligned} 1 &= (P_0 - P_1)^{-1}[L_{12} - L_{02}, P_2] \\ &= (P_0 - P_1)^{-1}(L_{12} - L_{02})P_2 - (P_0 - P_1)^{-1}P_2(L_{12} - L_{02}) \\ &= (P_0 - P_1)^{-1}(L_{12} - L_{02})P_2 - P_2(P_0 - P_1)^{-1}(L_{12} - L_{02}) \\ &= \left[(P_0 - P_1)^{-1}(L_{12} - L_{02}), P_2\right]. \end{aligned}$$

Thus for $\hat{p}_2, \hat{q}_2 \in \mathfrak{D}(\mathfrak{p}_3)$ defined by

$$\hat{p}_2 := (P_0 - P_1)^{-1}(L_{12} - L_{02}), \tag{3.9}$$

$$\hat{q}_2 := P_2, \tag{3.10}$$

we have $[\hat{p}_2, \hat{q}_2] = 1$. Moreover, $\hat{q}_2^* = -\hat{q}_2$ is trivial, and since $[L_{12} - L_{02}, P_0 - P_1] = 0$, (1.17) implies

$$\hat{p}_2^* = (L_{12}^* - L_{02}^*)\,(P_0^* - P_1^*)^{-1} = (L_{12} - L_{02})\,(P_0 - P_1)^{-1} = \hat{p}_2.$$

Furthermore, both commutators $[\hat{q}_1, \hat{q}_2]$ and $[\hat{p}_1, \hat{q}_2]$ are clearly zero as well as

$$[\hat{p}_2, \hat{q}_1] = (P_0 - P_1)^{-1}(L_{12} - L_{02})(P_0 - P_1) - L_{12} + L_{02} = L_{12} - L_{02} - L_{12} + L_{02} = 0$$

and

$$
\begin{aligned}
[\hat{p}_1, \hat{p}_2] = {} & (P_0 - P_1)^{-1} \left(L_{01} - \frac{1}{2} \right) (P_0 - P_1)^{-1} (L_{12} - L_{02}) \\
& - (P_0 - P_1)^{-1} (L_{12} - L_{02}) (P_0 - P_1)^{-1} \left(L_{01} - \frac{1}{2} \right) \\
= {} & (P_0 - P_1)^{-1} \left(L_{01} - \frac{1}{2} \right) (L_{12} - L_{02}) (P_0 - P_1)^{-1} \\
& - (P_0 - P_1)^{-1} (L_{12} - L_{02}) \left(L_{01} - \frac{1}{2} \right) (P_0 - P_1)^{-1} \\
& + (P_0 - P_1)^{-1} (L_{12} - L_{02}) \left[L_{01} - \frac{1}{2}, (P_0 - P_1)^{-1} \right] \\
= {} & (P_0 - P_1)^{-1} \left[L_{01}, L_{12} - L_{02} \right] (P_0 - P_1)^{-1} \\
& - (P_0 - P_1)^{-1} (L_{12} - L_{02}) (P_0 - P_1)^{-1} \left[L_{01}, P_0 - P_1 \right] (P_0 - P_1)^{-1} \\
= {} & (P_0 - P_1)^{-1} (L_{12} - L_{02}) (P_0 - P_1)^{-1} - (P_0 - P_1)^{-1} (L_{12} - L_{02}) (P_0 - P_1)^{-1} \\
= {} & 0.
\end{aligned}
$$

Let us rewrite the Casimir operators into for us more convenient forms. First,

$$M^2 = (P_1 - P_0)(P_1 + P_0) + P_2^2. \tag{3.11}$$

Second, as $[L_{02}, P_1] = [L_{12}, P_0] = 0$,

$$C = (P_1 - P_0) L_{12} - P_1 (L_{12} - L_{02}) - L_{01} P_2. \tag{3.12}$$

Like in the case of $\mathfrak{p}_2$, the relations (3.7) - (3.12) are to be inverted. Namely,

$$
\begin{aligned}
L_{01} &= \hat{q}_1 \hat{p}_1 + \frac{1}{2}, && (3.13)\\
P_0 &= \frac{\hat{q}_1^{-1}}{2} (\hat{q}_1^2 + \hat{q}_2^2 - M^2), && (3.14)\\
P_1 &= \frac{\hat{q}_1^{-1}}{2} (-\hat{q}_1^2 + \hat{q}_2^2 - M^2), && (3.15)\\
P_2 &= \hat{q}_2, && (3.16)\\
L_{12} - L_{02} &= \hat{q}_1 \hat{p}_2, && (3.17)\\
L_{12} + L_{02} &= -2 \hat{q}_1^{-1} \left[C + \left(\hat{q}_1 \hat{p}_1 + \frac{1}{2} \right) \hat{q}_2 + \frac{1}{2} \left(\hat{q}_2^2 - M^2 \right) \hat{p}_2 \right]. && (3.18)
\end{aligned}
$$

The last relation (3.18) was obtained from

$$L_{12} = -\hat{q}_1^{-1} \left[C + \left(\hat{q}_1 \hat{p}_1 + \frac{1}{2} \right) \hat{q}_2 + \frac{1}{2} \left(-\hat{q}_1^2 + \hat{q}_2^2 - M^2 \right) \hat{p}_2 \right]. \tag{3.19}$$

Now it is obvious that the linear mapping $\Psi : \mathfrak{p}_3 \to \mathfrak{D}_{2,2}(\mathbb{R})$ defined by

$$
\begin{aligned}
\Psi(L_{01}) &:= q_1 p_1 + \frac{1}{2}, && (3.20)\\
\Psi(P_0) &:= \frac{q_1^{-1}}{2} \left(q_1^2 + q_2^2 - \theta_1 \right), && (3.21)\\
\Psi(P_1) &:= \frac{q_1^{-1}}{2} \left(-q_1^2 + q_2^2 - \theta_1 \right), && (3.22)\\
\Psi(P_2) &:= q_2, && (3.23)\\
\Psi(L_{12} - L_{02}) &:= q_1 p_2, && (3.24)\\
\Psi(L_{12} + L_{02}) &:= -2 q_1^{-1} \left[\theta_2 + \left(q_1 p_1 + \frac{1}{2} \right) q_2 + \frac{1}{2} \left(q_2^2 - \theta_1 \right) p_2 \right], && (3.25)
\end{aligned}
$$

preserves the commutator and thus extends to a homomorphism $\Psi : \mathfrak{U}(\mathfrak{p}_3) \to \mathfrak{D}_{2,2}(\mathbb{R})$.

Now we would like to prove an analogue of Lemma 2.3 to show that Ψ is in fact an isomorphism of skew fields. Although the task is a little bit more complicated now, the procedure is just a repetition of the previous case.

Proposition 3.1. *In $\mathfrak{W}_2(\mathbb{R})$ we have $(q_1 p_1)^n = q_1^n p_1^n + f_n(q_1, p_1)$, with $f_n \in \mathfrak{W}_2$ containing p_1 at most to the power of $n-1$, for any $n \in \mathbb{N}$*

Proof. A trivial consequence of Proposition 2.2. □

Lemma 3.2. *For $\Psi : \mathfrak{U}(\mathfrak{p}_3) \to \mathfrak{D}_{2,2}(\mathbb{R})$ and $x \in \mathfrak{U}(\mathfrak{p}_3)$ one has $\Psi(x) = 0$ only if $x = 0$.*

Proof. Due to PBW theorem we have

$$x = \sum_{j_1,\dots,j_6=0}^{N} \alpha_{j_1,\dots,j_6} P_0^{j_1} P_1^{j_2} P_2^{j_3} L_{01}^{j_4} (L_{12} - L_{02})^{j_5} (L_{12} + L_{02})^{j_6}$$

for some $N \in \mathbb{N}$ and $\alpha_{j_1,\dots,j_6} \in \mathbb{R}$, $0 \leq j_1, \dots, j_6 \leq N$. Hence

$$\begin{aligned}
0 &= \Psi(x) \\
&= \sum_{j_6=0}^{N} \left[\sum_{j_1,\dots,j_5=0}^{N} \alpha_{j_1,\dots,j_6} \Psi(P_0)^{j_1} \Psi(P_1)^{j_2} \Psi(P_2)^{j_3} \Psi(L_{01})^{j_4} \Psi(L_{12} - L_{02})^{j_5} \right] \\
&\qquad \times \Psi(L_{12} + L_{02})^{j_6} \\
&= f_1(q_1, q_2, q_1^{-1}, p_1, p_2, \theta_1, \theta_2) \\
&\qquad + \left[\sum_{j_1,\dots,j_5=0}^{N} \alpha_{j_1,\dots,j_5,N} \Psi(P_0)^{j_1} \Psi(P_1)^{j_2} \Psi(P_2)^{j_3} \Psi(L_{01})^{j_4} \Psi(L_{12} - L_{02})^{j_5} \right] \\
&\qquad\quad \times (-2)^N q_1^{-N} \theta_2^N,
\end{aligned}$$

where $f_(q_1, q_2, q_1^{-1}, p_1, p_2, \theta_1, \theta_2) \in \mathfrak{D}_{2,2}$ contains at most θ_2^{N-1}. Consequently,

$$\sum_{j_1,\dots,j_5=0}^{N} \alpha_{j_1,\dots,j_5,N} \Psi(P_0)^{j_1} \Psi(P_1)^{j_2} \Psi(P_2)^{j_3} \Psi(L_{01})^{j_4} \Psi(L_{12} - L_{02})^{j_5} = 0.$$

In the same way as in the case $\mathfrak{p}_2$, we conclude that, for any $0 \leq j_6 \leq N$,

$$\sum_{j_1,\dots,j_5=0}^{N} \alpha_{j_1,\dots,j_6} \Psi(P_0)^{j_1} \Psi(P_1)^{j_2} \Psi(P_2)^{j_3} \Psi(L_{01})^{j_4} \Psi(L_{12} - L_{02})^{j_5} = 0. \tag{3.26}$$

Further, since $[q_1, p_2] = 0$, (3.26) can be written as

$$0 = \sum_{j_5=0}^{N} \left[\sum_{j_1,\dots,j_4=0}^{N} \alpha_{j_1,\dots,j_6} \Psi(P_0)^{j_1} \Psi(P_1)^{j_2} \Psi(P_2)^{j_3} \Psi(L_{01})^{j_4} \right] q_1^{j_5} p_2^{j_5}$$

and because none of the sums $\sum_{j_1,\dots,j_4=0}^{N} \alpha_{j_1,\dots,j_6} \Psi(P_0)^{j_1} \Psi(P_1)^{j_2} \Psi(P_2)^{j_3} \Psi(L_{01})^{j_4}$ contains p_2 at all, we have

$$\sum_{j_1,\dots,j_4=0}^{N} \alpha_{j_1,\dots,j_6} \Psi(P_0)^{j_1} \Psi(P_1)^{j_2} \Psi(P_2)^{j_3} \Psi(L_{01})^{j_4} = 0 \tag{3.27}$$

for any $0 \leq j_5, j_6 \leq N$. Analogously, (3.27) together with Proposition 3.1 imply

$$\begin{aligned}
0 &= \sum_{j_4=0}^{N} \left[\sum_{j_1,\dots,j_3=0}^{N} \alpha_{j_1,\dots,j_6} \Psi(P_0)^{j_1} \Psi(P_1)^{j_2} \Psi(P_2)^{j_3} \right] \left(q_1 p_1 + \frac{1}{2} \right)^{j_4} \\
&= \tilde{f}(q_1, q_2, q_1^{-1}, \theta_1, p_1) + \left[\sum_{j_1,\dots,j_3=0}^{N} \alpha_{j_1,j_2,j_3,N,j_5,j_6} \Psi(P_0)^{j_1} \Psi(P_1)^{j_2} \Psi(P_2)^{j_3} \right] q_1^N p_1^N,
\end{aligned}$$

with $\tilde{f}(q_1, q_2, q_1^{-1}, \theta_1, p_1) \in \mathfrak{D}_{2,2}(\mathbb{R})$ containing at most p_1^{N-1}. As before, after N iterations this leads to the following equality, for any $0 \leq j_4, j_5, j_6 \leq N$:

$$\sum_{j_1,\dots,j_3=0}^{N} \frac{\alpha_{j_1,\dots,j_6}}{2^{j_1+j_2}} \left[q_1^{-1}(q_1^2 + q_2^2 - \theta_1)\right]^{j_1} \left[q_1^{-1}(-q_1^2 + q_2^2 - \theta_1)\right]^{j_2} q_2^{j_3}. \tag{3.28}$$

Similarly as in the case $\mathfrak{p}_2$, since all q_1, q_1^{-1}, q_2 and θ_1 commute, (3.28) in fact means that

$$\sum_{j_1,\dots,j_3=0}^{N} \frac{\alpha_{j_1,\dots,j_6}}{2^{j_1+j_2}} \left(x + \frac{y^2}{x} - \frac{z}{x}\right)^{j_1} \left(-x + \frac{y^2}{x} - \frac{z}{x}\right)^{j_2} y^{j_3} \tag{3.29}$$

for $(x, y, z) \in \mathbb{R}^3$, $x \neq 0$. Recall that also $0 \leq j_4, j_5, j_6 \leq N$ are arbitrary. Because the Jacobian of mapping defined by $u := \left(x + \frac{y^2}{x} - \frac{z}{x}\right)$, $v := \left(-x + \frac{y^2}{x} - \frac{z}{x}\right)$, $w := y$, is

$$\det \begin{pmatrix} 1 - \frac{y^2}{x^2} + \frac{z}{x^2} & \frac{2y}{x} & -\frac{1}{x} \\ -1 - \frac{y^2}{x^2} + \frac{z}{x^2} & \frac{2y}{x} & -\frac{1}{x} \\ 0 & 1 & 0 \end{pmatrix} = -\det \begin{pmatrix} 1 - \frac{y^2}{x^2} + \frac{z}{x^2} & -\frac{1}{x} \\ -1 - \frac{y^2}{x^2} + \frac{z}{x^2} & -\frac{1}{x} \end{pmatrix} = \frac{2}{x},$$

the mapping is regular on $\mathbb{R}^\times \times \mathbb{R} \times \mathbb{R}$, thus the polynomial $\sum_{j_1,\dots,j_3=0}^{N} \frac{\alpha_{j_1,\dots,j_6}}{2^{j_1+j_2}} u^{j_1} v^{j_2} w^{j_3}$ equals zero on an open subset of $\mathbb{R}^3$. Consequently, it is the zero polynomial with $\alpha_{j_1,\dots,j_6} = 0$ for any $0 \leq j_1, \dots, j_6 \leq N$. $\square$

Consequently, $\Psi : \mathfrak{U}(\mathfrak{p}_3) \to \mathfrak{D}_{2,2}(\mathbb{R})$ is injective and it can be extended uniquely to an (injective) homomorphism $\Psi : \mathfrak{D}(\mathfrak{p}_3) \to \mathfrak{D}_{2,2}(\mathbb{R})$ with $\Psi(x^{-1}) = \Psi(x)^{-1}$, $x \in \mathfrak{U}(\mathfrak{p}_3)$. Furthermore, it is again easily seen that the extended mapping is surjective:

$$\Psi^{(-1)}(p_j) = \hat{p}_j, \quad \Psi^{(-1)}(q_j) = \hat{q}_j, \quad \Psi^{(-1)}(\theta_1) = M^2, \quad \Psi^{(-1)}(\theta_2) = C,$$

$j = 1, 2$. Finally, since $\hat{p}_j^* = \hat{p}_j$ and $\hat{q}_j^* = -\hat{q}_j$, $j = 1, 2$, and $(M^2)^* = M^2$ as well as $C^* = C$, Ψ is moreover involutive. All in all, the following theorem has been proven.

Theorem 3.3. *The mapping $\Psi : \mathfrak{D}(\mathfrak{p}_3) \to \mathfrak{D}_{2,2}(\mathbb{R})$ is a $*$-isomorphism.*

3.1.2 Skew-symmetric Representations of $\mathfrak{p}_3$

It is easily seen from (3.20) - (3.25) that $\Psi(\mathfrak{p}_3) \subset \mathfrak{D}'_{2,2}(\mathbb{R})$, hence we may again apply the technique introduced in §1.1.4 in order to induce skew-symmetric representations of $\mathfrak{p}_3$ from $\mathfrak{W}_{2,2}(\mathbb{R})$. We use the family $\Phi_{m^2,c}$, $m^2, c \in \mathbb{R}$, of representations of $\mathfrak{W}_{2,2}(\mathbb{R})$ on $\mathcal{H}_2 \equiv L^2(\mathbb{R}^\times \times \mathbb{R}, \mathrm{d}^2 x)$ defined for $j = 1, 2$ and $x \equiv (x_1, x_2) \in \mathbb{R}^\times \times \mathbb{R}$ by,

$$\Phi_{m^2,c}(p_j)\psi(x) = -i\partial_{x_j}\psi(x), \tag{3.30}$$

$$\Phi_{m^2,c}(q_j)\psi(x) = ix_j\psi(x), \tag{3.31}$$

$$\Phi_{m^2,c}(\theta_1)\psi(x) = m^2\psi(x), \tag{3.32}$$

$$\Phi_{m^2,c}(\theta_2)\psi(x) = c\psi(x). \tag{3.33}$$

The restriction of Ψ to $\mathfrak{p}_3$, composed with $\Phi_{m^2,c}$, produces the following family $\Omega_{m^2,c}$, $m^2, c \in \mathbb{R}$, of skew-symmetric representations of $\mathfrak{p}_3$ on $\mathcal{H}_2$:

$$\Omega_{m^2,c}(L_{01})\psi(x) = \left(x_1\partial_{x_1} + \frac{1}{2}\right)\psi(x), \tag{3.34}$$

$$\Omega_{m^2,c}(P_0)\psi(x) = \frac{i}{2x_1}\left(x_1^2 + x_2^2 + m^2\right)\psi(x), \tag{3.35}$$

$$\Omega_{m^2,c}(P_1)\psi(x) = \frac{i}{2x_1}\left(-x_1^2 + x_2^2 + m^2\right)\psi(x), \tag{3.36}$$

$$\Omega_{m^2,c}(P_2)\psi(x) = ix_2\psi(x), \tag{3.37}$$

$$\Omega_{m^2,c}(L_{12}-L_{02})\psi(x) = x_1\partial_{x_2}\psi(x), \tag{3.38}$$

$$\Omega_{m^2,c}(L_{12}+L_{02})\psi(x) = -\frac{2}{x_1}\left[\left(x_1\partial_{x_1}+\frac{1}{2}\right)x_2+\frac{1}{2}\left(x_2^2+m^2\right)\partial_{x_2}-ic\right]\psi(x). \tag{3.39}$$

Remark 3.2. Recall that the domain of all operators considered here is assumed to be $C_0^\infty(\mathbb{R}^\times\times\mathbb{R})$, as discussed in §1.1.4.

3.1.3 Irreducible Unitary Representations of $\mathcal{P}_3$

One-parameter Subgroups

The operators (3.34) - (3.39) can be again integrated into one-parameter subgroups of unitary operators on $\mathcal{H}$. Recall the notation $x \equiv (x_1, x_2)$ is kept. For $t \in \mathbb{R}$ we define

$$U^{(1)}_{m^2,c}(t)\psi(x) \equiv \exp\left\{t\,\Omega_{m^2,c}(L_{01})\right\}\psi(x) = e^{\frac{t}{2}}\psi\left(e^t x_1, x_2\right), \tag{3.40}$$

$$U^{(2)}_{m^2,c}(t)\psi(x) \equiv \exp\left\{t\,\Omega_{m^2,c}(P_0)\right\}\psi(x) = e^{\frac{it}{2}\left(x_1+\frac{x_2^2+m^2}{x_1}\right)}\psi(x), \tag{3.41}$$

$$U^{(3)}_{m^2,c}(t)\psi(x) \equiv \exp\left\{t\,\Omega_{m^2,c}(P_1)\right\}\psi(x) = e^{-\frac{it}{2}\left(x_1-\frac{x_2^2+m^2}{x_1}\right)}\psi(x), \tag{3.42}$$

$$U^{(4)}_{m^2,c}(t)\psi(x) \equiv \exp\left\{t\,\Omega_{m^2,c}(P_2)\right\}\psi(x) = e^{itx_2}\psi(x), \tag{3.43}$$

$$U^{(5)}_{m^2,c}(t)\psi(x) \equiv \exp\left\{t\,\Omega_{m^2,c}(L_{12}-L_{02})\right\}\psi(x) = \psi(x_1, x_2+tx_1), \tag{3.44}$$

$$U^{(6)}_{m^2,c}(t)\psi(x) \equiv \exp\left\{t\,\Omega_{m^2,c}(L_{12}+L_{02})\right\}\psi(x) = \alpha^{(6)}(x;t)\,\psi\left(X_1^{(6)}(x;t), X_2^{(6)}(x;t)\right), \tag{3.45}$$

where

$$\alpha^{(6)}(x;t) = \begin{cases} \left(\frac{x_1-tx_2+it\sqrt{m^2}}{x_1-tx_2-it\sqrt{m^2}}\right)^{\frac{c}{\sqrt{m^2}}}\frac{\sqrt{X_1^{(6)}(x;t)}}{\sqrt{x_1}}, & m^2 \neq 0,\\[2ex] \left(1-\frac{tx_2}{x_1}\right)\exp\frac{2ict}{x_1-tx_2}, & m^2 = 0, \end{cases}$$

and

$$X_1^{(6)}(x;t) = x_1 - 2x_2 t + \frac{x_2^2+m^2}{x_1}t^2 = \frac{1}{x_1}\left[(x_1-x_2t)^2+m^2t^2\right],$$

$$X_2^{(6)}(x;t) = x_2 - \frac{x_2^2+m^2}{x_1}t.$$

Remark 3.3. Notice that strictly speaking, for $m^2 = 0$ and for given $t \in \mathbb{R}$, the coefficient $\alpha^{(6)}(x;t)$ is not well-defined when $x_1 = tx_2$. Nevertheless, $\alpha^{(6)}(x;t)$, as a function of x, is continuous except on the line $x_1 = tx_2$ and $\lim_{x_1\to tx2} = 0$. Therefore we may naturally put $\alpha(tx_2, x_2; t) := 0$ in order to make $\alpha^{(6)}$ continuous everywhere.

The first five "one-parameter subgroups" are easy to be guessed. The sixth one, however, is more difficult to obtain; in this case it is necessary to suppose $U^{(6)}_{m^2,c}(t)\psi(x)$ in the form $\alpha^{(6)}(x;t)\,\psi\left(X_1^{(6)}(x;t), X_2^{(6)}(x;t)\right)$, for sufficiently differentiable unknown functions α, X_1, X_2 and to solve the system of partial differential equations induced by requirements of additivity in t and having $\Omega_{m^2,c}(L_{12}+L_{02})$ as the generator.

As before, all the one-parameter sets of operators (3.40) - (3.45) need to be verified they are in fact one-parameter unitary subgroups with correct generators. From this reason we will have to exclude certain combinations of parameters m^2 and c. We shall return back to this issue immediately.

Proposition 3.4. *Let $m^2 \in \mathbb{R}$ and $c \in \mathbb{R}$ be such that $\frac{c}{\sqrt{m^2}} \in \mathbb{Z}$ provided $m^2 > 0$. Then for any $1 \leq j \leq 6$, $U^{(j)}_{m^2,c}(t)$ are one-parameter subgroups of unitary operators on $\mathcal{H}_2$.*

Proof. For $j \leq 5$, the proof is a trivial repetition of the proof of Proposition 2.6, thus the case $j = 6$ is the only non-trivial one to be proven here. Nonetheless, the method remains the same as before. Let us take any admissible $m^2, c \in \mathbb{R}$.

(a) Assume $m^2 \neq 0$. First, $U^{(6)}_{m^2,c}(t)\psi \in \mathcal{H}_2$ for any $t \in \mathbb{R}$ and $\psi \in \mathcal{H}_2$ because

$$\begin{aligned}
\left\| U^{(6)}_{m^2,c}(t)\psi \right\|^2 &= \int_{\mathbb{R}^2} \left| \left(\frac{x_1 - tx_2 + it\sqrt{m^2}}{x_1 - tx_2 - it\sqrt{m^2}} \right)^{\frac{c}{\sqrt{m^2}}} \frac{\sqrt{X^{(6)}_1}}{\sqrt{x_1}} \right|^2 \cdot \left| \psi\left(X^{(6)}_1, X^{(6)}_2 \right) \right|^2 \mathrm{d}^2 x \\
&= \int_{\mathbb{R}^2} \left| \frac{X^{(6)}_1}{x_1} \right| \cdot \left| \psi\left(X^{(6)}_1, X^{(6)}_2 \right) \right|^2 \mathrm{d}^2 x \\
&= \int_{\mathbb{R}^2} \left| \psi\left(X^{(6)}_1, X^{(6)}_2 \right) \right|^2 \cdot \left| \frac{\partial\left(X^{(6)}_1, X^{(6)}_2 \right)}{\partial(x_1, x_2)} \right| \mathrm{d}^2 x \\
&= \|\psi\|^2 .
\end{aligned}$$

The relation $\frac{\partial\left(X^{(6)}_1, X^{(6)}_2\right)}{\partial(x_1,x_2)} = \frac{X^{(6)}_1}{x_1}$ is proven in the Appendix. Second, for any $t, s \in \mathbb{R}$ we have

$$\begin{aligned}
&X^{(6)}_1\left(X^{(6)}_1(x;t), X^{(6)}_2(x;t); s \right) \\
&\quad = X^{(6)}_1(x;t) - 2sX^{(6)}_2(x;t) + \frac{\left(X^{(6)}_2(x;t) \right)^2 + m^2}{X^{(6)}_1(x;t)} \cdot s^2 \\
&\quad = x_1 - 2x_2(t+s) + \frac{x_2^2 + m^2}{x_1}(t+s)^2 - \frac{x_2^2+m^2}{x_1}s^2 + \frac{\left(x_2 - \frac{x_2^2+m^2}{x_1}t \right)^2 + m^2}{x_1 - 2x_2 t + \frac{x_2^2+m^2}{x_1}t^2} \cdot s^2 \\
&\quad = X^{(6)}_1(x;t+s) - \frac{x_2^2+m^2}{x_1}s^2 + \frac{x_1x_2^2 + x_1 m^2 - 2x_2(x_2^2+m^2)t + \frac{(x_2^2+m^2)^2}{x_1}t^2}{x_1\left(x_1 - 2x_2 t + \frac{x_2^2+m^2}{x_1}t^2 \right)} \cdot s^2 \\
&\quad = X^{(6)}_1(x;t+s),
\end{aligned}$$

similarly

$$\begin{aligned}
&X^{(6)}_2\left(X^{(6)}_1(x;t), X^{(6)}_2(x;t); s \right) \\
&\quad = X^{(6)}_2(x;t) - \frac{\left(X^{(6)}_2(x;t) \right)^2 + m^2}{X^{(6)}_1(x;t)} \cdot s \\
&\quad = x_2 - \frac{x_2^2+m^2}{x_1}(t+s) + \frac{x_2^2+m^2}{x_1}s - \frac{\left(x_2 - \frac{x_2^2+m^2}{x_1}t \right)^2 + m^2}{x_1 - 2x_2 t + \frac{x_2^2+m^2}{x_1}t^2} \cdot s \\
&\quad = X^{(6)}_2(x;t+s),
\end{aligned}$$

and also

$$\alpha^{(6)}(x;t) \cdot \alpha^{(6)}\left(X^{(6)}_1(x;t), X^{(6)}_2(x;t); s \right)$$

$$
\begin{aligned}
&= \left[\left(\frac{x_1 - tx_2 + it\sqrt{m^2}}{x_1 - tx_2 - it\sqrt{m^2}} \right) \left(\frac{1 - \frac{sx_2 - \frac{x_2^2+m^2}{x_1} ts - is\sqrt{m^2}}{x_1 - 2x_2 t + \frac{x_2^2+m^2}{x_1} t^2}}{1 - \frac{sx_2 - \frac{x_2^2+m^2}{x_1} ts + is\sqrt{m^2}}{x_1 - 2x_2 t + \frac{x_2^2+m^2}{x_1} t^2}} \right) \right]^{\frac{c}{\sqrt{m^2}}} \\
&\quad \times \frac{\sqrt{X_1^{(6)}(x;t)}}{\sqrt{x_1}} \cdot \frac{\sqrt{X_1^{(6)}\left(X_1^{(6)}(x;t), X_2^{(6)}(x;t); s\right)}}{\sqrt{X_1^{(6)}(x;t)}} \\
&\equiv \frac{\sqrt{X_1^{(6)}(x;t+s)}}{\sqrt{x_1}} \cdot \left(\frac{A + iB\sqrt{m^2}}{A - iB\sqrt{m^2}} \right)^{\frac{c}{\sqrt{m^2}}},
\end{aligned}
$$

(here we made use of the condition $\frac{c}{\sqrt{m^2}} \in \mathbb{Z}$ whenever $m^2 > 0$), where

$$
\begin{aligned}
A &= (x_1 - tx_2) \left(1 - \frac{sx_2 - \frac{x_2^2+m^2}{x_1} ts}{x_1 - 2x_2 t + \frac{x_2^2+m^2}{x_1} t^2} \right) - \frac{tsm^2}{x_1 - 2x_2 t + \frac{x_2^2+m^2}{x_1} t^2} \\
&= x_1 - tx_2 - \frac{sx_1x_2 - tsx_2^2 - (x_2^2 + m^2)ts + \frac{x_2}{x_1}(x_2^2 + m^2)t^2 s + tsm^2}{x_1 - 2x_2 t + \frac{x_2^2+m^2}{x_1} t^2} \\
&= x_1 - (t+s)x_2
\end{aligned}
$$

and

$$
\begin{aligned}
B &= t \left(1 - \frac{sx_2 - \frac{x_2^2+m^2}{x_1} ts}{x_1 - 2x_2 t + \frac{x_2^2+m^2}{x_1} t^2} \right) + \frac{(x_1 - tx_2)s}{x_1 - 2x_2 t + \frac{x_2^2+m^2}{x_1} t^2} \\
&= t + \frac{-tsx_2 + \frac{x_2^2+m^2}{x_1} t^2 s + sx_1 - tsx_2}{x_1 - 2x_2 t + \frac{x_2^2+m^2}{x_1} t^2} \\
&= t + s.
\end{aligned}
$$

Hence $\alpha^{(6)}(x;t) \cdot \alpha^{(6)}\left(X_1^{(6)}(x;t), X_2^{(6)}(x;t); s\right) = \alpha^{(6)}(x;t+s)$, and, altogether,

$$
U_{m^2,c}^{(6)}(t) U_{m^2,c}^{(6)}(s) = U_{m^2,c}^{(6)}(t+s).
$$

Third, for any $\phi, \psi \in \mathcal{H}_2$ and $t \in \mathbb{R}$ we can write, $X_j \equiv X_j^{(6)}(x;t)$, $j = 1, 2$,

$$
\begin{aligned}
&\left(\phi, U_{m^2,c}^{(6)}(t)\psi\right) \\
&= \int_{\mathbb{R}^2} \overline{\phi(x_1, x_2)} \left(\frac{x_1 - tx_2 + it\sqrt{m^2}}{x_1 - tx_2 - it\sqrt{m^2}} \right)^{\frac{c}{\sqrt{m^2}}} \cdot \frac{\sqrt{X_1}}{\sqrt{x_1}}\, \psi(X_1, X_2)\, \mathrm{d}^2 x \\
&= \int_{\mathbb{R}^2} \overline{\phi(x_1, x_2) \left(\frac{x_1 - tx_2 - it\sqrt{m^2}}{x_1 - tx_2 + it\sqrt{m^2}} \right)^{\frac{c}{\sqrt{m^2}}}} \cdot \left| \frac{\partial(X_1, X_2)}{\partial(x_1, x_2)} \right| \cdot \overline{\left(\frac{\sqrt{x_1}}{\sqrt{X_1}} \right)}\, \psi(X_1, X_2)\, \mathrm{d}^2 x \\
&= \int_{\mathbb{R}^2} \left\{ \overline{\phi\left(X_1^{(6)}(X;-t), X_1^{(6)}(X;-t) \right) \left(\frac{X_1 + tX_2 - it\sqrt{m^2}}{X_1 - tX_2 + it\sqrt{m^2}} \right)^{\frac{c}{\sqrt{m^2}}} \sqrt{\frac{X_1^{(6)}(X;-t)}{X_1}}} \right. \\
&\qquad \left. \times\, \psi(X_1, X_2) \right\} \mathrm{d}^2 X \\
&= \left(U_{m^2,c}^{(6)}(-t)\phi, \psi \right),
\end{aligned}
$$

as $x_j = X_j^{(6)}\left(X_1^{(6)}(x;t), X_1^{(6)}(x;t); -t\right)$, $j = 1,2$, and $X_1^{(6)}(x;t) + tX_2^{(6)}(x;t) = x_1 - tx_2$. This proves unitarity. Finally, strong continuity results from Lemma 2.5.

(b) When $m^2 = 0$, the relations for composition of $X_j^{(6)}$ remain unchanged. Further,

$$\frac{\partial\left(X_1^{(6)}, X_2^{(6)}\right)}{\partial(x_1, x_2)} = \frac{X_1^{(6)}}{x_1} = 1 - \frac{2x_2 t}{x_1} + \frac{x_2^2 t^2}{x_1^2} = \left(1 - \frac{tx_2}{x_1}\right)^2 \geq 0.$$

Then it is easily seen $\left\|U_{0,c}^{(6)}(t)\psi\right\|^2 = \|\psi\|^2$ and

$$\begin{aligned}
&\alpha^{(6)}(x;t) \cdot \alpha^{(6)}\left(X_1^{(6)}(x;t), X_2^{(6)}(x;t); s\right) \\
&= \sqrt{\frac{X_1^{(6)}\left(X_1^{(6)}(x;t), X_2^{(6)}(x;t); s\right)}{x_1}} \cdot \exp\left[\frac{2ict}{x_1 - tx_2} + \frac{2icsx_1}{(x_1 - tx_2)(x_1 - tx_2 - sx_2)}\right] \\
&= \sqrt{\frac{X_1^{(6)}(x;t+s)}{x_1}} \cdot \exp\left[\frac{2ic(x_1 - tx_2)(t+s)}{(x_1 - tx_2)(x_1 - tx_2 - sx_2)}\right] \\
&= \alpha^{(6)}(x;t+s).
\end{aligned}$$

Since $X_1^{(6)}(x;t) + tX_2^{(6)}(x;t) = x_1 - tx_2$ still holds true, also unitarity is proven in very much the same way as in the previous case. Finally, strong continuity is a consequence of Lemma 2.5 again. $\square$

Notice that also in this case, assumptions of Lemma 2.5 (b) are fulfilled for any considered $m^2, c \in \mathbb{R}$, and hence operators $\Omega_{m^2,c}$ given by (3.34) - (3.39) are restrictions of generators for respective one-parameter subgroups $U_{m^2,c}$.

Unitary Representations

Take some $m^2, c \in \mathbb{R}$ such that $\frac{c}{\sqrt{m^2}} \in \mathbb{Z}$ provided $m^2 > 0$. As a consequence of Proposition 3.4, the mapping

$$(t_1, \ldots, t_6) \mapsto U_{m^2,c}(t_1, \ldots, t_6) \equiv U_{m^2,c}^{(2)}(t_2) U_{m^2,c}^{(3)}(t_3) U_{m^2,c}^{(4)}(t_4) U_{m^2,c}^{(5)}(t_5) U_{m^2,c}^{(1)}(t_1) U_{m^2,c}^{(6)}(t_6)$$

maps from $\mathbb{R}^6$ to $\mathcal{U}(\mathcal{H}_2)$ and it is unitary and strongly continuous. Explicitly,

$$U_{m^2,c}(t_1, \ldots, t_6)\psi(x) = \alpha(x; t_1, \ldots, t_6)\, \psi\left(X_1(x; t_1, t_5, t_6), X_2(x; t_1, t_5, t_6)\right) \tag{3.46}$$

where

$$\alpha(x;t) = \begin{cases} \exp\left\{\frac{t_1}{2} + \frac{it_2}{2}\left(x_1 + \frac{x_2^2 + m^2}{x_1}\right) - \frac{it_3}{2}\left(x_1 - \frac{x_2^2 + m^2}{x_1}\right) + it_4 x_2\right\} & \\ \quad \times \left(\frac{e^{t_1}x_1 - t_6 x_2 - t_5 t_6 x_1 + it_6\sqrt{m^2}}{e^{t_1}x_1 - t_6 x_2 - t_5 t_6 x_1 - it_6\sqrt{m^2}}\right)^{\frac{c}{\sqrt{m^2}}} \frac{\sqrt{X_1(x;t_1,t_5,t_6)}}{\sqrt{e^{t_1}x_1}}, & m^2 \neq 0, \\ \exp\left\{\frac{t_1}{2} + \frac{it_2}{2}\left(x_1 + \frac{x_2^2}{x_1}\right) - \frac{it_3}{2}\left(x_1 - \frac{x_2^2}{x_1}\right) + it_4 x_2 + \frac{2ict_6}{e^{t_1}x_1 - t_6(x_2 + t_5 x_1)}\right\} & \\ \quad \times \left(1 - t_6 \frac{x_2 + t_5 x_1}{e^{t_1}x_1}\right), & m^2 = 0, \end{cases}$$

$t \equiv (t_1, \ldots, t_6) \in \mathbb{R}^6$, and

$$\begin{aligned}
X_1(x; t_1, t_5, t_6) &= e^{t_1}x_1 - 2(x_2 + t_5 x_1)t_6 + \frac{(x_2 + t_5 x_1)^2 + m^2}{e^{t_1}x_1} t_6^2 \\
&= \frac{1}{e^{t_1}x_1}\left[\left(e^{t_1}x_1 - (x_2 + t_5 x_1)t_6\right)^2 + m^2 t_6^2\right], \\
X_2(x; t_1, t_5, t_6) &= x_2 + t_5 x_1 - \frac{(x_2 + t_5 x_1)^2 + m^2}{e^{t_1}x_1} t_6.
\end{aligned}$$

As discussed in Remark 3.3, we again define $\alpha(x;t) := 0$ if $e^{t_1}x_1 = t_6(x_2 + t_6 x_1)$.

Consequently, for any $m^2, c \in \mathbb{R}$ such that $\frac{c}{\sqrt{m^2}} \in \mathbb{Z}$ provided $m^2 > 0$, the group, denote it by $\mathcal{G}_{m^2,c}$, generated by $\{U_{m^2,c}(t) | t \in \mathbb{R}^6\}$ is a Lie group. In order to prove that it is locally isomorphic to $\mathcal{P}_3$, the (local) multiplication rule (3.2) could be verified directly, similarly as in the case $\mathcal{P}_2$. Since this task would be too complicated here to be done by hand, an alternative approach shall be called for. Namely, we shall show that the product of, twelve in fact, unitary operators $U_{m^2,c}(t)U_{m^2,c}(t')$ can be after finitely many steps reordered into the form $U_{m^2,c}(t'')$, where $t''_j = \hat{f}_j(t,t')$, $1 \le j \le 6$, are continuous function on $\hat{\mathbb{T}}_6 \times \hat{\mathbb{T}}_6$, for a certain neighbourhood $\hat{\mathbb{T}}_6$ of $0 \in \mathbb{R}^6$, cf. (1.28). Then, since they share isomorphic Lie algebras, the Lie groups $\mathcal{G}_{m^2,c}$ and $\mathcal{P}_3$ have to be locally isomorphic.

Clearly, it is sufficient to restrict ourselves to reordering of all pairs of one-parameter subgroups composed in "wrong" order to $U_{m^2,c}(t'')$, where $t''_j = f_j(t,t')$, $1 \le j \le 6$, are continuous functions defined for $t, t' \in \mathbb{T}_6 \equiv \mathbb{R}^4 \times (0,1) \times (0,1)$. This is left to the Appendix. Then the previous requirement of (1.28) is fulfilled from continuity of the considered parameter functions and from finiteness of the number of steps needed to reordering of the whole $U_{m^2,c}(t)U_{m^2,c}(t')$.

Thus, the following theorem holds:

Theorem 3.5. *For any $m^2, c \in \mathbb{R}$ such that $\frac{c}{\sqrt{m^2}} \in \mathbb{Z}$ provided $m^2 > 0$, the group $\mathcal{G}_{m^2,c}$ is a Lie group, locally isomorphic to $\mathcal{P}_3$.*

Let us explain why certain pairs of $m^2, c \in \mathbb{R}$ were excluded from our consideration. Suppose temporarily, that the previous theorem holds for any $m^2, c \in \mathbb{R}$. First, we look at the product of two special elements of $\mathcal{P}_3$; for

$$R_0\left(\frac{\pi}{2}\right) := g(\ln 2, 0, 0, 0, 1, 1) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \tag{3.47}$$

and

$$R_0\left(\frac{3\pi}{2}\right) := g(\ln 2, 0, 0, 0, -1, -1) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \tag{3.48}$$

we have

$$R_0\left(\frac{\pi}{2}\right) R_0\left(\frac{3\pi}{2}\right) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}. \tag{3.49}$$

Therefore the necessary condition imposed on $\mathcal{G}_{m^2,c}$ to be isomorphic with $\mathcal{P}_3$ is

$$U_{m^2,c}(\ln 2, 0, 0, 0, 1, 1) \circ U_{m^2,c}(\ln 2, 0, 0, 0, -1, -1) = \mathbb{1}. \tag{3.50}$$

In particular, for the respective pre-factors α this requires

$$\alpha(x; \ln 2, 0, 0, 0, 1, 1)\, \alpha[X_1(x; \ln 2, 1, 1), X_2(x; \ln 2, 1, 1); \ln 2, 0, 0, 0, -1, -1] = 1 \tag{3.51}$$

for almost any $x \in \mathbb{R}$. At first, we have

$$X_1 \equiv X_1(x; \ln 2, 1, 1) = -2x_2 + \frac{(x_2+x_1)^2+m^2}{2x_1} = \frac{(x_2-x_1)^2+m^2}{2x_1}, \tag{3.52}$$

$$X_2 \equiv X_2(x; \ln 2, 1, 1 = x_2 + x_1 - \frac{(x_2+x_1)^2+m^2}{2x_1}, \tag{3.53}$$

and thus $X_1 + X_2 = x_1 - x_2$ and $X_2 - X_1 = \frac{x_2(x_1-x_2)-m^2}{x_1}$.

First, for $m^2 > 0$, the product on the left-hand side of (3.51) takes form

$$\begin{aligned}
&\sqrt{2}\left(\frac{x_1-x_2+i\sqrt{m^2}}{x_1-x_2-i\sqrt{m^2}}\right)^{\frac{c}{\sqrt{m^2}}} \frac{\sqrt{X_1}}{\sqrt{2x_1}} \sqrt{2}\left(\frac{X_1+X_2-i\sqrt{m^2}}{X_1+X_2+i\sqrt{m^2}}\right)^{\frac{c}{\sqrt{m^2}}} \frac{\sqrt{\frac{(X_2+X_1)^2+m^2}{2X_1}}}{\sqrt{2X_1}} \\
&= \left(\frac{x_1-x_2+i\sqrt{m^2}}{x_1-x_2-i\sqrt{m^2}}\right)^{\frac{c}{\sqrt{m^2}}} \left(\frac{x_1-x_2-i\sqrt{m^2}}{x_1-x_2+i\sqrt{m^2}}\right)^{\frac{c}{\sqrt{m^2}}} \frac{\sqrt{2X_1}\cdot\sqrt{2x_1}}{\sqrt{2x_1}\cdot\sqrt{2X_1}} \\
&\equiv \left[e^{i\beta(x;m^2)}\right]^{\frac{c}{\sqrt{m^2}}} \left[e^{i\left(2\pi-\beta(x;m^2)\right)}\right]^{\frac{c}{\sqrt{m^2}}} \\
&= \exp\left\{ i\beta(x;m^2)\frac{c}{\sqrt{m^2}} + i2\pi\frac{c}{\sqrt{m^2}} - i\beta(x;m^2)\frac{c}{\sqrt{m^2}} \right\} \\
&= \exp\left(i2\pi\frac{c}{\sqrt{m^2}} \right),
\end{aligned}$$

where $\beta(x; m^2) := \arg\left(\frac{x_1-x_2+i\sqrt{m^2}}{x_1-x_2-i\sqrt{m^2}}\right)$. Now it is clear that the condition (3.51) is fulfilled if and only if $\frac{c}{\sqrt{m^2}} \in \mathbb{Z}$. Second, for $m^2 < 0$, there is no restriction on $m^2, c \in \mathbb{R}$ since the product in (3.51) is

$$\begin{aligned}
&\left(\frac{x_1-x_2-\sqrt{|m^2|}}{x_1-x_2+\sqrt{|m^2|}}\right)^{-\frac{ic}{\sqrt{|m^2|}}} \left(\frac{x_1-x_2+\sqrt{|m^2|}}{x_1-x_2-\sqrt{|m^2|}}\right)^{-\frac{ic}{\sqrt{|m^2|}}} \frac{\sqrt{2X_1}\cdot\sqrt{2x_1}}{\sqrt{2x_1}\cdot\sqrt{2X_1}} \\
&= \exp\left\{ -\frac{ic}{\sqrt{|m^2|}} \left(\ln\left|\frac{x_1-x_2-\sqrt{|m^2|}}{x_1-x_2+\sqrt{|m^2|}}\right| + \ln\left|\frac{x_1-x_2+\sqrt{|m^2|}}{x_1-x_2-\sqrt{|m^2|}}\right| \right) \right\} \\
&= 1
\end{aligned}$$

because $\arg\left(\frac{x_1-x_2-\sqrt{|m^2|}}{x_1-x_2+\sqrt{|m^2|}}\right) = \arg\left(\frac{x_1-x_2+\sqrt{|m^2|}}{x_1-x_2-\sqrt{|m^2|}}\right) = \frac{\pi}{2} \pm \frac{\pi}{2}$. Similarly, if $m^2 = 0$,

$$\sqrt{2}\, e^{\frac{2ic}{x_1-x_2}} \left(\frac{x_1-x_2}{2x_1}\right) \sqrt{2}\, e^{-\frac{2ic}{X_1+X_2}} \left(\frac{X_1+X_2}{2X_1}\right) = 2\frac{(x_1-x_2)^2}{2x_1\frac{(x_2-x_1)^2}{x_1}} e^{\frac{2ic}{x_1-x_2}-\frac{2ic}{x_1-x_2}} = 1.$$

In each case we also have (in fact, the first relation was already used)

$$\begin{aligned}
X_1(X_1, X_2; \ln 2, -1, -1) &= \frac{(X_1+X_2)^2+m^2}{2X_1} = \frac{(x_1-x_2)^2+m^2}{2\frac{(x_1-x_2)^2+m^2}{2x_1}} = x_1, \\
X_2(X_1, X_2; \ln 2, -1, -1) &= \frac{X_2^2 - X_1^2 + m^2}{2X_1} = \frac{(X_2+X_1)(X_2-X_1)+m^2}{2X_1} \\
&= \frac{(x_1-x_2)\frac{x_2(x_1-x_2)-m^2}{x_1} + m^2}{\frac{(x_1-x_2)^2+m^2}{x_1}} = x_2.
\end{aligned}$$

All in all, (3.50) holds for any $m^2, c \in \mathbb{R}$ such that $\frac{c}{\sqrt{m^2}} \in \mathbb{Z}$ provided $m^2 > 0$ and obviously fails for the others. Thus, if $U^{(6)}_{m^2,c}$, $\frac{c}{\sqrt{m^2}} \notin \mathbb{Z}$ were one-parameter subgroups indeed, they would correspond to certain non-trivial coverings of $\mathcal{P}_3$, not to $\mathcal{P}_3$ itself.

It is not difficult to see that the necessary condition (3.50) is in fact also sufficient. Namely, that the coordinates in the "semisimple part" $\mathrm{SO}_0(1,2)$ of $\mathcal{P}_3$ could be chosen as $e^{\tilde{t}_5 \mathbf{L}_{12}} e^{\tilde{t}_1 \mathbf{L}_{01}} e^{\tilde{t}_6 \mathbf{L}_{02}}$, $(\tilde{t}_1, \tilde{t}_2, \tilde{t}_3) \in \mathbb{R}^3$, where

$$e^{\tilde{t}_6 \mathbf{L}_{02}} = \begin{pmatrix} \cosh \tilde{t}_6 & 0 & -\sinh \tilde{t}_6 & 0 \\ 0 & 1 & 0 & 0 \\ -\sinh \tilde{t}_6 & 0 & \cosh \tilde{t}_6 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \quad \text{and} \quad e^{\tilde{t}_5 \mathbf{L}_{12}} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos \tilde{t}_5 & -\sin \tilde{t}_5 & 0 \\ 0 & \sin \tilde{t}_5 & \cos \tilde{t}_5 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

Then it is clear that, topologically, $\mathcal{P}_3 \cong \mathrm{SO}(2, \mathbb{R}) \times \mathbb{R}^5$. Since the topological space $\mathbb{R}^5$ is simply connected, the universal covering group $\widetilde{\mathcal{P}}_3$ of $\mathcal{P}_3$ has to be homeomorphic to $\widetilde{\mathrm{SO}}(2, \mathbb{R}) \times \mathbb{R}^5$, where $\widetilde{\mathrm{SO}}(2, \mathbb{R})$ is the universal cover of the rotation group $\mathrm{SO}(2, \mathbb{R})$.

Altogether, Theorem 3.5 was strengthen as follows:

Theorem 3.6. *For any $m^2, c \in \mathbb{R}$ such that $\frac{c}{\sqrt{m^2}} \in \mathbb{Z}$ provided $m^2 > 0$, we have $\mathcal{G}_{m^2,c} \cong \mathcal{P}_3$.*

Thus, for any pair of the parameters $m^2, c \in \mathbb{R}$ specified in the theorem, the formula (3.46) defines a unitary representations of $\mathcal{P}_3$. Repeat that not every element of $\mathcal{P}_3$ is represented directly by (3.46), in principle it has to be decomposed into a product of elements for which (3.46) is defined. For sake of brevity we shall denote the resulting representation (of whole $\mathcal{P}_3$) also by $U_{m^2,c}$.

Irreducibility

First, it is again well-visible which of the representations (3.46) are reducible. Namely, one can see that if $m^2 \geq 0$ than $\operatorname{sgn} X_1(x; t_1, t_5, t_6) = \operatorname{sgn} x_1$ and therefore the subspaces $\mathcal{H}_2^+ \equiv L^2(\mathbb{R}^+ \times \mathbb{R}, \mathrm{d}^2 x)$ and $\mathcal{H}_2^- \equiv L^2(\mathbb{R}^- \times \mathbb{R}, \mathrm{d}^2 x)$ are invariant, and $\mathcal{H}_2 = \mathcal{H}_2^+ \oplus \mathcal{H}_2^-$. As before, let us denote

$$U^{\pm}_{m^2,c}(t) := U_{m^2,c}(t)\big|_{\mathcal{H}_2^{\pm}} \tag{3.54}$$

whenever $m^2 = 0$, or $m^2 > 0$ and $\frac{c}{\sqrt{m^2}} \in \mathbb{Z}$. Again, no further refinement is admissible.

Proposition 3.7. *The representations*
(a) $U^{\pm}_{0,c}$, $c \in \mathbb{R}$,
(b) $U^{\pm}_{m^2,c}$, $m^2 > 0$, $c \in \mathbb{R}$ such that $\frac{c}{\sqrt{m^2}} \in \mathbb{Z}$,
(c) $U_{m^2,c}$, $m^2 < 0$ and $c \in \mathbb{R}$,
are irreducible.

Proof. As discussed in the proof of Proposition 2.8, any $T \in \mathcal{B}(\mathcal{H})$ commuting with all images under a representations on $\mathcal{H}$ commutes also with the respective generators. Here the Hilbert space $\mathcal{H}$ varies with $\operatorname{sgn} m^2$.

First, assume $m^2 \geq 0$. Commutativity of T with $\Omega^{\pm}_{m^2,c}(z) := \Omega_{m^2,c}(z)\big|_{C_0^\infty(\mathbb{R}^{\pm} \times \mathbb{R})}$ for z equals to $P_0 - P_1$ and P_2, respectively, forces it to be of the form $T\psi(x_1, x_2) = \tau(x_1, x_2)\psi(x_1, x_2)$, $\psi \in C_0^\infty(\mathbb{R}^{\pm} \times \mathbb{R})$, for some bounded function $\tau : \mathbb{R}^{\pm} \times \mathbb{R} \to \mathbb{C}$. Commutativity with $U^{(1)}_{m^2,c}(t_1)$ further implies that $\tau(x_1, x_2) \equiv \tilde{\tau}(x_2)$ is independent of x_1. Finally, commuting T with the fifth one-parameter subgroup, we obtain that $\tilde{\tau}$ is constant in fact and hence T is a multiple of the identity.

Second, if $m^2 < 0$, analogous arguments lead to $T\psi(x_1, x_2) = \tau(x_1, x_2)\psi(x_1, x_2)$, $\psi \in C_0^\infty(\mathbb{R}^\times \times \mathbb{R})$, where $\tau : \mathbb{R}^\times \times \mathbb{R} \to \mathbb{C}$ is constant on $\mathbb{R}^+ \times \mathbb{R}$ and $\mathbb{R}^- \times \mathbb{R}$. In this case we have to make use of the sixth one-parameter subgroup in order to "connect" these two component; clearly $\operatorname{sgn} X_1^{(6)}(x; t) = -\operatorname{sgn} x_1 = -1$ for any $x_1 \in \mathbb{R}^+$ and $x_2 \in \mathbb{R}$ such that $(x_1 - t x_2)^2 + m^2 t^2 < 0$. Thus, τ has to be constant everywhere. $\square$

Mutual Non-equivalence

Now the only remaining task is to show that none of the representations are equivalent with each other. Also in this case we shall make use of Lemma 2.9 in order to focus on spectra of representatives of certain elements from $\mathfrak{U}(\mathfrak{p}_3)$.

Again the Lie algebra representations related to $U^{\pm}_{m^2,c}$, m^2, c specified above, are

$$\Omega^{\pm}_{m^2,c}(x) := \Omega_{m^2,c}(x)\big|_{C_0^\infty(\mathbb{R}^\pm\times\mathbb{R})}, \tag{3.55}$$

$x \in \mathfrak{p}_3$. Extension to $\mathfrak{U}(\mathfrak{p}_3)$ is straightforward. According to part (a) of Lemma 2.9, the representations corresponding to distinct values of the parameters m^2 and c cannot be equivalent. Thus, if $m^2 < 0$, the question of non-equivalence is answered. Let us now fix $m^2 \geq 0$ and $c \in \mathbb{R}$ such that $\frac{c}{\sqrt{m^2}} \in \mathbb{Z}$ provided $m^2 > 0$, and let us look at the two representations $U^{\pm}_{m^2,c}$ in some detail.

In all admissible cases, for any $\psi \in C_0^\infty(\mathbb{R}^\pm \times \mathbb{R})$ we have

$$\Omega^{\pm}_{m^2,c}(P_0)\psi(x) = \frac{i}{2x_1}(x_1^2 + x_2^2 + m^2)\psi(x).$$

Again, both operators $\Omega^{\pm}_{m^2,c}(P_0)$ have non-empty spectrum (cf. [6]) and simultaneously $\sigma\left[\Omega^{+}_{m^2,c}(P_0)\right] \subset i\mathbb{R}^+$ while $\sigma\left[\Omega^{-}_{m^2,c}(P_0)\right] \subset i\mathbb{R}^-$, regardless of m^2 and c. Consequently, $U^{+}_{m^2,c} \not\cong U^{+}_{m^2,c}$ by Lemma 2.9 (b).

Summary

To conclude,

Theorem 3.8. *The set*

$$\left\{U^{\pm}_{0,c}\middle|c \in \mathbb{R}\right\} \cup \left\{U^{\pm}_{m^2,c}\middle|m^2 > 0, c \in \mathbb{R}, \frac{c}{\sqrt{m^2}} \in \mathbb{Z}\right\} \cup \left\{U_{m^2,c}\middle|m^2 < 0, c \in \mathbb{R}\right\},$$

where $U^{(\pm)}_{m^2,c}$ are given by (3.46) *and* (3.54)*, is a family of pairwise non-equivalent irreducible unitary representations of the Lie group $\mathcal{P}_3$.*

Above all, we shall see below, by comparison with the representations constructed within the frame of Mackey theory, that our construction exhausts the whole list of all irreducible unitary representations of the Lie group $\mathcal{P}_3$.

3.2 Mackey's Technique

Again we shall construct the set of irreducible unitary representations of the Lie group $\mathcal{P}_3$ also in frame of Mackey theory, in order to compare and verify our results.

The dual group to T^3 is

$$\hat{T}^3 = \left\{ \begin{pmatrix} \chi_0 \\ \chi_1 \\ \chi_2 \end{pmatrix} \middle| \chi_0, \chi_1, \chi_2 \in \mathbb{R} \right\}. \tag{3.56}$$

Any (non-zero) orbit can be parametrized as follows:

$$\mathcal{O}_\xi = \left\{\Lambda^{-1}\xi\middle|\Lambda \in \mathrm{SO}_0(1,2)\right\} = \{\chi(x_1, x_2) := R_1(-x_1)R_5(-x_2)\xi | x_1, x_2 \in \mathbb{R}\}, \tag{3.57}$$

and therefore the mapping $h : \mathcal{O}_\xi \to \mathrm{SO}_0(1,2)$ considered in §1.3.2 can be naturally chosen as $\chi(x_1, x_2) \mapsto \Lambda(x_1, x_2, 0)$. In order to determine the action of $\mathrm{SO}_0(1,2)$ on $\mathcal{O}_\xi$, we have to find $X_1 = X_1(x_1, x_2; t_1, t_5, t_6)$ and $X_2 = X_2(x_1, x_2; t_1, t_5, t_6)$ such that

$$\chi(X_1, X_2) = \Lambda(t_5, t_1, t_6)^{-1}\chi(x_1, x_2) \equiv R_6(-t_6)R_1(-t_1)R_5(-t_5)\chi(x_1, x_2). \tag{3.58}$$

Since the action is more complicated now, the solution of (3.58) has to be discussed for different types of orbits separately. Neither the stabilizer groups are trivial any more; to find the stabilizer S_ξ of ξ, one has to solve the vector equation

$$\xi = \Lambda(t_5, t_1, t_6)^{-1}\xi \equiv R_6(-t_6)R_1(-t_1)R_5(-t_5)\xi. \tag{3.59}$$

Of course, also the solution of (3.59) varies among different types of orbits. As before, we shall denote, for an orbit $\mathcal{O}_\xi$ and a representation W of S_ξ, the resulting representation of $\mathcal{P}_3$ as follows:

$$U_{\mathcal{O}_\xi,W}(t) \equiv U_{\mathcal{O}_\xi,W}(t_1, \ldots, t_6) \equiv U_{\mathcal{O}_\xi,W}\left(\Lambda(t_5, t_1, t_6), a(t_2, t_3, t_4)\right).$$

3.2.1 Orbits of Type I

First, consider an orbit of type $\mathrm{I}^\pm$. In this case the origin is $\xi = \pm\begin{pmatrix}1\\1\\0\end{pmatrix}$, hence

$$\chi(x_1, x_2) = \pm\begin{pmatrix}\cosh x_1 & \sinh x_1 & 0\\ \sinh x_1 & \cosh x_1 & 0\\ 0 & 0 & 1\end{pmatrix}\begin{pmatrix}1+\frac{x_2^2}{2} & \frac{x_2^2}{2} & -x_2\\ -\frac{x_2^2}{2} & 1-\frac{x_2^2}{2} & x_2\\ -x_2 & -x_2 & 1\end{pmatrix}\begin{pmatrix}1\\1\\0\end{pmatrix} = \pm\begin{pmatrix}e^{x_1}+\frac{x_2^2}{e^{x_1}}\\ e^{x_1}-\frac{x_2^2}{e^{x_1}}\\ -2x_2\end{pmatrix}.$$

In this case the vector equation (3.58) has solution

$$\begin{aligned} X_1^{\mathrm{I}} = x_1 + t_1 + \ln\big(1 &+ t_5^2t_6^2e^{-2t_1} + 2t_5t_6^2x_2e^{-x_1-2t_1}\\ &-2t_5t_6e^{-t_1} + t_6^2x_2^2e^{-2x_1-2t_1} - 2t_6x_2e^{-x_1-t_1}\big), \end{aligned} \tag{3.60}$$

$$X_2^{\mathrm{I}} = x_2 - e^{x_1-t_1}t_5^2t_6 + e^{x_1}t_5 - 2e^{-t_1}t_5t_6x_2 - t_6e^{-t_1-x_1}x_2^2. \tag{3.61}$$

Further, we need the matrix $k_{(\chi,\Lambda)} \in S_\xi$ satisfying $h(\chi)\Lambda = k_{(\chi,\Lambda)}h(\Lambda^{-1}\chi)$. Since the equation (3.59) is solved by $t_1 = t_5 = 0$ and t_6 arbitrary, $S_\xi = \{R_6(\varphi)|\varphi \in \mathbb{R}\} \cong \mathrm{E}_1$ and hence we are searching for $k_{(\chi,\Lambda)}$ in the form of $R_6(\varphi)$. In other words, we are solving the following matrix equation for $\varphi = \varphi(x_1, x_2; t_1, t_5, t_6)$:

$$R_5(x_2)R_1(x_1)R_5(t_5)R_1(t_1)R_6(t_6) = R_6(\varphi)R_5(X_2^{\mathrm{I}})R_1(X_1^{\mathrm{I}}), \tag{3.62}$$

with X_1^{I} and X_2^{I} given by (3.60) and (3.61), respectively. The solution is

$$\varphi = \frac{t_6}{e^{t_1+x_1} - t_5t_6e^{x_1} - t_6x_2} \tag{3.63}$$

and, because all irreducible unitary representations W_s of E_1 are one-dimensional (cf. [4], p. 159) and therefore of the form $R_6(\varphi) \mapsto e^{is\varphi}$, $s \in \mathbb{R}$, we have

$$W_s\left[k_{(\chi(x_1,x_2),\Lambda(t_5,t_1,t_6))}\right] = \exp\left(\frac{ist_6}{e^{t_1+x_1} - t_5t_6e^{x_1} - t_6x_2}\right), \quad s \in \mathbb{R}. \tag{3.64}$$

The "character" part $\exp(i\chi \cdot a)$ is completely analogical to the $n = 2$ case and hence it only remains to determine the Radon-Nikodym derivative of a quasi-invariant measure on $\mathcal{O}_\xi$. Such a measure is given, up to an inessential multiplicative factor, by

$$\mathrm{d}\mu(\chi) = \frac{\mathrm{d}\chi_1\mathrm{d}\chi_2}{|\chi_0|} \tag{3.65}$$

(cf. [4], p. 131). In our parametrization we have

$$\mathrm{d}\mu(\chi(x_1, x_2)) = \left|\frac{\partial(\chi_1(x_1, x_2), \chi_2(x_1, x_2))}{\partial(x_1, x_2)}\right| \cdot \frac{\mathrm{d}x_1\mathrm{d}x_2}{|\chi_0(x_1, x_2)|}$$

$$= \left|\det\begin{pmatrix} e^{x_1}+\frac{x_2^2}{e^{x_1}} & -\frac{2x_2}{e^{x_1}} \\ 0 & -2 \end{pmatrix}\right| \cdot \frac{\mathrm{d}x_1\mathrm{d}x_2}{\left|e^{x_1}+\frac{x_2^2}{e^{x_1}}\right|}$$
$$= 2\,\mathrm{d}x_1\mathrm{d}x_2.$$

Further, it is shown in the Appendix that $\left|\frac{\partial(X_1^{\mathrm{I}},X_2^{\mathrm{I}})}{\partial(x_1,x_2)}\right| = 1$. Consequently we have

$$\mathrm{d}\mu\left[\Lambda(t_5,t_1,t_6)^{-1}\chi(x_1,x_2)\right] = \mathrm{d}\mu(X_1^{\mathrm{I}},X_2^{\mathrm{I}}) = 2\,\mathrm{d}X_1^{\mathrm{I}}\mathrm{d}X_2^{\mathrm{I}} = \frac{2}{\left|\frac{\partial(X_1^{\mathrm{I}},X_2^{\mathrm{I}})}{\partial(x_1,x_2)}\right|}\mathrm{d}x_1\mathrm{d}x_2 = 2\,\mathrm{d}x_1\mathrm{d}x_2$$

and therefore $\rho(\chi,\Lambda)\equiv 1$.

All in all, the representations corresponding to the orbits $\mathrm{I}^{\pm}$ are

$$U^{\mathrm{I}}_{s,\pm}(t)\psi(x_1,x_2) = e^{\pm i\left[t_2\left(e^{x_1}+\frac{x_2^2}{e^{x_1}}\right)-t_3\left(e^{x_1}-\frac{x_2^2}{e^{x_1}}\right)+2t_4x_2\right]+\frac{ist_6}{e^{t_1+x_1}-t_5t_6e^{x_1}-t_6x_2}}\,\psi(X_1^{\mathrm{I}},X_2^{\mathrm{I}}), \tag{3.66}$$

with $s\in\mathbb{R}$ and X_1^{I}, X_2^{I} given by (3.60), (3.61). Notice that, similarly as in the case of $\mathcal{P}_2$, we abused the notation to identify $\psi(\chi(x_1,x_2))\equiv\psi(x_1,x_2)$ and hence the representation space is $L^2(\mathbb{R}^2,\mathrm{d}^2x)$.

3.2.2 Orbits of Type II

Second, for an orbit of type $\mathrm{II}^{\pm}_{|m|}$, $|m|>0$, we have $\xi = \pm|m|\begin{pmatrix}1\\0\\0\end{pmatrix}$. Then

$$\chi(x_1,x_2) = R_1(-x_1)R_5(-x_2)\xi = \pm|m|\begin{pmatrix}\cosh x_1+\frac{x_2^2}{2e^{x_1}}\\ \sinh x_1-\frac{x_2^2}{2e^{x_1}}\\ -x_2\end{pmatrix}.$$

Putting this expression back to (3.58), the solution of the equation is

$$\begin{aligned} X_1^{\mathrm{II}} = x_1+t_1+\ln\big(1&+t_5^2t_6^2e^{-2t_1}+2t_5t_6^2x_2e^{-x_1-2t_1}-2t_5t_6e^{-t_1}\\ &+t_6^2x_2^2e^{-2x_1-2t_1}-2t_6x_2e^{-x_1-t_1}+t_6^2e^{-2x_1-2t_1}\big),\end{aligned} \tag{3.67}$$

$$X_2^{\mathrm{II}} = x_2-e^{x_1-t_1}t_5^2t_6+e^{x_1}t_5-2e^{-t_1}t_5t_6x_2-t_6e^{-t_1-x_1}x_2^2-t_6e^{-t_1-x_1}. \tag{3.68}$$

Further, the stabilizer equation (3.59) is now solved by $t_5=t_6$ and $t_1=\ln(1+t_6^2)$, with t_6 arbitrary. Since

$$\Lambda(t_6,\ln(1+t_6^2),t_6) = \begin{pmatrix}1&0&0\\0&\frac{1-t_6^2}{1+t_6^2}&-\frac{2t_6}{1+t_6^2}\\0&\frac{2t_6}{1+t_6^2}&\frac{1-t_6^2}{1+t_6^2}\end{pmatrix} \equiv \begin{pmatrix}1&0&0\\0&\cos\varphi&-\sin\varphi\\0&\sin\varphi&\cos\varphi\end{pmatrix} \equiv R_0(\varphi),$$

$S_\xi\cong\mathrm{SO}(2,\mathbb{R})$, as expected. In order to determine the element $(k_{\Lambda,\chi})$, we have to solve

$$R_5(x_2)R_1(x_1)R_5(t_5)R_1(t_1)R_6(t_6) = R_0(\varphi)R_5(X_2^{\mathrm{II}})R_1(X_1^{\mathrm{II}}), \tag{3.69}$$

with X_1^{II} and X_2^{II} given by (3.67) and (3.68), respectively. The solution is

$$e^{i\varphi} = \frac{1-e^{-t_1}t_5t_6-e^{-t_1-x_1}t_6x_2+ie^{-t_1-x_1}t_6}{1-e^{-t_1}t_5t_6-e^{-t_1-x_1}t_6x_2-ie^{-t_1-x_1}t_6} \tag{3.70}$$

and since the irreducible unitary representations W_s of $\mathrm{SO}(2,\mathbb{R})$ are $\varphi\mapsto e^{is\varphi}$, $s\in\mathbb{Z}$, (cf. [4], p. 159), we have

$$W_s\left[k_{(\chi(x_1,x_2),\Lambda(t_5,t_1,t_6))}\right] = \left(\frac{1-e^{-t_1}t_5t_6-e^{-t_1-x_1}t_6x_2+ie^{-t_1-x_1}t_6}{1-e^{-t_1}t_5t_6-e^{-t_1-x_1}t_6x_2-ie^{-t_1-x_1}t_6}\right)^s, \quad s\in\mathbb{Z}. \tag{3.71}$$

Finally,

$$d\mu(\chi(x_1,x_2)) = \frac{d\chi_1 d\chi_2}{|\chi_0|} = \left| |m|^2 \det \begin{pmatrix} \cosh x_1 + \frac{x_2^2}{2e^{x_1}} & -\frac{x_2}{e^{x_1}} \\ 0 & -1 \end{pmatrix} \right| \cdot \frac{dx_1 dx_2}{\left| |m| \left(\cosh x_1 + \frac{x_2^2}{2e^{x_1}} \right) \right|}$$
$$= |m|\, dx_1 dx_2.$$

and $\left| \frac{\partial(X_1^{\mathrm{II}}, X_2^{\mathrm{II}})}{\partial(x_1,x_2)} \right| = 1$ (see the Appendix) together imply $\rho \equiv 1$ again.

Altogether, we obtain the following family of representations:

$$U^{\mathrm{II}}_{|m|,s,\pm}(t)\psi(x_1,x_2) = e^{\pm i|m| \left[t_2 \left(\cosh x_1 + \frac{x_2^2}{2e^{x_1}} \right) - t_3 \left(\sinh x_1 - \frac{x_2^2}{2e^{x_1}} \right) + t_4 x_2 \right]}$$
$$\times \left(\frac{1 - e^{-t_1} t_5 t_6 - e^{-t_1 - x_1} t_6 x_2 + i e^{-t_1 - x_1} t_6}{1 - e^{-t_1} t_5 t_6 - e^{-t_1 - x_1} t_6 x_2 - i e^{-t_1 - x_1} t_6} \right)^s \psi(X_1^{\mathrm{II}}, X_2^{\mathrm{II}}), \tag{3.72}$$

where $\psi \in L^2(\mathbb{R}^2, d^2x)$, $|m| > 0$, $s \in \mathbb{Z}$ and X_1^{II}, X_2^{II} are given by (3.67), (3.68).

3.2.3 Orbits of Type III

Finally, the procedure is completely analogous if the orbit is of type $\mathrm{III}_{|m|}$, $|m| > 0$. In this case we have $\xi = |m| \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$ and $\chi(x) = |m| \begin{pmatrix} \sinh x_1 + \frac{x_2^2}{2e^{x_1}} \\ \cosh x_1 - \frac{x_2^2}{2e^{x_1}} \\ -x_2 \end{pmatrix}$. Further,

$$X_1^{\mathrm{III}} = x_1 + t_1 + \ln\left(1 + t_5^2 t_6^2 e^{-2t_1} + 2t_5 t_6^2 x_2 e^{-x_1 - 2t_1} - 2t_5 t_6 e^{-t_1} + t_6^2 x_2^2 e^{-2x_1 - 2t_1} - 2t_6 x_2 e^{-x_1 - t_1} - t_6^2 e^{-2x_1 - 2t_1}\right), \tag{3.73}$$

$$X_2^{\mathrm{III}} = x_2 - e^{x_1 - t_1} t_5^2 t_6 + e^{x_1} t_5 - 2e^{-t_1} t_5 t_6 x_2 - t_6 e^{-t_1 - x_1} x_2^2 + t_6 e^{-t_1 - x_1}. \tag{3.74}$$

A general element of the stabilizer $S_\xi \cong \mathrm{SO}(1,1)$ is now, with $t_6 \in (-1,1)$ arbitrary,

$$\Lambda\left(-t_6, \ln(1 - t_6^2), t_6\right) = \begin{pmatrix} \frac{1+t_6^2}{1-t_6^2} & 0 & -\frac{2t_6}{1-t_6^2} \\ 0 & 1 & 0 \\ -\frac{2t_6}{1-t_6^2} & 0 & \frac{1+t_6^2}{1-t_6^2} \end{pmatrix} \equiv \begin{pmatrix} \cosh\varphi & 0 & -\sinh\varphi \\ 0 & 1 & 0 \\ -\sinh\varphi & 0 & \cosh\varphi \end{pmatrix} \equiv R(\varphi),$$

and $k_{(\chi,\Lambda)} = R(\varphi)$ for $e^\varphi = \frac{1 - e^{-t_1} t_5 t_6 - e^{-t_1 - x_1} t_6 x_2 + e^{-t_1 - x_1} t_6}{1 - e^{-t_1} t_5 t_6 - e^{-t_1 - x_1} t_6 x_2 - e^{-t_1 - x_1} t_6}$. Because $\mathrm{SO}(1,1) \cong \mathrm{E}_1$,

$$W_s\left[k_{(\chi(x_1,x_2), \Lambda(t_5,t_1,t_6))}\right] = \left(\frac{1 - e^{-t_1} t_5 t_6 - e^{-t_1 - x_1} t_6 x_2 + e^{-t_1 - x_1} t_6}{1 - e^{-t_1} t_5 t_6 - e^{-t_1 - x_1} t_6 x_2 - e^{-t_1 - x_1} t_6} \right)^{is}, \quad s \in \mathbb{R}. \tag{3.75}$$

Finally, exactly as in the previous case we find $d\mu(\chi(x_1,x_2)) = |m|\, dx_1 dx_2$ and $\rho \equiv 1$.

Therefore, the resulting representations are

$$U^{\mathrm{III}}_{|m|,s}(t)\psi(x_1,x_2) = e^{|m| \left[t_2 \left(\sinh x_1 + \frac{x_2^2}{2e^{x_1}} \right) - t_3 \left(\cosh x_1 - \frac{x_2^2}{2e^{x_1}} \right) + t_4 x_2 \right]}$$
$$\times \left(\frac{1 - e^{-t_1} t_5 t_6 - e^{-t_1 - x_1} t_6 x_2 + e^{-t_1 - x_1} t_6}{1 - e^{-t_1} t_5 t_6 - e^{-t_1 - x_1} t_6 x_2 - e^{-t_1 - x_1} t_6} \right)^{is} \psi(X_1^{\mathrm{III}}, X_2^{\mathrm{III}}), \tag{3.76}$$

with $\psi \in L^2(\mathbb{R}^2, d^2x)$, $|m| > 0$, $s \in \mathbb{R}$ and X_1^{III}, X_2^{III} given by (3.73), (3.74).

Remark 3.4. Notice that the solutions of above matrix equations were found with help MAPLE CAS since the matrices involved are to complicated to solve the relations by hand. This is one of the advantage of the construction technique we suggested and conducted above, namely that we deal with much simpler algebraical tasks.

3.3 Comparison of Results

As in the case of $\mathcal{P}_2$, we shall show that our approach to construction of irreducible unitary representations of the Lie group $\mathcal{P}_3$ is completely equivalent to the Mackey's technique.

3.3.1 Spectra of Generators and Casimir Operators

And again, we shall first investigate spectra of certain represented elements of $\mathfrak{p}_3$ and $\mathfrak{U}(\mathfrak{p}_3)$ within the representations Θ on $\mathcal{H}_2$ induced by the Lie group representations U constructed in the previous section.

First, for the representations of type I we easily have

$$\Theta^{\mathrm{I}}_{s,\pm}(P_0) \equiv \frac{\mathrm{d}}{\mathrm{d}t_2} U^{\mathrm{I}}_{s,\pm}(t_2)\Big|_{t_2=0} = \pm i \left(e^{x_1} + \frac{x_2^2}{e^{x_1}}\right) \mathbb{1}, \tag{3.77}$$

$$\Theta^{\mathrm{I}}_{s,\pm}(P_1) \equiv \frac{\mathrm{d}}{\mathrm{d}t_3} U^{\mathrm{I}}_{s,\pm}(t_3)\Big|_{t_3=0} = \mp i \left(e^{x_1} - \frac{x_2^2}{e^{x_1}}\right) \mathbb{1}, \tag{3.78}$$

$$\Theta^{\mathrm{I}}_{s,\pm}(P_2) \equiv \frac{\mathrm{d}}{\mathrm{d}t_4} U^{\mathrm{I}}_{s,\pm}(t_4)\Big|_{t_4=0} = \pm 2 i x_2 \mathbb{1}, \tag{3.79}$$

$$\Theta^{\mathrm{I}}_{s,\pm}(L_{01}) \equiv \frac{\mathrm{d}}{\mathrm{d}t_1} U^{\mathrm{I}}_{s,\pm}(t_1)\Big|_{t_1=0} = \partial_{x_1}, \tag{3.80}$$

$$\Theta^{\mathrm{I}}_{s,\pm}(L_{12} - L_{02}) \equiv \frac{\mathrm{d}}{\mathrm{d}t_5} U^{\mathrm{I}}_{s,\pm}(t_5)\Big|_{t_5=0} = e^{x_1}\partial_{x_2}, \tag{3.81}$$

$$\Theta^{\mathrm{I}}_{s,\pm}(L_{12} + L_{02}) \equiv \frac{\mathrm{d}}{\mathrm{d}t_6} U^{\mathrm{I}}_{s,\pm}(t_6)\Big|_{t_6=0} = i s e^{-x_1}\mathbb{1} - 2x_2 e^{-x_1}\partial_{x_1} - x_2^2 e^{-x_1}\partial_{x_2}, \tag{3.82}$$

(omitted parameters t_j in argument of $U^{\mathrm{I}}_{s,\pm}$ equal zero), hence

$$\Theta^{\mathrm{I}}_{s,\pm}(L_{02}) = \frac{is}{2e^{x_1}}\mathbb{1} - \frac{x_2}{e^{x_1}}\partial_{x_1} - \frac{1}{2}\left(e^{x_1} + \frac{x_2^2}{e^{x_1}}\right)\partial_{x_2}, \tag{3.83}$$

$$\Theta^{\mathrm{I}}_{s,\pm}(L_{12}) = \frac{is}{2e^{x_1}}\mathbb{1} - \frac{x_2}{e^{x_1}}\partial_{x_1} + \frac{1}{2}\left(e^{x_1} - \frac{x_2^2}{e^{x_1}}\right)\partial_{x_2}. \tag{3.84}$$

Then

$$\Theta^{\mathrm{I}}_{s,\pm}(M^2) = \left[\left(e^{x_1} + \frac{x_2^2}{e^{x_1}}\right)^2 - \left(e^{x_1} - \frac{x_2^2}{e^{x_1}}\right)^2 - 4x_2^2\right]\mathbb{1} = 0,$$

and similarly, cf. (3.6),

$$\begin{aligned}
\Theta^{\mathrm{I}}_{s,\pm}(C) &= \mp i \left(e^{x_1} - \frac{x_2^2}{e^{x_1}}\right)\left[\frac{is}{2e^{x_1}}\mathbb{1} - \frac{x_2}{e^{x_1}}\partial_{x_1} - \frac{1}{2}\left(e^{x_1} + \frac{x_2^2}{e^{x_1}}\right)\partial_{x_2}\right] \\
&\quad \mp i \left(e^{x_1} + \frac{x_2^2}{e^{x_1}}\right)\left[\frac{is}{2e^{x_1}}\mathbb{1} - \frac{x_2}{e^{x_1}}\partial_{x_1} + \frac{1}{2}\left(e^{x_1} - \frac{x_2^2}{e^{x_1}}\right)\partial_{x_2}\right] \\
&\quad \mp 2 i x_2 \partial_{x_1} \\
&= \mp i \left[2e^{x_1}\left(\frac{is}{2e^{x_1}}\mathbb{1} - \frac{x_2}{e^{x_1}}\partial_{x_1} - \frac{x_2^2}{2e^{x_1}}\partial_{x_2}\right) + 2\frac{x_2^2}{e^{x_1}}\left(\frac{e^{x_1}}{2}\partial_{x_2}\right) + 2x_2\partial_{x_1}\right] \\
&= \pm s\,\mathbb{1}.
\end{aligned}$$

Analogously, for the type II we have

$$\Theta^{\mathrm{II}}_{|m|,s,\pm}(P_0) = \pm i\,|m|\left(\cosh x_1 + \frac{x_2^2}{2e^{x_1}}\right)\mathbb{1}, \tag{3.85}$$

$$\Theta^{\mathrm{II}}_{|m|,s,\pm}(P_1) = \mp i\,|m| \left(\sinh x_1 - \frac{x_2^2}{2e^{x_1}}\right)\mathbb{1}, \tag{3.86}$$

$$\Theta^{\mathrm{II}}_{|m|,s,\pm}(P_2) = \pm i\,|m|\,x_2\mathbb{1}, \tag{3.87}$$

$$\Theta^{\mathrm{II}}_{|m|,s,\pm}(L_{01}) = \partial_{x_1}, \tag{3.88}$$

$$\Theta^{\mathrm{II}}_{|m|,s,\pm}(L_{02}) = \frac{is}{e^{x_1}}\mathbb{1} - \frac{x_2}{e^{x_1}}\partial_{x_1} - \left(\cosh x_1 + \frac{x_2^2}{2e^{x_1}}\right)\partial_{x_2}, \tag{3.89}$$

$$\Theta^{\mathrm{II}}_{|m|,s,\pm}(L_{12}) = \frac{is}{e^{x_1}}\mathbb{1} - \frac{x_2}{e^{x_1}}\partial_{x_1} + \left(\sinh x_1 - \frac{x_2^2}{2e^{x_1}}\right)\partial_{x_2}. \tag{3.90}$$

Then

$$\Theta^{\mathrm{II}}_{|m|,s,\pm}(M^2) = |m|^2\left[\left(\cosh x_1 + \frac{x_2^2}{2e^{x_1}}\right)^2 - \left(\sinh x_1 - \frac{x_2^2}{2e^{x_1}}\right)^2 - x_2^2\right]\mathbb{1} = |m|^2\,\mathbb{1},$$

and

$$\begin{aligned}\Theta^{\mathrm{II}}_{|m|,s,\pm}(C) = &\mp i\,|m|\left(\sinh x_1 - \frac{x_2^2}{2e^{x_1}}\right)\left[\frac{is}{e^{x_1}}\mathbb{1} - \frac{x_2}{e^{x_1}}\partial_{x_1} - \left(\cosh x_1 + \frac{x_2^2}{2e^{x_1}}\right)\partial_{x_2}\right]\\ &\pm i\,|m|\left(\cosh x_1 + \frac{x_2^2}{2e^{x_1}}\right)\left[\frac{is}{e^{x_1}}\mathbb{1} - \frac{x_2}{e^{x_1}}\partial_{x_1} + \left(\sinh x_1 - \frac{x_2^2}{2e^{x_1}}\right)\partial_{x_2}\right]\\ &\mp i\,|m|\,x_2\partial_{x_1}\\ = &\mp i\,|m|\left[e^{x_1}\left(\frac{is}{e^{x_1}}\mathbb{1} - \frac{x_2}{e^{x_1}}\partial_{x_1}\right) + x_2\partial_{x_1}\right]\\ = &\pm s\,|m|\,\mathbb{1}.\end{aligned}$$

Finally, the representations of type III induce

$$\Theta^{\mathrm{III}}_{|m|,s}(P_0) = i\,|m|\left(\sinh x_1 + \frac{x_2^2}{2e^{x_1}}\right)\mathbb{1}, \tag{3.91}$$

$$\Theta^{\mathrm{III}}_{|m|,s}(P_1) = -i\,|m|\left(\cosh x_1 - \frac{x_2^2}{2e^{x_1}}\right)\mathbb{1}, \tag{3.92}$$

$$\Theta^{\mathrm{III}}_{|m|,s}(P_2) = i\,|m|\,x_2\mathbb{1}, \tag{3.93}$$

$$\Theta^{\mathrm{III}}_{|m|,s}(L_{01}) = \partial_{x_1}, \tag{3.94}$$

$$\Theta^{\mathrm{III}}_{|m|,s}(L_{02}) = \frac{is}{e^{x_1}}\mathbb{1} - \frac{x_2}{e^{x_1}}\partial_{x_1} - \left(\sinh x_1 + \frac{x_2^2}{2e^{x_1}}\right)\partial_{x_2}, \tag{3.95}$$

$$\Theta^{\mathrm{III}}_{|m|,s}(L_{12}) = \frac{is}{e^{x_1}}\mathbb{1} - \frac{x_2}{e^{x_1}}\partial_{x_1} + \left(\cosh x_1 - \frac{x_2^2}{2e^{x_1}}\right)\partial_{x_2}. \tag{3.96}$$

Then

$$\Theta^{\mathrm{III}}_{|m|,s}(M^2) = |m|^2\left[\left(\sinh x_1 + \frac{x_2^2}{2e^{x_1}}\right)^2 - \left(\cosh x_1 - \frac{x_2^2}{2e^{x_1}}\right)^2 - x_2^2\right]\mathbb{1} = -|m|^2\,\mathbb{1},$$

$$\begin{aligned}\Theta^{\mathrm{III}}_{|m|,s}(C) = &-i\,|m|\left(\cosh x_1 - \frac{x_2^2}{2e^{x_1}}\right)\left[\frac{is}{e^{x_1}}\mathbb{1} - \frac{x_2}{e^{x_1}}\partial_{x_1} - \left(\sinh x_1 + \frac{x_2^2}{2e^{x_1}}\right)\partial_{x_2}\right]\\ &- i\,|m|\left(\sinh x_1 + \frac{x_2^2}{2e^{x_1}}\right)\left[\frac{is}{e^{x_1}}\mathbb{1} - \frac{x_2}{e^{x_1}}\partial_{x_1} + \left(\cosh x_1 - \frac{x_2^2}{2e^{x_1}}\right)\partial_{x_2}\right]\\ &- i\,|m|\,x_2\partial_{x_1}\\ = &-i\,|m|\left[e^{x_1}\left(\frac{is}{e^{x_1}}\mathbb{1} - \frac{x_2}{e^{x_1}}\partial_{x_1}\right) + x_2\partial_{x_1}\right]\\ = &\;s\,|m|\,\mathbb{1}.\end{aligned}$$

Similarly as in the case of $\mathcal{P}_2$, comparing how the Casimir operators are represented, we have the following correspondences of irreducible unitary representations of $\mathcal{P}_3$:

$$\left\{U^{\pm}_{0,s}\right\} \longleftrightarrow \left\{U^{\mathrm{I}}_{s,+}, U^{\mathrm{I}}_{-s,-}\right\}, \qquad s \in \mathbb{R}, \tag{3.97}$$

$$\left\{U^{\pm}_{|m|^2,s|m|}\right\} \longleftrightarrow \left\{U^{\mathrm{II}}_{|m|,s,+}, U^{\mathrm{II}}_{|m|,-s,-}\right\}, \quad |m| \in \mathbb{R}^+, s \in \mathbb{Z}, \tag{3.98}$$

$$\left\{U_{-|m|^2,s|m|}\right\} \longleftrightarrow \left\{U^{\mathrm{III}}_{|m|,s}\right\}, \qquad |m| \in \mathbb{R}^+, c \in \mathbb{R}. \tag{3.99}$$

The corresponding sets (3.99) contains, for each $|m|$ and s precisely one element, therefore the respective representations obviously correspond with each other. In the other two cases more work needs to be done. As before, we shall compare spectra or represented operator P_0. Namely, we have $\sigma\left[\Omega^{\pm}_{0,s}(P_0)\right] \subset i\mathbb{R}^{\pm}$ and $\sigma\left[\Theta^{\mathrm{I}}_{\pm s,\pm}(P_0)\right] \subset i\mathbb{R}^{\pm}$, and $\sigma\left[\Omega^{\pm}_{|m|^2,s|m|}(P_0)\right] \subset i\mathbb{R}^{\pm}$ and $\sigma\left[\Theta^{\mathrm{II}}_{|m|,s,\pm}(P_0)\right] \subset i\mathbb{R}^{\pm}$. Since all the considered spectra are non-empty again, we finally have

$$U^{\pm}_{0,s} \longleftrightarrow U^{\mathrm{I}}_{\pm s,\pm}, \qquad s \in \mathbb{R}, \tag{3.100}$$

$$U^{\pm}_{|m|^2,s|m|} \longleftrightarrow U^{\mathrm{II}}_{|m|,\pm s,\pm}, \quad |m| \in \mathbb{R}^+, s \in \mathbb{Z}, \tag{3.101}$$

$$U_{-|m|^2,s|m|} \longleftrightarrow U^{\mathrm{III}}_{|m|,s}, \qquad |m| \in \mathbb{R}^+, s \in \mathbb{R}. \tag{3.102}$$

Again, taking the fact the the "Mackey's" list of irreducible unitary representations of $\mathcal{P}_3$ is exhaustive into account, "$\longleftrightarrow$" means *equivalence* here in fact. Explicit isometries that realize the equivalences are given below.

Remark 3.5. Similarly as in Remark 2.3, the independent parameters involved in our method are to be related with those appearing in Mackey construction. As before, we obviously have "$|m|^2 = |m^2|$". Furthermore, we have seen that the parameter c is related with "Mackey's" *spin* s. Namely we have $c = \pm s$ in the *massless* case $m^2 = 0$, $c = \pm s\,|m|$ in the *real-mass* case $m^2 > 0$ and $c = s\,|m|$ in the *imaginary-mass* case $m^2 < 0$.

3.3.2 Explicit Isometries

Let us define, for any $|m| > 0$, the mappings $\mathcal{R}^{\pm}_{|m|} : L^2(\mathbb{R}^{\pm} \times \mathbb{R}, \mathrm{d}^2x) \to L^2(\mathbb{R}^2, \mathrm{d}^2x)$:

$$\mathcal{R}^{\pm}_{|m|}\psi(x_1, x_2) := |m|\, e^{\frac{x_1}{2}}\, \psi(\pm\,|m|\, e^{x_1}, \pm\,|m|\, x_2). \tag{3.103}$$

Proposition 3.9. *Each $\mathcal{R}^{\pm}_{|m|}$ is an isometry.*

Proof. For any $m > 0$ and $\psi, \phi \in L^2(\mathbb{R}^{\pm} \times \mathbb{R}, \mathrm{d}^2x)$ we have

$$\begin{aligned}\left(\mathcal{R}^{\pm}_{|m|}\phi, \mathcal{R}^{\pm}_{|m|}\psi\right)_{L^2(\mathbb{R}^2,\mathrm{d}^2x)} &= \int_{\mathbb{R}^2} |m|^2\, e^{x_1}\, \overline{\phi(\pm\,|m|\, e^{x_1}, \pm\,|m|\, x_2)}\, \psi(\pm\,|m|\, e^{x_1}, \pm\,|m|\, x_2)\, \mathrm{d}^2x \\ &= \pm \int_0^{\pm\infty} \mathrm{d}x_1 \int_{-\infty}^{\infty} \mathrm{d}x_2\, \overline{\phi(y_1, y_2)}\, \psi(y_1, y_2) \\ &= (\phi, \psi)_{L^2(\mathbb{R}^{\pm}\times\mathbb{R},\mathrm{d}^2x)}.\end{aligned}$$

□

Now we are ready to prove the concluding theorem:

Theorem 3.10. *With the above notation, we have*

$$U^{\pm}_{0,s} \cong U^{\mathrm{I}}_{\pm s,\pm}, \qquad s \in \mathbb{R}, \tag{3.104}$$

$$U^{\pm}_{|m|^2,s|m|} \cong U^{\mathrm{II}}_{|m|,\pm s,\pm}, \quad |m| \in \mathbb{R}^+, s \in \mathbb{Z}, \tag{3.105}$$

$$U_{-|m|^2,s|m|} \cong U^{\mathrm{III}}_{|m|,s}, \qquad |m| \in \mathbb{R}^+, s \in \mathbb{R}. \tag{3.106}$$

Proof. Take arbitrary $|m| \in \mathbb{R}^+$, $t \equiv (t_1, \dots, t_6) \in \mathbb{T}_6$ and $\psi \in L^2(\mathbb{R}^\pm \times \mathbb{R})$.
(a) First, let $s \in \mathbb{R}$. On the one hand we have

$$\begin{aligned}
&\mathcal{R}_2^\pm U^\pm_{0,s}(t)\psi(x_1, x_2)\\
&\quad = \mathcal{R}_2^\pm e^{\frac{t_1}{2}+\frac{it_2}{2}\left(x_1+\frac{x_2^2}{x_1}\right)-\frac{it_3}{2}\left(x_1-\frac{x_2^2}{x_1}\right)+it_4x_2+\frac{2ist_6}{e^{t_1}x_1-t_6(x_2+t_5x_1)}}\left(1-t_6\frac{x_2+t_5x_1}{e^{t_1}x_1}\right)\psi(X_1,X_2)\\
&\quad = 2\,e^{\frac{t_1+x_1}{2}\pm i\left[t_2\left(e^{x_1}+\frac{x_2^2}{e^{x_1}}\right)-t_3\left(e^{x_1}-\frac{x_2^2}{e^{x_1}}\right)+2t_4x_2\right]\pm\frac{ist_6}{e^{t_1+x_1}-t_6(x_2+t_5e^{x_1})}}\\
&\qquad\times\left(1-t_6\frac{x_2+t_5e^{x_1}}{e^{t_1+x_1}}\right)\psi(\widetilde{X}_1,\widetilde{X}_2),
\end{aligned}$$

where

$$\begin{aligned}
\widetilde{X}_1 &= \pm\frac{2}{e^{t_1+x_1}}\left[e^{t_1+x_1}-(x_2+t_5e^{x_1})t_6\right]^2\\
&= \pm 2\left(e^{t_1+x_1}-2x_2t_6-2t_5t_6e^{x_1}+x_2^2t_6^2e^{-t_1-x_1}+2x_2t_5t_6e^{-t_1}+t_5^2t_6^2e^{x_1-t_1}\right)
\end{aligned}$$

and

$$\begin{aligned}
\widetilde{X}_2 &= \pm 2\left[x_2+t_5e^{x_1}-\frac{t_6}{e^{t_1+x_1}}(x_2+t_5e^{x_1})^2\right]\\
&= \pm 2\left(x_2+t_5e^{x_1}-t_6x_2^2e^{-t_1-x_1}-2t_6t_5x_2e^{-t_1}-t_6t_5^2e^{x_1-t_1}\right).
\end{aligned}$$

On the other hand,

$$\begin{aligned}
&U^{\mathrm{I}}_{\pm s,\pm}(t)\mathcal{R}_2^\pm\psi(x)\psi(x_1,x_2)\\
&\quad = U^{\mathrm{I}}_{\pm s,\pm}(t)\left[2\,e^{\frac{x_1}{2}}\,\psi(\pm 2e^{x_1},\pm 2x_2)\right]\\
&\quad = 2\,e^{\frac{1}{2}\left[x_1+t_1+\ln\left(1+t_5^2t_6^2e^{-2t_1}+2t_5t_6^2x_2e^{-x_1-2t_1}-2t_5t_6e^{-t_1}+t_6^2x_2^2e^{-2x_1-2t_1}-2t_6x_2e^{-x_1-t_1}\right)\right]}\\
&\qquad\times e^{\pm i\left[t_2\left(e^{x_1}+\frac{x_2^2}{e^{x_1}}\right)-t_3\left(e^{x_1}-\frac{x_2^2}{e^{x_1}}\right)+2t_4x_2\right]\pm\frac{ist_6}{e^{t_1+x_1}-t_5t_6e^{x_1}-t_6x_2}}\psi(\widetilde{X}^{\mathrm{I}}_1,\widetilde{X}^{\mathrm{I}}_2)\\
&\quad = 2\left(1+t_5^2t_6^2e^{-2t_1}+2t_5t_6^2x_2e^{-x_1-2t_1}-2t_5t_6e^{-t_1}+t_6^2x_2^2e^{-2x_1-2t_1}-2t_6x_2e^{-x_1-t_1}\right)^{\frac{1}{2}}\\
&\qquad\times e^{\frac{x_1+t_1}{2}\pm i\left[t_2\left(e^{x_1}+\frac{x_2^2}{e^{x_1}}\right)-t_3\left(e^{x_1}-\frac{x_2^2}{e^{x_1}}\right)+2t_4x_2\right]\pm\frac{ist_6}{e^{t_1+x_1}-t_5t_6e^{x_1}-t_6x_2}}\psi(\widetilde{X}^{\mathrm{I}}_1,\widetilde{X}^{\mathrm{I}}_2)\\
&\quad = 2\,e^{\frac{x_1+t_1}{2}\pm i\left[t_2\left(e^{x_1}+\frac{x_2^2}{e^{x_1}}\right)-t_3\left(e^{x_1}-\frac{x_2^2}{e^{x_1}}\right)+2t_4x_2\right]\pm\frac{ist_6}{e^{t_1+x_1}-t_5t_6e^{x_1}-t_6x_2}}\\
&\qquad\times\left(1-\frac{t_6x_2+t_5t_6e^{x_1}}{e^{t_1+x_1}}\right)\psi(\widetilde{X}^{\mathrm{I}}_1,\widetilde{X}^{\mathrm{I}}_2),
\end{aligned}$$

where

$$\begin{aligned}
\widetilde{X}^{\mathrm{I}}_1 &= \pm 2e^{x_1+t_1+\ln\left(1+t_5^2t_6^2e^{-2t_1}+2t_5t_6^2x_2e^{-x_1-2t_1}-2t_5t_6e^{-t_1}+t_6^2x_2^2e^{-2x_1-2t_1}-2t_6x_2e^{-x_1-t_1}\right)}\\
&= \pm 2\left(e^{x_1+t_1}+t_5^2t_6^2e^{x_1-t_1}+2t_5t_6^2x_2e^{-t_1}-2t_5t_6e^{x_1}+t_6^2x_2^2e^{-x_1-t_1}-2t_6x_2\right)
\end{aligned}$$

and

$$\widetilde{X}^{\mathrm{I}}_2 = \pm 2\left(x_2-e^{x_1-t_1}t_5^2t_6+e^{x_1}t_5-2e^{-t_1}t_5t_6x_2-t_6e^{-t_1-x_1}x_2^2\right).$$

Hence $U^\pm_{0,s} \cong U^{\mathrm{I}}_{\pm s,\pm}$.
(b) Second, take an arbitrary $s \in \mathbb{Z}$. Then

$$\begin{aligned}
\mathcal{R}^\pm_{|m|}U^\pm_{|m|^2,s|m|}(t)\psi(x_1,x_2) = \mathcal{R}^\pm_{|m|}\Bigg\{&\left(\frac{e^{t_1}x_1-t_6x_2-t_5t_6x_1+it_6|m|}{e^{t_1}x_1-t_6x_2-t_5t_6x_1-it_6|m|}\right)^s\frac{\sqrt{X_1}}{\sqrt{e^{t_1}x_1}}\\
&\times e^{\frac{t_1}{2}+\frac{it_2}{2}\left(x_1+\frac{x_2^2+|m|^2}{x_1}\right)-\frac{it_3}{2}\left(x_1-\frac{x_2^2+|m|^2}{x_1}\right)+it_4x_2}\psi(X_1,X_2)\Bigg\}
\end{aligned}$$

$$
\begin{aligned}
&= |m| \left(\frac{e^{t_1+x_1} - t_6 x_2 - t_5 t_6 e^{x_1} \pm i t_6}{e^{t_1+x_1} - t_6 x_2 - t_5 t_6 e^{x_1} \mp i t_6} \right)^s \sqrt{\frac{\widetilde{X}_1}{\pm |m| e^{t_1+x_1}}} \\
&\quad \times e^{\frac{t_1+x_1}{2} \pm \frac{i|m|t_2}{2}\left(e^{x_1}+\frac{x_2^2+1}{e^{x_1}}\right) \mp \frac{i|m|t_3}{2}\left(e^{x_1}-\frac{x_2^2+1}{e^{x_1}}\right) \pm i|m|t_4 x_2} \psi(\widetilde{X}_1, \widetilde{X}_2) \\
&= |m| \left(\frac{1 - e^{-x_1-t_1} t_6 x_2 - e^{-t_1} t_5 t_6 + i e^{-x_1-t_1} t_6}{1 - e^{-x_1-t_1} t_6 x_2 - e^{-t_1} t_5 t_6 - i e^{-x_1-t_1} t_6} \right)^{\pm s} \\
&\quad \times \big(1 + e^{-2t_1-2x_1} t_6^2 x_2^2 + e^{-2t_1} t_5^2 t_6^2 - 2e^{-t_1-x_1} t_6 x_2 \\
&\qquad\quad -2e^{-t_1} t_5 t_6 + 2e^{-2t_1-x_1} t_5 t_6^2 x_2 + e^{-2t_1-2x_1} t_6^2 \big)^{\frac{1}{2}} \\
&\quad \times e^{\frac{t_1+x_1}{2} \pm i|m| \left[t_2 \left(\cosh x_1 + \frac{x_2^2}{2e^{x_1}} \right) - t_3 \left(\sinh x_1 - \frac{x_2^2}{2e^{x_1}} \right) + t_4 x_2 \right]} \psi(\widetilde{X}_1, \widetilde{X}_2),
\end{aligned}
$$

where

$$
\begin{aligned}
\widetilde{X}_1 &= \pm \frac{|m|}{e^{t_1+x_1}} \left[\left(e^{t_1+x_1} - (x_2 + t_5 e^{x_1}) t_6 \right)^2 + t_6^2 \right] \\
&= \pm |m| \big(e^{t_1+x_1} + e^{-t_1-x_1} t_6^2 x_2^2 + e^{x_1-t_1} t_5^2 t_6^2 - 2t_6 x_2 - 2e^{x_1} t_5 t_6 + 2e^{-t_1} t_5 t_6^2 x_2 + e^{-t_1-x_1} t_6^2 \big)
\end{aligned}
$$

and

$$
\begin{aligned}
\widetilde{X}_2 &= \pm |m| \left\{ x_2 + t_5 e^{x_1} - \frac{t_6}{e^{t_1+x_1}} \left[(x_2 + t_5 e^{x_1})^2 + 1 \right] \right\} \\
&= \pm |m| \left(x_2 + e^{x_1} t_5 - e^{-t_1-x_1} t_6 x_2^2 - e^{-t_1+x_1} t_5^2 t_6 - 2e^{-t_1} t_5 t_6 x_2 - e^{-t_1-x_1} t_6 \right),
\end{aligned}
$$

equals to

$$
\begin{aligned}
& U^{\mathrm{II}}_{|m|,\pm s,\pm}(t) \mathcal{R}^{\pm}_{|m|} \psi(x_1, x_2) \\
&\quad = U^{\mathrm{II}}_{|m|,\pm s,\pm}(t) \left[|m| e^{\frac{x_1}{2}} \psi(\pm |m| e^{x_1}, \pm |m| x_2) \right] \\
&\quad = |m| e^{\frac{1}{2}\left[x_1 + t_1 + \ln\left(1 + t_5^2 t_6^2 e^{-2t_1} + 2t_5 t_6^2 x_2 e^{-x_1-2t_1} - 2t_5 t_6 e^{-t_1} + t_6^2 x_2^2 e^{-2x_1-2t_1} - 2t_6 x_2 e^{-x_1-t_1} + t_6^2 e^{-2x_1-2t_1} \right) \right]} \\
&\qquad \times e^{\pm i|m| \left[t_2 \left(\cosh x_1 + \frac{x_2^2}{2e^{x_1}} \right) - t_3 \left(\sinh x_1 - \frac{x_2^2}{2e^{x_1}} \right) + t_4 x_2 \right]} \\
&\qquad\quad \times \left(\frac{1 - e^{-t_1} t_5 t_6 - e^{-t_1-x_1} t_6 x_2 + i e^{-t_1-x_1} t_6}{1 - e^{-t_1} t_5 t_6 - e^{-t_1-x_1} t_6 x_2 - i e^{-t_1-x_1} t_6} \right)^{\pm s} \psi(\widetilde{X}_1^{\mathrm{II}}, \widetilde{X}_2^{\mathrm{II}}) \\
&\quad = |m| e^{\frac{x_1+t_1}{2} \pm i|m| \left[t_2 \left(\cosh x_1 + \frac{x_2^2}{2e^{x_1}} \right) - t_3 \left(\sinh x_1 - \frac{x_2^2}{2e^{x_1}} \right) + t_4 x_2 \right]} \\
&\qquad \times \big(1 + t_5^2 t_6^2 e^{-2t_1} + 2t_5 t_6^2 x_2 e^{-x_1-2t_1} - 2t_5 t_6 e^{-t_1} + t_6^2 x_2^2 e^{-2x_1-2t_1} \\
&\qquad\qquad -2t_6 x_2 e^{-x_1-t_1} + t_6^2 e^{-2x_1-2t_1} \big)^{\frac{1}{2}} \\
&\qquad\quad \times \left(\frac{1 - e^{-t_1} t_5 t_6 - e^{-t_1-x_1} t_6 x_2 + i e^{-t_1-x_1} t_6}{1 - e^{-t_1} t_5 t_6 - e^{-t_1-x_1} t_6 x_2 - i e^{-t_1-x_1} t_6} \right)^{\pm s} \psi(\widetilde{X}_1^{\mathrm{II}}, \widetilde{X}_2^{\mathrm{II}}),
\end{aligned}
$$

with

$$
\begin{aligned}
\widetilde{X}_1^{\mathrm{II}} &= \pm |m| e^{\frac{1}{2}\left[x_1 + t_1 + \ln\left(1 + t_5^2 t_6^2 e^{-2t_1} + 2t_5 t_6^2 x_2 e^{-x_1-2t_1} - 2t_5 t_6 e^{-t_1} + t_6^2 x_2^2 e^{-2x_1-2t_1} - 2t_6 x_2 e^{-x_1-t_1} + t_6^2 e^{-2x_1-2t_1} \right) \right]} \\
&= \pm |m| e^{x_1+t_1} \big(1 + t_5^2 t_6^2 e^{-2t_1} + 2t_5 t_6^2 x_2 e^{-x_1-2t_1} - 2t_5 t_6 e^{-t_1} + t_6^2 x_2^2 e^{-2x_1-2t_1} \\
&\qquad\qquad -2t_6 x_2 e^{-x_1-t_1} + t_6^2 e^{-2x_1-2t_1} \big) \\
&= \pm |m| \big(e^{x_1+t_1} + t_5^2 t_6^2 e^{x_1-t_1} + 2t_5 t_6^2 x_2 e^{-t_1} - 2t_5 t_6 e^{x_1} + t_6^2 x_2^2 e^{-x_1-t_1} - 2t_6 x_2 + t_6^2 e^{-x_1-t_1} \big)
\end{aligned}
$$

and

$$
\widetilde{X}_2^{\mathrm{II}} = \pm |m| \left(x_2 - e^{x_1-t_1} t_5^2 t_6 + e^{x_1} t_5 - 2e^{-t_1} t_5 t_6 x_2 - t_6 e^{-t_1-x_1} x_2^2 - t_6 e^{-t_1-x_1} \right).
$$

Hence, $U^{\pm}_{|m|^2, s|m|} \cong U^{\mathrm{II}}_{|m|,\pm s,\pm}$.

(c) Finally,

$$
\begin{aligned}
&\mathcal{R}^{+}_{|m|} U_{-|m|^2,s|m|}(t)\psi(x_1,x_2)\\
&\quad= \mathcal{R}^{\pm}_{|m|}\left\{ \left(\frac{e^{t_1}x_1 - t_6x_2 - t_5t_6x_1 - t_6\,|m|}{e^{t_1}x_1 - t_6x_2 - t_5t_6x_1 + t_6\,|m|}\right)^{-is} \frac{\sqrt{X_1}}{\sqrt{e^{t_1}x_1}} \right.\\
&\qquad\qquad \left. \times\, e^{\frac{t_1}{2}+\frac{it_2}{2}\left(x_1+\frac{x_2^2-|m|^2}{x_1}\right)-\frac{it_3}{2}\left(x_1-\frac{x_2^2-|m|^2}{x_1}\right)+it_4x_2}\ \psi(X_1,X_2)\right\}\\
&\quad= |m|\left(\frac{e^{t_1+x_1} - t_6x_2 - t_5t_6e^{x_1} + t_6}{e^{t_1+x_1} - t_6x_2 - t_5t_6e^{x_1} - t_6}\right)^{is} \frac{\sqrt{\widetilde{X}_1}}{\sqrt{|m|\,e^{t_1+x_1}}}\\
&\qquad \times\, e^{\frac{t_1+x_1}{2}+\frac{i|m|t_2}{2}\left(e^{x_1}+\frac{x_2^2-1}{e^{x_1}}\right)-\frac{i|m|t_3}{2}\left(e^{x_1}-\frac{x_2^2-1}{e^{x_1}}\right)+i|m|t_4x_2}\ \psi(\widetilde{X}_1,\widetilde{X}_2)\\
&\quad= |m|\left(\frac{1-e^{-x_1-t_1}t_6x_2 - e^{-t_1}t_5t_6 + e^{-x_1-t_1}t_6}{1-e^{-x_1-t_1}t_6x_2 - e^{-t_1}t_5t_6 - e^{-x_1-t_1}t_6}\right)^{is}\\
&\qquad \times \big(1+e^{-2t_1-2x_1}t_6^2x_2^2 + e^{-2t_1}t_5^2t_6^2 - 2e^{-t_1-x_1}t_6x_2 - 2e^{-t_1}t_5t_6\\
&\qquad\qquad +2e^{-2t_1-x_1}t_5t_6^2x_2 - e^{-2t_1-2x_1}t_6^2\big)^{\frac{1}{2}}\\
&\qquad\quad \times\, e^{\frac{t_1+x_1}{2}+i|m|\left[t_2\left(\sinh x_1+\frac{x_2^2}{2e^{x_1}}\right)-t_3\left(\cosh x_1-\frac{x_2^2}{2e^{x_1}}\right)+t_4x_2\right]}\ \psi(\widetilde{X}_1,\widetilde{X}_2),
\end{aligned}
$$

since $|m|\,e^{t_1+x_1} > 0$, where

$$
\begin{aligned}
\widetilde{X}_1 &= \frac{|m|}{e^{t_1+x_1}}\left[\left(e^{t_1+x_1} - (x_2+t_5e^{x_1})t_6\right)^2 - t_6^2\right]\\
&= |m|\big(e^{t_1+x_1} + e^{-t_1-x_1}t_6^2x_2^2 + e^{x_1-t_1}t_5^2t_6^2 - 2t_6x_2 - 2e^{x_1}t_5t_6 + 2e^{-t_1}t_5t_6^2x_2 - e^{-t_1-x_1}t_6^2\big)
\end{aligned}
$$

and

$$
\begin{aligned}
\widetilde{X}_2 &= |m|\left\{x_2 + t_5e^{x_1} - \frac{t_6}{e^{t_1+x_1}}\left[(x_2+t_5e^{x_1})^2 - 1\right]\right\}\\
&= |m|\left(x_2 + e^{x_1}t_5 - e^{-t_1-x_1}t_6x_2^2 - e^{-t_1+x_1}t_5^2t_6 - 2e^{-t_1}t_5t_6x_2 + e^{-t_1-x_1}t_6\right),
\end{aligned}
$$

is identical to

$$
\begin{aligned}
&U^{\mathrm{III}}_{|m|,s}(t)\mathcal{R}^{+}_{|m|}\psi(x_1,x_2)\\
&\quad= U^{\mathrm{III}}_{|m|,s}(t)\left[|m|\,e^{\frac{x_1}{2}}\ \psi(|m|\,e^{x_1}, |m|\,x_2)\right]\\
&\quad= |m|\,e^{\frac{1}{2}\left[x_1+t_1+\ln\left(1+t_5^2t_6^2e^{-2t_1}+2t_5t_6^2x_2e^{-x_1-2t_1}-2t_5t_6e^{-t_1}+t_6^2x_2^2e^{-2x_1-2t_1}-2t_6x_2e^{-x_1-t_1}-t_6^2e^{-2x_1-2t_1}\right)\right]}\\
&\qquad \times\, e^{|m|\left[t_2\left(\sinh x_1+\frac{x_2^2}{2e^{x_1}}\right)-t_3\left(\cosh x_1-\frac{x_2^2}{2e^{x_1}}\right)+t_4x_2\right]}\\
&\qquad\quad \times \left(\frac{1-e^{-t_1}t_5t_6 - e^{-t_1-x_1}t_6x_2 + e^{-t_1-x_1}t_6}{1-e^{-t_1}t_5t_6 - e^{-t_1-x_1}t_6x_2 - e^{-t_1-x_1}t_6}\right)^{is}\ \psi(\widetilde{X}_1^{\mathrm{III}}, \widetilde{X}_2^{\mathrm{III}})\\
&\quad= |m|\,e^{\frac{x_1+t_1}{2}\pm i|m|\left[t_2\left(\sinh x_1+\frac{x_2^2}{2e^{x_1}}\right)-t_3\left(\cosh x_1-\frac{x_2^2}{2e^{x_1}}\right)+t_4x_2\right]}\\
&\qquad \times \big(1+t_5^2t_6^2e^{-2t_1} + 2t_5t_6^2x_2e^{-x_1-2t_1} - 2t_5t_6e^{-t_1} + t_6^2x_2^2e^{-2x_1-2t_1}\\
&\qquad\qquad -2t_6x_2e^{-x_1-t_1} - t_6^2e^{-2x_1-2t_1}\big)^{\frac{1}{2}}\\
&\qquad\quad \times \left(\frac{1-e^{-t_1}t_5t_6 - e^{-t_1-x_1}t_6x_2 + e^{-t_1-x_1}t_6}{1-e^{-t_1}t_5t_6 - e^{-t_1-x_1}t_6x_2 - e^{-t_1-x_1}t_6}\right)^{is}\ \psi(\widetilde{X}_1^{\mathrm{III}}, \widetilde{X}_2^{\mathrm{III}}),
\end{aligned}
$$

with

$$
\begin{aligned}
\widetilde{X}_1^{\mathrm{III}} &= |m|\, e^{\frac{1}{2}\left[x_1+t_1+\ln\left(1+t_5^2t_6^2e^{-2t_1}+2t_5t_6^2x_2e^{-x_1-2t_1}-2t_5t_6e^{-t_1}+t_6^2x_2^2e^{-2x_1-2t_1}-2t_6x_2e^{-x_1-t_1}-t_6^2e^{-2x_1-2t_1}\right)\right]} \\
&= |m|\, e^{x_1+t_1}\left(1+t_5^2t_6^2e^{-2t_1}+2t_5t_6^2x_2e^{-x_1-2t_1}-2t_5t_6e^{-t_1}+t_6^2x_2^2e^{-2x_1-2t_1}\right. \\
&\qquad\qquad \left.-2t_6x_2e^{-x_1-t_1}-t_6^2e^{-2x_1-2t_1}\right) \\
&= |m|\left(e^{x_1+t_1}+t_5^2t_6^2e^{x_1-t_1}+2t_5t_6^2x_2e^{-t_1}-2t_5t_6e^{x_1}+t_6^2x_2^2e^{-x_1-t_1}-2t_6x_2-t_6^2e^{-x_1-t_1}\right)
\end{aligned}
$$

and

$$
\widetilde{X}_2^{\mathrm{III}} = |m|\left(x_2-e^{x_1-t_1}t_5^2t_6+e^{x_1}t_5-2e^{-t_1}t_5t_6x_2-t_6e^{-t_1-x_1}x_2^2+t_6e^{-t_1-x_1}\right).
$$

Therefore, $U_{-|m|^2,s|m|} \cong U^{\mathrm{III}}_{|m|,s}$. $\square$

Chapter 4

Discussion on Representations of $\mathcal{P}_4$

Finally, let us consider the case $n = 4$, i.e. the "famous" and physically interesting ten-dimensional Poincaré group $\mathcal{P}_4$. It is not surprising that each step of the construction presented in the previous chapters is even more complicated at this instance, thus the discussion is rather a brief and informal one. Nevertheless, we shall see that the essential part of the procedure has been in fact already done.

The second-kind canonical coordinates are naturally chosen to extend the chart introduced in $\mathcal{P}_3$. Namely, for all $t \equiv (t_1, \ldots, t_{10}) \in \mathbb{R}^{10}$ we define

$$g(t) \equiv e^{t_2 \mathbf{P}_0} e^{t_3 \mathbf{P}_1} e^{t_4 \mathbf{P}_2} e^{t_7 \mathbf{P}_3} e^{t_5 (\mathbf{L}_{12} - \mathbf{L}_{02})} e^{t_9 (\mathbf{L}_{13} - \mathbf{L}_{03})} e^{t_1 \mathbf{L}_{01}} e^{t_8 \mathbf{L}_{23}} e^{t_6 (\mathbf{L}_{12} + \mathbf{L}_{02})} e^{t_{10} (\mathbf{L}_{13} + \mathbf{L}_{03})}. \tag{4.1}$$

There is certainly no need to state explicit forms of the matrices in product (4.1), this could be done exactly in the same way as for $\mathcal{P}_3$. Notice that neither these coordinates are global; certainly the same counterexample (3.3) as before can be used to support this assertion.

4.1 Lie Field Technique

The Lie algebra $\mathfrak{p}_4$ is a ten-dimensional real Lie algebra, generated by P_0, P_1, P_2, P_3, L_{01}, L_{02}, L_{03}, L_{12}, L_{13} and L_{23} subject to the following non-trivial commutation relations:

$$\begin{array}{llll}
[L_{01}, L_{02}] = -L_{12}, & [L_{01}, L_{03}] = -L_{13}, & [L_{01}, L_{12}] = -L_{02}, & [L_{01}, L_{13}] = -L_{03}, \\
[L_{02}, L_{03}] = -L_{23}, & [L_{02}, L_{12}] = L_{01}, & [L_{02}, L_{23}] = -L_{03}, & [L_{03}, L_{13}] = L_{01}, \\
[L_{03}, L_{23}] = L_{02}, & [L_{12}, L_{13}] = L_{23}, & [L_{12}, L_{23}] = -L_{13}, & [L_{13}, L_{23}] = L_{12}, \\
[L_{01}, P_0] = -P_1, & [L_{01}, P_1] = -P_0, & [L_{02}, P_0] = -P_2, & [L_{02}, P_2] = -P_0, \\
[L_{03}, P_0] = -P_3, & [L_{03}, P_3] = -P_0, & [L_{12}, P_1] = P_2, & [L_{12}, P_2] = -P_1, \\
[L_{13}, P_1] = P_3, & [L_{13}, P_3] = -P_1, & [L_{23}, P_2] = P_3, & [L_{23}, P_3] = -P_2.
\end{array}$$

Since for the matrix $\mathbf{S}(\mathfrak{p}_4)$ constructed with respect to the above basis we have

$$\operatorname{rank}_{\mathfrak{S}(\mathfrak{p}_4)} \begin{pmatrix}
0 & -L_{12} & -L_{13} & -L_{02} & -L_{03} & 0 & -P_1 & -P_0 & 0 & 0 \\
L_{12} & 0 & -L_{23} & L_{01} & 0 & -L_{03} & -P_2 & 0 & -P_0 & 0 \\
L_{13} & L_{23} & 0 & 0 & L_{01} & L_{02} & -P_3 & 0 & 0 & -P_0 \\
L_{02} & -L_{01} & 0 & 0 & L_{23} & -L_{13} & 0 & P_2 & -P_1 & 0 \\
L_{03} & 0 & -L_{01} & -L_{23} & 0 & L_{12} & 0 & P_3 & 0 & -P_1 \\
0 & L_{03} & -L_{02} & L_{13} & -L_{12} & 0 & 0 & 0 & P_3 & -P_2 \\
P_1 & P_2 & P_3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
P_0 & 0 & 0 & -P_2 & -P_3 & 0 & 0 & 0 & 0 & 0 \\
0 & P_0 & 0 & P_1 & 0 & -P_3 & 0 & 0 & 0 & 0 \\
0 & 0 & P_0 & 0 & P_1 & P_2 & 0 & 0 & 0 & 0
\end{pmatrix} = 8,$$

index$(\mathfrak{p}_4) = 10 - 8 = 2$ and the centre $\mathfrak{Z}(\mathfrak{p}_4)$ is generated by two independent Casimir elements; explicitly (cf. [7], p. 3, but also e.g. [8] or [39]),

$$M^2 := P_3^2 + P_2^2 + P_1^2 - P_0^2 = -P_\mu P^\mu \tag{4.2}$$

and

$$W := P_\mu P^\nu L_{\nu\rho} L^{\rho\mu} - \frac{1}{2} P_\rho P^\rho L_{\mu\nu} L^{\mu\nu}. \tag{4.3}$$

Here the Einstein summation convention is used together with the rule $L_{\mu\nu} = -L_{\nu\mu}$.

Although we have $\frac{1}{2}(10-2) = 4$ and hence the Gelfand-Kirillov conjecture (§1.1.5) suggests to relate $\mathfrak{D}(\mathfrak{p}_4)$ with $\mathfrak{D}_{4,2}(\mathbb{R})$, it turns out to be more convenient in this case to modify the procedure of construction partially.

To be precise, we shall embed $\mathfrak{p}_4$ in algebra $\mathfrak{D}'_{3,1;s}(\mathbb{R})$ defined as follows. First, let $\mathfrak{W}_{3,1;s}(\mathbb{R})$ be the real unital associative algebra generated by 11 abstract elements p_1, p_2, p_3, q_1, q_2, q_3, θ_1, θ_2, s_{12}, s_{13} and s_{23} subject to the following relations:

$$[p_j, q_k] = \delta_{jk}, \quad [s_{12}, s_{13}] = -4\theta s_{23}, \quad [s_{13}, s_{23}] = s_{12}, \quad [s_{23}, s_{12}] = s_{13}, \tag{4.4}$$

$j = 1, 2, 3$. Otherwise, the generators commute. $\mathfrak{W}_{3,1;s}(\mathbb{R})$ is also a *G-algebra* (cf. [19]), so it possesses the PBW basis and is embedded in its field of fractions, $\mathfrak{D}_{3,1;s}(\mathbb{R})$. It makes therefore sense to define the localization $\mathfrak{D}'_{3,1;s}(\mathbb{R}) := \mathfrak{D}_{\mathfrak{Q}}(\mathfrak{W}_{3,1;s}(\mathbb{R}))$, where $\mathfrak{Q}$ is, similarly as in §1.1.4, the subalgebra of $\mathfrak{W}_{3,1;s}(\mathbb{R})$ generated by q_1.

Let us define the following involution on $\mathfrak{W}_{3,1;s}(\mathbb{R})$:

$$p_j^* := p_j, \quad q_j^* := -q_j, \quad \theta^* := \theta, \quad s_{12}^* := s_{12}, \quad s_{13}^* := s_{13}, \quad s_{23}^* := -s_{23}, \tag{4.5}$$

$j = 1, 2, 3$. It extends to $\mathfrak{D}_{3,1;s}(\mathbb{R})$, and thus to $\mathfrak{D}'_{3,1;s}(\mathbb{R})$, as usual.

4.1.1 Isomorphism of $\mathfrak{D}(\mathfrak{p}_4)$ and $\mathfrak{D}_{3,1;s}(\mathbb{R})$

Since both the subalgebras of $\mathfrak{p}_4$ generated by L_{01}, L_{0j}, L_{1j}, P_0, P_1 and P_j, $j = 2, 3$, respectively, are in an obvious way isomorphic to $\mathfrak{p}_3$, it is reasonable to put

$$\hat{p}_1 := (P_0 - P_1)^{-1} \left(L_{01} - \frac{1}{2} \right), \tag{4.6}$$

$$\hat{q}_1 := P_0 - P_1, \tag{4.7}$$

$$\hat{p}_j := (P_0 - P_1)^{-1} (L_{1j} - L_{0j}), \tag{4.8}$$

$$\hat{q}_j := P_j, \tag{4.9}$$

$j = 2, 3$. Almost all the relations $\hat{p}_j^* = \hat{p}_j$, $\hat{q}_k^* = -\hat{q}_k$, $[\hat{p}_j, \hat{q}_k] = \delta_{jk}$ were already proven. The only exceptions are $[\hat{p}_2, \hat{q}_3] = [\hat{p}_3, \hat{q}_2] = 0$, but they are obvious. Further, let us pick

$$\hat{s}_{23} := L_{23} + \hat{q}_2 \hat{p}_3 - \hat{q}_3 \hat{p}_2, \tag{4.10}$$

$$\hat{s}_{12} := \hat{q}_1 (L_{12} + L_{02}) + 2 \left(\hat{q}_1 \hat{p}_1 + \frac{1}{2} \right) \hat{q}_2 + \left(\hat{q}_2^2 + \hat{q}_3^2 - M^2 \right) \hat{p}_2 + 2 \hat{q}_3 \hat{s}_{23}, \tag{4.11}$$

$$\hat{s}_{13} := \hat{q}_1 (L_{13} + L_{03}) + 2 \left(\hat{q}_1 \hat{p}_1 + \frac{1}{2} \right) \hat{q}_3 + \left(\hat{q}_2^2 + \hat{q}_3^2 - M^2 \right) \hat{p}_3 - 2 \hat{q}_2 \hat{s}_{23}. \tag{4.12}$$

It is straightforward to show that these elements satisfy exactly the same commutation and involutive relations as the respective elements of $\mathfrak{D}_{3,1;s}(\mathbb{R})$, i.e. s_{jk} without hats, with $\theta \mapsto M^2$. For precise calculations, we refer the reader to the Appendix.

Since

$$M^2 = P_3^2 + P_2^2 - (P_0 - P_1)(P_0 + P_1) = \left(M^2 \right)^*, \tag{4.13}$$

the above relations are to be easily inverted into

$$
\begin{aligned}
L_{01} &= \hat{q}_1\hat{p}_1 + \frac{1}{2}, && (4.14)\\
P_0 &= \frac{\hat{q}_1^{-1}}{2}\left(\hat{q}_1^2 + \hat{q}_2^2 + \hat{q}_3^2 - M^2\right), && (4.15)\\
P_1 &= \frac{\hat{q}_1^{-1}}{2}\left(-\hat{q}_1^2 + \hat{q}_2^2 + \hat{q}_3^2 - M^2\right), && (4.16)\\
P_2 &= \hat{q}_2, && (4.17)\\
L_{12} - L_{02} &= \hat{q}_1\hat{p}_2, && (4.18)\\
L_{12} + L_{02} &= -\hat{q}_1^{-1}\left[2\left(\hat{q}_1\hat{p}_1 + \frac{1}{2}\right)\hat{q}_2 + \left(\hat{q}_2^2 + \hat{q}_3^2 - M^2\right)\hat{p}_2 + 2\hat{q}_3\hat{s}_{23} - \hat{s}_{12}\right], && (4.19)\\
P_3 &= \hat{q}_3, && (4.20)\\
L_{23} &= \hat{q}_3\hat{p}_2 - \hat{q}_2\hat{p}_3 + \hat{s}_{23}, && (4.21)\\
L_{13} - L_{03} &= \hat{q}_1\hat{p}_3, && (4.22)\\
L_{13} + L_{03} &= -\hat{q}_1^{-1}\left[2\left(\hat{q}_1\hat{p}_1 + \frac{1}{2}\right)\hat{q}_3 + \left(\hat{q}_2^2 + \hat{q}_3^2 - M^2\right)\hat{p}_3 - 2\hat{q}_2\hat{s}_{23} - \hat{s}_{13}\right]. && (4.23)
\end{aligned}
$$

Thus the linear mapping $\Psi : \mathfrak{U}(\mathfrak{p}_4) \to \mathfrak{D}_{3,1;s}(\mathbb{R})$ defined by

$$
\begin{aligned}
\Psi(L_{01}) &= q_1 p_1 + \frac{1}{2}, && (4.24)\\
\Psi(P_0) &= \frac{q_1^{-1}}{2}\left(q_1^2 + q_2^2 + q_3^2 - \theta\right), && (4.25)\\
\Psi(P_1) &= \frac{q_1^{-1}}{2}\left(-q_1^2 + q_2^2 + q_3^2 - \theta\right), && (4.26)\\
\Psi(P_2) &= q_2, && (4.27)\\
\Psi(L_{12} - L_{02}) &= q_1 p_2, && (4.28)\\
\Psi(L_{12} + L_{02}) &= -q_1^{-1}\left[2\left(q_1 p_1 + \frac{1}{2}\right) q_2 + \left(q_2^2 + q_3^2 - \theta\right) p_2 + 2 q_3 s_{23} - s_{12}\right], && (4.29)\\
\Psi(P_3) &= q_3, && (4.30)\\
\Psi(L_{23}) &= q_3 p_2 - q_2 p_3 + s_{23}, && (4.31)\\
\Psi(L_{13} - L_{03}) &= q_1 p_3, && (4.32)\\
\Psi(L_{13} + L_{03}) &= -q_1^{-1}\left[2\left(q_1 p_1 + \frac{1}{2}\right) q_3 + \left(q_2^2 + q_3^2 - \theta\right) p_3 - 2 q_2 s_{23} - s_{13}\right], && (4.33)
\end{aligned}
$$

is a homomorphism. Furthermore,

Lemma 4.1. *For $\Psi : \mathfrak{U}(\mathfrak{p}_4) \to \mathfrak{D}_{3,1;s}(\mathbb{R})$ and $x \in \mathfrak{U}(\mathfrak{p}_4)$ one has $\Psi(x) = 0$ only if $x = 0$.*

Proof. We only sketch the proof since it is done in exactly the same way as for Lemma 3.2. Here assume that for the following element of $\mathfrak{U}(\mathfrak{p}_4)$:

$$
x = \sum_{j_1,\ldots,j_{10}}^{N} \alpha_{j_1,\ldots,j_{10}} P_0^{j_1} P_1^{j_2} P_2^{j_3} P_3^{j_4} L_{01}^{j_5} (L_{12} - L_{02})^{j_6} (L_{13} - L_{03})^{j_7} L_{23}^{j_8} (L_{12} + L_{02})^{j_9} (L_{13} + L_{03})^{j_{10}},
$$

we have $\Psi(x) = 0$. Now the coefficients $\alpha_{j_1,\ldots,j_{10}} \in \mathbb{R}$ will be eliminated in successive steps. First, since the term with $\Psi(L_{13} + L_{03})^N$ is the only one in $\Psi(x)$ containing s_{13}^N, its coefficient has to be zero. Iterating the same argument, we obtain

$$
0 = \sum_{j_1,\ldots,j_9}^{N} \alpha_{j_1,\ldots,j_{10}} \Psi(P_0)^{j_1} \cdots \Psi(L_{12} + L_{02})^{j_9},
$$

for all $0 \le j_{10} \le N$. Repeating the same procedure for $\Psi(L_{12}+L_{02})$ exclusively containing s_{12}, for $\Psi(L_{23})$ containing s_{23}, for $\Psi(L_{13}-L_{03})$ containing p_3, for $\Psi(L_{12}-L_{02})$ containing p_2 and for $\Psi(L_{01})$ containing p_1, respectively, we end with

$$\begin{aligned} 0 &= \sum_{j_1,\dots,j_4}^{N} \alpha_{j_1,\dots,j_{10}} \Psi(P_0)^{j_1}\Psi(P_1)^{j_2}\Psi(P_2)^{j_3}\Psi(P_3)^{j_4} \\ &= \sum_{j_1,\dots,j_4}^{N} \frac{\alpha_{j_1,\dots,j_{10}}}{2^{j_1+j_2}} \left[q_1^{-1}(q_1^2+q_2^2+q_3^2-\theta)\right]^{j_1} \left[q_1^{-1}(-q_1^2+q_2^2+q_3^2-\theta)\right]^{j_2} q_2^{j_3} q_3^{j_4}, \end{aligned}$$

$0 \le j_5,\dots,j_{10} \le N$. Similarly as in the cases $\mathfrak{p}_2$ and $\mathfrak{p}_3$, this means

$$0 = \sum_{j_1,\dots,j_4}^{N} \frac{\alpha_{j_1,\dots,j_{10}}}{2^{j_1+j_2}} \left(x + \frac{y^2}{x} + \frac{z^2}{x} - \frac{u}{x}\right)^{j_1} \left(-x + \frac{y^2}{x} + \frac{z^2}{x} - \frac{u}{x}\right)^{j_2} y^{j_3} z^{j_4},$$

$(x,y,z,u) \in \mathbb{R}^\times \times \mathbb{R}^3$, and since the Jacobian of mapping

$$x' := x + \frac{y^2}{x} + \frac{z^2}{x} - \frac{u}{x}, \quad y' := -x + \frac{y^2}{x} + \frac{z^2}{x} - \frac{u}{x}, \quad z' := y, \quad u' := z,$$

is

$$\det \begin{pmatrix} 1 - \frac{y^2}{x^2} - \frac{z^2}{x^2} + \frac{u}{x^2} & \frac{2y}{x} & \frac{2z}{x} & -\frac{1}{x} \\ -1 - \frac{y^2}{x^2} - \frac{z^2}{x^2} + \frac{u}{x^2} & \frac{2y}{x} & \frac{2z}{x} & -\frac{1}{x} \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix} = \det \begin{pmatrix} 1 - \frac{y^2}{x^2} - \frac{z^2}{x^2} + \frac{u}{x^2} & \frac{2y}{x} & \frac{2z}{x} & -\frac{1}{x} \\ -2 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix} = \frac{2}{x},$$

hence non-zero on $\mathbb{R}^\times \times \mathbb{R}^3$, we finally have $\alpha_{j_1,\dots,j_{10}} = 0$ for any $0 \le j_1,\dots,j_{10} \le N$. $\square$

Thus Ψ is injective and extends to $\mathfrak{D}(\mathfrak{p}_4)$. Surjectivity and the involutive property were in fact already proven as well. Altogether we have

Theorem 4.2. *The mapping* $\Psi : \mathfrak{D}(\mathfrak{p}_4) \to \mathfrak{D}_{3,1;s}(\mathbb{R})$ *is a* $*$*-isomorphism.*

4.1.2 Skew-symmetric Representations of the Lie Algebra $\mathfrak{p}_4$

Recall that $\mathcal{H}_3 \equiv L^2(\mathbb{R}^\times \times \mathbb{R}^2, \mathrm{d}^3x)$. Let V be a non-specified complex Hilbert space and let us consider the following representations $\Phi_{m^2,S}$ of $\mathfrak{W}_{3,1;s}$ on $\mathcal{H}_3 \otimes V$:

$$\Phi_{m^2,S}(p_j)\,[\psi(x)\otimes v] = \left[-i\partial_{x_j}\psi(x)\right]\otimes v, \tag{4.34}$$

$$\Phi_{m^2,S}(q_j)\,[\psi(x)\otimes v] = \left[ix_j\psi(x)\right]\otimes v, \tag{4.35}$$

$$\Phi_{m^2,S}(\theta)\,[\psi(x)\otimes v] = \left[m^2\psi(x)\right]\otimes v, \tag{4.36}$$

$$\Phi_{m^2,S}(s_{12})\,[\psi(x)\otimes v] = \psi(x)\otimes(iS_{12}v)\,, \tag{4.37}$$

$$\Phi_{m^2,S}(s_{13})\,[\psi(x)\otimes v] = \psi(x)\otimes(iS_{13}v)\,, \tag{4.38}$$

$$\Phi_{m^2,S}(s_{23})\,[\psi(x)\otimes v] = \psi(x)\otimes(S_{23}v)\,, \tag{4.39}$$

where $m^2 \in \mathbb{R}$, and $S \equiv \{S_{12}, S_{13}, S_{23}\}$ are skew-symmetric operators on V satisfying

$$[S_{12}, S_{13}] = 4m^2 S_{23}, \quad [S_{13}, S_{23}] = S_{12}, \quad [S_{23}, S_{12}] = S_{13}. \tag{4.40}$$

It is easily seen that such defined mapping preserves commutator as well as the involution (cf. Remark 1.6 for discussion on "involution" on $\mathcal{L}(\mathcal{H}_3 \otimes V)$). Nonetheless, it remains to specify a dense subset of $\mathcal{H}_3 \otimes V$ on which the representation is well-defined. For the "$\mathcal{H}_3$" part, the usual set $C_0^\infty(\mathbb{R}^\times \times \mathbb{R}^2)$ is used, while the appropriate subset of V must be specified together with V itself.

Remark 4.1. The set S can be obviously also viewed as a skew-symmetric representation of one of the following three Lie algebras $\mathfrak{s}_1$, $\mathfrak{s}_0$ and $\mathfrak{s}_{-1}$, depending upon $\operatorname{sgn} m^2$:

$$\mathfrak{s}_{\operatorname{sgn} m^2} \text{ is defined to be } \begin{cases} \textit{orthogonal} \text{ Lie algebra } \mathfrak{so}(3,\mathbb{R}), & \text{if } m^2 > 0, \\ \textit{Euclidean} \text{ Lie algebra } \mathfrak{e}_2, & \text{if } m^2 = 0, \\ \textit{pseudo-orthogonal} \text{ Lie algebra } \mathfrak{so}(1,2), & \text{if } m^2 < 0. \end{cases} \quad (4.41)$$

(cf. [34]). To see this in case $m^2 \neq 0$, one has to scale $S_{1j} \mapsto 2\sqrt{|m^2|}S_{1j}$, $j = 2,3$. For $m^2 = 0$ the correspondence is obvious.

Since the above general representation can be again easily extended to $\mathfrak{D}'_{3,1;s}(\mathbb{R})$ and the isomorphism Ψ from the previous section satisfy $\Psi(\mathfrak{p}_4) \subset \mathfrak{D}'_{3,1;s}(\mathbb{R})$, the mappings can be composed together into a skew-symmetric representation of $\mathfrak{p}_4$. To simplify the notation, notice that $\mathcal{H}_3 \otimes V \equiv L^2(\mathbb{R}^\times \times \mathbb{R}^2, \mathrm{d}^3x; V)$ and let us denote $\boldsymbol{\psi}(x) = \psi(x) \otimes v$. The tensor-product sign between operators on $\mathcal{H}_3$ and V shall be omitted as well.

For any $m^2 \in \mathbb{R}$ and any skew-symmetric representation S of $\mathfrak{s}_{\operatorname{sgn} m^2}$, the relations

$$\Omega_{m^2,S}(L_{01})\boldsymbol{\psi}(x) = \left(x_1\partial_{x_1} + \frac{1}{2}\right)\boldsymbol{\psi}(x), \quad (4.42)$$

$$\Omega_{m^2,S}(P_0)\boldsymbol{\psi}(x) = \frac{i}{2x_1}\left(x_1^2 + x_2^2 + x_3^2 + m^2\right)\boldsymbol{\psi}(x), \quad (4.43)$$

$$\Omega_{m^2,S}(P_1)\boldsymbol{\psi}(x) = \frac{i}{2x_1}\left(-x_1^2 + x_2^2 + x_3^2 + m^2\right)\boldsymbol{\psi}(x), \quad (4.44)$$

$$\Omega_{m^2,S}(P_2)\boldsymbol{\psi}(x) = ix_2\boldsymbol{\psi}(x), \quad (4.45)$$

$$\Omega_{m^2,S}(L_{12} - L_{02})\boldsymbol{\psi}(x) = x_1\partial_{x_2}\boldsymbol{\psi}(x), \quad (4.46)$$

$$\Omega_{m^2,S}(L_{12} + L_{02})\boldsymbol{\psi}(x) = -\frac{1}{x_1}\left[2\left(x_1\partial_{x_1} + \frac{1}{2}\right)x_2 + \left(x_2^2 + x_3^2 + m^2\right)\partial_{x_2} + 2x_3S_{23} - S_{12}\right]\boldsymbol{\psi}(x), \quad (4.47)$$

$$\Omega_{m^2,S}(P_3)\boldsymbol{\psi}(x) = ix_3\boldsymbol{\psi}(x), \quad (4.48)$$

$$\Omega_{m^2,S}(L_{23})\boldsymbol{\psi}(x) = \left(x_3\partial_{x_2} - x_2\partial_{x_3} + S_{23}\right)\boldsymbol{\psi}(x), \quad (4.49)$$

$$\Omega_{m^2,S}(L_{13} - L_{03})\boldsymbol{\psi}(x) = x_1\partial_{x_3}\boldsymbol{\psi}(x), \quad (4.50)$$

$$\Omega_{m^2,S}(L_{13} + L_{03})\boldsymbol{\psi}(x) = -\frac{1}{x_1}\left[2\left(x_1\partial_{x_1} + \frac{1}{2}\right)x_3 + \left(x_2^2 + x_3^2 + m^2\right)\partial_{x_3} - 2x_2S_{23} - S_{13}\right]\boldsymbol{\psi}(x), \quad (4.51)$$

define a skew-symmetric representation of $\mathfrak{p}_4$ on $\mathcal{H}_3 \otimes V$, provided there is a common dense invariant subset of V for operators S_{12}, S_{13}, S_{23} of the representation S.

4.1.3 Irreducible Unitary Representations of the Lie Group $\mathcal{P}_4$

One-parameter Subgroups

With help of the results we have already had, "formal" integration of the operators (4.42) - (4.42) is comparatively easy. Namely, the first six operators agree with those in the previous case, up to (formal) substitutions $m^2 \mapsto m^2 + x_3^2$ and $2ic \mapsto S_{12} - 2x_3S_{23}$. Three of the four remaining ones could be obtained from those already discussed only by intertwining indices $2 \leftrightarrow 3$. Finally, the last one, $\Omega_{m^2,S}(L_{23})$, is easy to be integrated since it is a sum of two commuting operators acting non-trivially on the opposite parts of $\mathcal{H}_3 \otimes V$.

Therefore,

$$U^{(1)}_{m^2,S}(t)\boldsymbol{\psi}(x) \equiv \exp\{t\,\Omega_{m^2,S}(L_{01})\}\boldsymbol{\psi}(x) = e^{\frac{t}{2}}\boldsymbol{\psi}\left(e^t x_1, x_2, x_3\right), \tag{4.52}$$

$$U^{(2)}_{m^2,S}(t)\boldsymbol{\psi}(x) \equiv \exp\{t\,\Omega_{m^2,S}(P_0)\}\boldsymbol{\psi}(x) = e^{\frac{it}{2}\left(x_1+\frac{x_2^2+x_3^2+m^2}{x_1}\right)}\boldsymbol{\psi}(x), \tag{4.53}$$

$$U^{(3)}_{m^2,S}(t)\boldsymbol{\psi}(x) \equiv \exp\{t\,\Omega_{m^2,S}(P_1)\}\boldsymbol{\psi}(x) = e^{-\frac{it}{2}\left(x_1-\frac{x_2^2+x_3^2+m^2}{x_1}\right)}\boldsymbol{\psi}(x), \tag{4.54}$$

$$U^{(4)}_{m^2,S}(t)\boldsymbol{\psi}(x) \equiv \exp\{t\,\Omega_{m^2,S}(P_2)\}\boldsymbol{\psi}(x) = e^{itx_2}\boldsymbol{\psi}(x), \tag{4.55}$$

$$U^{(5)}_{m^2,S}(t)\boldsymbol{\psi}(x) \equiv \exp\{t\,\Omega_{m^2,S}(L_{12}-L_{02})\}\boldsymbol{\psi}(x) = \boldsymbol{\psi}(x_1, x_2+tx_1, x_3), \tag{4.56}$$

$$U^{(6)}_{m^2,S}(t)\boldsymbol{\psi}(x) \equiv \exp\{t\,\Omega_{m^2,S}(L_{12}+L_{02})\}\boldsymbol{\psi}(x) = \alpha^{(6)}(x;t)\,\boldsymbol{\psi}\Big(X_1^{(6)}(x;t), X_2^{(6)}(x;t), x_3\Big), \tag{4.57}$$

$$U^{(7)}_{m^2,S}(t)\boldsymbol{\psi}(x) \equiv \exp\{t\,\Omega_{m^2,S}(P_3)\}\boldsymbol{\psi}(x) = e^{itx_3}\boldsymbol{\psi}(x), \tag{4.58}$$

$$U^{(8)}_{m^2,S}(t)\boldsymbol{\psi}(x) \equiv \exp\{t\,\Omega_{m^2,S}(L_{23})\}\boldsymbol{\psi}(x) = e^{tS_{23}}\boldsymbol{\psi}(x_1, x_2\cos t + x_3\sin t, x_3\cos t - x_2\sin t), \tag{4.59}$$

$$U^{(9)}_{m^2,S}(t)\boldsymbol{\psi}(x) \equiv \exp\{t\,\Omega_{m^2,S}(L_{13}-L_{03})\}\boldsymbol{\psi}(x) = \boldsymbol{\psi}(x_1, x_2, x_3+tx_1), \tag{4.60}$$

$$U^{(10)}_{m^2,S}(t)\boldsymbol{\psi}(x) \equiv \exp\{t\,\Omega_{m^2,S}(L_{13}+L_{03})\}\boldsymbol{\psi}(x) = \alpha^{(10)}(x;t)\,\boldsymbol{\psi}\Big(X_1^{(10)}(x;t), x_2, X_3^{(10)}(x;t)\Big), \tag{4.61}$$

$x \equiv (x_1, x_2, x_3) \in \mathbb{R}^\times \times \mathbb{R}^2$, $t \in \mathbb{R}$, where

$$\alpha^{(6)}(x;t) = \begin{cases} \frac{\sqrt{X_1^{(6)}(x;t)}}{\sqrt{x_1}} \left(\frac{x_1-tx_2+it\sqrt{m^2}}{x_1-tx_2-it\sqrt{m^2}}\right)^{\frac{i}{\sqrt{m^2}}\left(x_3S_{23}-\frac{1}{2}S_{12}\right)}, & m^2 \neq 0, \\ \frac{\sqrt{X_1^{(6)}(x;t)}}{\sqrt{x_1}} \exp\left\{\frac{t}{x_1-tx_2}\left(S_{12}-2x_3S_{23}\right)\right\}, & m^2 = 0, \end{cases}$$

and

$$X_1^{(6)}(x;t) = x_1 - 2x_2t + \frac{x_2^2+x_3^2+m^2}{x_1}t^2 = \frac{1}{x_1}\left[(x_1-x_2t)^2 + x_3^2t^2 + m^2t^2\right],$$
$$X_2^{(6)}(x;t) = x_2 - \frac{x_2^2+x_3^2+m^2}{x_1}t,$$

and similarly

$$\alpha^{(10)}(x;t) = \begin{cases} \frac{\sqrt{X_1^{(10)}(x;t)}}{\sqrt{x_1}} \left(\frac{x_1-tx_3+it\sqrt{m^2}}{x_1-tx_3-it\sqrt{m^2}}\right)^{\frac{i}{\sqrt{m^2}}\left(-x_2S_{23}-\frac{1}{2}S_{13}\right)}, & m^2 \neq 0, \\ \frac{\sqrt{X_1^{(10)}(x;t)}}{\sqrt{x_1}} \exp\left\{\frac{t}{x_1-tx_3}\left(S_{13}+2x_2S_{23}\right)\right\}, & m^2 = 0, \end{cases}$$

and

$$X_1^{(10)}(x;t) = x_1 - 2x_3t + \frac{x_2^2+x_3^2+m^2}{x_1}t^2 = \frac{1}{x_1}\left[(x_1-x_3t)^2 + x_2^2t^2 + m^2t^2\right],$$
$$X_3^{(10)}(x;t) = x_3 - \frac{x_2^2+x_3^2+m^2}{x_1}t.$$

Remark 4.2. One has to understand

$$\left(\frac{x_1-tx_2+it\sqrt{m^2}}{x_1-tx_2-it\sqrt{m^2}}\right)^{\frac{i}{\sqrt{m^2}}\left(x_3S_{23}-\frac{1}{2}S_{12}\right)} \equiv e^{\frac{i}{\sqrt{m^2}}\ln\left(\frac{x_1-tx_2+it\sqrt{m^2}}{x_1-tx_2-it\sqrt{m^2}}\right)\left(x_3S_{23}-\frac{1}{2}S_{12}\right)}. \tag{4.62}$$

Again, such "one-parameter subgroups" would have to be verified to satisfy all the desired properties. Here, in general, it has to be done for each possible representation S (taken from the list of all mutually non-equivalent representations) separately. We do not therefore go into detail here. Notice only, that the verification is really needed only for the subgroups $U^{(j)}_{m^2,S}$, for $j = 6, 8, 10$, i.e. for those containing operators of S. The rest was in fact already verified in Chapter 3.

Unitary Representations

Similarly, one has to show that the set of all products taken with respect to the chosen coordinates (4.1), i.e.

$$\begin{aligned} U_{m^2,S}(t_1, \dots, t_{10}) \equiv {} & U^{(2)}_{m^2,S}(t_2) U^{(3)}_{m^2,S}(t_3) U^{(4)}_{m^2,S}(t_4) U^{(7)}_{m^2,S}(t_7) U^{(5)}_{m^2,S}(t_5) \\ & \times U^{(9)}_{m^2,S}(t_9) U^{(1)}_{m^2,S}(t_1) U^{(8)}_{m^2,S}(t_8) U^{(6)}_{m^2,S}(t_6) U^{(10)}_{m^2,S}(t_{10}), \end{aligned} \tag{4.63}$$

$t_j \in \mathbb{R}$, $1 \le j \le 10$, forms a Lie group. Again, this is possible to be done by "local, continuous" reordering of $U_{m^2,S}(t_1, \dots, t_{10}) U_{m^2,S}(t'_1, \dots, t'_{10})$ back into $U_{m^2,S}(t''_1, \dots, t''_{10})$. Further, a discussion on global isomorphism of such group with $\mathcal{P}_4$, based on decision whether the group is isomorphic to $\mathcal{P}_4$ itself, or one of its non-trivial coverings, would be in principle needed. Nonetheless, we shall see below, comparing our results with the Mackey theory, that in fact no such case really occurs. As before, $U_{m^2,S}$ shall refer to the resulting representation of the whole Poincaré group, generated by (3.46).

Irreducibility

Because we are interested entirely in irreducible representations of $\mathcal{P}_4$, it is reasonable to require the representation S to have the property as well. Then irreducibility of constructed representations could be discussed in exactly the same manner as before, regardless of the concrete choice of S.

First, it is obvious that the representation $U_{m^2,S}(t_1, \dots, t_{10})$ is reducible whenever $m^2 > 0$; in that case the complementary subspaces $\mathcal{H}_3^\pm \otimes V$ are invariant. Second, using exactly the same argument as for Proposition 3.7, one easily proves that no further reducibility is admissible.

Mutual Non-equivalence

As before, the representations corresponding to distinct values of the real parameter m^2 are non-equivalent. The same obviously applies to representations depending on non-equivalent representations S.

Moreover, the irreducible representations obtained by restricting to $\mathcal{H}_3^\pm \otimes V$ could not be equivalent either, as easily seen from comparison of respective spectra of the operators $\Omega^\pm_{m^2,S}(P_0) := \Omega_{m^2,S}(P_0)\big|_{C_0^\infty(\mathbb{R}^\pm \times \mathbb{R}^2) \otimes V}$.

Summary

Altogether, with the notation kept as above we claim that

Conjecture 4.3. *The set*

$$\left\{ U^\pm_{0,S} \middle| S \in \mathfrak{A}(\mathfrak{s}_0) \right\} \cup \left\{ U^\pm_{m^2,S} \middle| m^2 > 0, S \in \mathfrak{A}(\mathfrak{s}_1) \right\} \cup \left\{ U_{m^2,S} \middle| m^2 < 0, S \in \mathfrak{A}(\mathfrak{s}_{-1}) \right\},$$

where $\mathfrak{A}(\mathfrak{s}_\varepsilon)$ is the set of mutually non-equivalent irreducible skew-symmetric representations of the Lie algebra $\mathfrak{s}_\varepsilon$, is the family all of pairwise non-equivalent irreducible unitary representations of the Poincaré Lie group $\mathcal{P}_4$.

Since no rigorous proof has been done, our result is stated as a conjecture in this case. Nevertheless, there is a strong, yet heuristic evidence for the validity of the assertion based on quantitative comparison with Mackey theory approach.

Namely, according to §1.3.3, the set of all irreducible unitary representations of $\mathcal{P}_4$ has the following form:

$$\left\{ U^{\mathrm{I}}_{W,\pm} \middle| W \in \mathcal{A}(\mathcal{S}_0) \right\} \cup \left\{ U^{\mathrm{II}}_{|m|,W,\pm} \middle| |m| \in \mathbb{R}^+, W \in \mathcal{A}(\mathcal{S}_1) \right\} \cup \\ \cup \left\{ U^{\mathrm{III}}_{|m|,W} \middle| |m| \in \mathbb{R}^+, W \in \mathcal{A}(\mathcal{S}_{-1}) \right\},$$

where $\mathcal{A}(\mathcal{S}_\varepsilon)$ is the set of mutually non-equivalent irreducible unitary representations of the Lie group $\mathcal{S}_\varepsilon$ that is defined

$$\mathcal{S}_\varepsilon := \begin{cases} \mathrm{SO}(3,\mathbb{R}), & \text{if } \varepsilon = 1, \\ \mathrm{E}_2, & \text{if } \varepsilon = 0, \\ \mathrm{SO}_0(1,2), & \text{if } \varepsilon = -1. \end{cases} \tag{4.64}$$

Because each $\mathfrak{s}_\varepsilon$ is the Lie algebra of the respective Lie group $\mathcal{S}_\varepsilon$, $\varepsilon = -1, 0, 1$, there is a one-to-one correspondence between the sets $\mathfrak{A}(\mathfrak{s}_\varepsilon)$ and $\mathcal{A}(\mathcal{S}_\varepsilon)$ for each ε. Consequently, there is an obvious one-to-one correspondence between the two presented families of representations of $\mathcal{P}_4$.

Conclusion

In the thesis we have focused on the construction of irreducible unitary representations for the Poincaré groups $\mathcal{P}_2$, $\mathcal{P}_3$ and $\mathcal{P}_4$. For this purpose, the relationship between fields of fractions corresponding to the respective universal enveloping algebras and suitably extended Weyl algebras has been used.

In the first chapter we have summarized the theoretical fundamentals needed for the construction and described the construction technique in detail. Further, we have recalled the standard framework of Mackey theory of induced representations.

In the second chapter we have made use of our method in order to obtain all irreducible unitary representations of the Lie group $\mathcal{P}_2$. First, a $*$-isomorphism between the fields $\mathfrak{D}(\mathfrak{p}_2)$ and $\mathfrak{D}_{1,1}(\mathbb{R})$ has been found. Notice that this has verified the Gelfand-Kirillov conjecture (or its analogue over $\mathbb{R}$) for this case. Second, the isomorphism has been used to induce skew-symmetric representations of the Lie algebra $\mathfrak{p}_2$. Third, we have integrated these representations into unitary representations of the Lie group $\mathcal{P}_2$ and finally, we have discussed their irreducibility and mutual non-equivalence. After all, the set of all irreducible unitary representations has been constructed also with respect to Mackey theory and it has been shown that both approaches led to the same results. This fact has been explicitly demonstrated by transition isometries.

In the third chapter we have dealt with the six-dimensional Poincaré group $\mathcal{P}_3$. The procedure of the preceding chapter has been repeated in order to obtain the complete set of irreducible unitary representations for this case. Within the scope of the construction, the Gelfand-Kirillov conjecture has been verified for the Lie algebra $\mathfrak{p}_3$, since we have introduced a $*$-isomorphism between $\mathfrak{D}(\mathfrak{p}_3)$ and $\mathfrak{D}_{2,2}(\mathbb{R})$. Also in this case our method has been proven to be completely equivalent to Mackey's approach.

Finally, the possibility of application of our method to the physically interesting Poincaré group $\mathcal{P}_4$ has been discussed in the fourth chapter. We have modified the suggested technique slightly, namely we have embedded the Lie field $\mathfrak{D}(\mathfrak{p}_4)$ in $\mathfrak{D}_{3,1;s}(\mathbb{R})$ rather that in one of $\mathfrak{D}_{m,r}(\mathbb{R})$. Unlike $\mathfrak{D}_{m,r}(\mathbb{R})$, the field $\mathfrak{D}_{3,1;s}(\mathbb{R})$ corresponds to the Weyl algebra extended moreover by certain non-commuting elements. Thus, we have not concerned with the Gelfand-Kirillov conjecture here. Though, we have again made use of the $*$-isomorphism to produce skew-symmetric representations of $\mathfrak{p}_4$. Due to its complexity, the discussion on integration of the representations into unitary representations of $\mathcal{P}_4$ has not been completely rigorous and the completed representations have not been strictly verified to satisfy all the desired properties. Nonetheless, we have seen by a casual comparison with Mackey theory that our technique applied to this case as well and that in principle we were able to reproduce Wigner's classification of irreducible unitary representations for $\mathcal{P}_4$.

Appendix A

Auxiliary Calculations

Throughout the Appendix, the notation from the main text is kept.

A.1 Coordinates in $\mathcal{P}_2$

First of all, we shall convince ourselves that the canonical coordinates in $\mathcal{P}_2$ we established at the beginning of the second chapter are global.

Proposition A.1. *The coordinates in $\mathcal{P}_2$ defined by* (2.1) *are global.*

Proof. Clearly, it is enough to verify that

$$\mathrm{SO}_0(1,1) = \left\{ \Lambda(t_1) \equiv \begin{pmatrix} \cosh t_1 & -\sinh t_1 \\ -\sinh t_1 & \cosh t_1 \end{pmatrix} \middle| t_1 \in \mathbb{R} \right\}. \tag{A.1}$$

First, for any $t_1 \in \mathbb{R}$ we have

$$\begin{aligned}
\Lambda(t_1)^T \eta \Lambda(t_1) &= \begin{pmatrix} \cosh t_1 & -\sinh t_1 \\ -\sinh t_1 & \cosh t_1 \end{pmatrix}^T \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} \cosh t_1 & -\sinh t_1 \\ -\sinh t_1 & \cosh t_1 \end{pmatrix} \\
&= \begin{pmatrix} \cosh t_1 & -\sinh t_1 \\ -\sinh t_1 & \cosh t_1 \end{pmatrix} \begin{pmatrix} \cosh t_1 & -\sinh t_1 \\ \sinh t_1 & -\cosh t_1 \end{pmatrix} \\
&= \begin{pmatrix} \cosh^2 t_1 - \sinh^2 t_1 & 0 \\ 0 & \sinh^2 t_1 - \cosh^2 t_1 \end{pmatrix} \\
&= \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \\
&= \eta
\end{aligned}$$

as well as $\det \Lambda(t_1) = \cosh^2 t_1 - \sinh^2 t_1 = 1$ and $\Lambda(t_1)_{00} = \cosh t_1 \geq 1$.

On the other hand, any $\Lambda = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \in \mathrm{SO}_0(1,1)$ must satisfy

$$\begin{aligned}
\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} &= \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}^T \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \\
&= \begin{pmatrix} \alpha & \gamma \\ \beta & \delta \end{pmatrix} \begin{pmatrix} \alpha & \beta \\ -\gamma & -\delta \end{pmatrix} \\
&= \begin{pmatrix} \alpha^2 - \gamma^2 & \alpha\beta - \gamma\delta \\ \alpha\beta - \gamma\delta & \beta^2 - \delta^2 \end{pmatrix},
\end{aligned}$$

therefore $\alpha\beta = \gamma\delta$, $\alpha^2 = \gamma^2 + 1$ and $\beta^2 = \delta^2 - 1$, as well as $\alpha\delta - \beta\gamma = 1$ and $\alpha \geq 1$. From the first three relations we have

$$\gamma^2\delta^2 = \alpha^2\beta^2 = (\gamma^2+1)(\delta^2-1) = \gamma^2\delta^2 - \gamma^2 + \delta^2 - 1 = \gamma^2\delta^2 + \beta^2 - \gamma^2.$$

Thus $\beta = \pm\gamma$. If $\beta = -\gamma$, then also $\alpha = -\delta$ and $\det\Lambda = -\alpha^2 + \gamma^2 = -1$. Therefore it is necessary that $\beta = \gamma$ and $\alpha = \delta$. Certainly, there is $t \in \mathbb{R}$ such that $\gamma = \beta = \sinh t$. Then $\alpha^2 = \sinh^2 t + 1 = \frac{e^{2t}}{4} - \frac{1}{2} + \frac{e^{-2t}}{4} + 1 = \cosh^2 t$. The requirement $\alpha \geq 1$ finally choose $\alpha = \cosh t$ and hence $\Lambda = \Lambda(t)$. $\square$

A.2 One-parameter Subgroups in $\mathcal{P}_3$

Let us shift to the Lie group $\mathcal{P}_3$ now. First, an auxiliary assertion needed in proof of Proposition 3.4 is verified.

Lemma A.2. *We have*

$$\frac{\partial\left(X_1^{(6)}(x;t), X_2^{(6)}(x;t)\right)}{\partial(x_1,x_2)} = \frac{X_1^{(6)}(x;t)}{x_1}.$$

Proof. By direct computation we have

$$\begin{aligned}
\frac{\partial\left(X_1^{(6)}(x;t), X_2^{(6)}(x;t)\right)}{\partial(x_1,x_2)} &= \det\begin{pmatrix} 1 - \frac{x_2^2+m^2}{x_1^2}t^2 & -2t + \frac{2t^2x_2}{x_1} \\ \frac{x_2^2+m^2}{x_1^2}t & 1 - \frac{2tx_2}{x_1} \end{pmatrix} \\
&= \det\begin{pmatrix} 1 - \frac{x_2^2+m^2}{x_1^2}t^2 & -2t + \frac{2t^2x_2}{x_1} \\ \frac{x_2^2+m^2}{x_1^2}t & 1 - \frac{2tx_2}{x_1} \end{pmatrix} \\
&= \det\begin{pmatrix} 1 & -t \\ \frac{x_2^2+m^2}{x_1^2}t & 1 - \frac{2tx_2}{x_1} \end{pmatrix} \\
&= 1 - \frac{2tx_2}{x_1} + \frac{x_2^2+m^2}{x_1^2}t^2 \\
&= \frac{X_1^{(6)}(x;t)}{x_1}.
\end{aligned}$$

$\square$

Second, we shall show how the one-parameter subgroups (3.40) - (3.45) commute (locally) with each other. As discussed in §2.1.3, the relations help to simplify the proof of Theorem 3.5 significantly.

Lemma A.3. *For $(t_1,\dots,t_6) \in \mathbb{T}_6 \equiv \mathbb{R}^4 \times (-1,1) \times (-1,1)$, the following relations hold:*

$$U^{(1)}_{m^2,c}(t_1)\, U^{(5)}_{m^2,c}(t_5') = U^{(5)}_{m^2,c}(t_5' e^{t_1})\, U^{(1)}_{m^2,c}(t_1), \tag{A.2}$$

$$U^{(6)}_{m^2,c}(t_6)\, U^{(1)}_{m^2,c}(t_1') = U^{(1)}_{m^2,c}(t_1')\, U^{(6)}_{m^2,c}(t_6 e^{t_1'}), \tag{A.3}$$

$$U^{(6)}_{m^2,c}(t_6)\, U^{(5)}_{m^2,c}(t_5') = U^{(5)}_{m^2,c}\left(\frac{t_5'}{1-t_5't_6}\right) U^{(1)}_{m^2,c}[-2\ln(1-t_5't_6)]\, U^{(6)}_{m^2,c}\left(\frac{t_6}{1-t_5't_6}\right), \tag{A.4}$$

$$U^{(1)}_{m^2,c}(t_1)\, U^{(2)}_{m^2,c}(t_2') = U^{(2)}_{m^2,c}(t_2'\cosh t_1)\, U^{(3)}_{m^2,c}(-t_2'\sinh t_1)\, U^{(1)}_{m^2,c}(t_1), \tag{A.5}$$

$$U^{(1)}_{m^2,c}(t_1)\, U^{(3)}_{m^2,c}(t_3') = U^{(2)}_{m^2,c}(-t_3'\sinh t_1)\, U^{(3)}_{m^2,c}(t_3'\cosh t_1)\, U^{(1)}_{m^2,c}(t_1), \tag{A.6}$$

$$U^{(1)}_{m^2,c}(t_1)\, U^{(4)}_{m^2,c}(t_4') = U^{(4)}_{m^2,c}(t_4')\, U^{(1)}_{m^2,c}(t_1), \tag{A.7}$$

$$U^{(5)}_{m^2,c}(t_5)\,U^{(2)}_{m^2,c}(t'_2) = U^{(2)}_{m^2,c}\left[t'_2\left(1+\frac{t_5^2}{2}\right)\right]\,U^{(3)}_{m^2,c}\left(-\frac{t'_2t_5^2}{2}\right)\,U^{(4)}_{m^2,c}(t'_2t_5)\,U^{(5)}_{m^2,c}(t_5), \quad \text{(A.8)}$$

$$U^{(5)}_{m^2,c}(t_5)\,U^{(3)}_{m^2,c}(t'_3) = U^{(2)}_{m^2,c}\left(\frac{t_5^2t'_3}{2}\right)\,U^{(3)}_{m^2,c}\left[t'_3\left(1-\frac{t_5^2}{2}\right)\right]\,U^{(4)}_{m^2,c}(t'_3t_5)\,U^{(5)}_{m^2,c}(t_5), \quad \text{(A.9)}$$

$$U^{(5)}_{m^2,c}(t_5)\,U^{(4)}_{m^2,c}(t'_4) = U^{(2)}_{m^2,c}(t'_4t_5)\,U^{(3)}_{m^2,c}(-t'_4t_5)\,U^{(4)}_{m^2,c}(t'_4)\,U^{(5)}_{m^2,c}(t_5), \quad \text{(A.10)}$$

$$U^{(6)}_{m^2,c}(t_6)\,U^{(2)}_{m^2,c}(t'_2) = U^{(2)}_{m^2,c}\left[t'_2\left(1+\frac{t_6^2}{2}\right)\right]\,U^{(3)}_{m^2,c}\left(\frac{t'_2t_6^2}{2}\right)\,U^{(4)}_{m^2,c}(-t'_2t_6)\,U^{(6)}_{m^2,c}(t_6), \quad \text{(A.11)}$$

$$U^{(6)}_{m^2,c}(t_6)\,U^{(3)}_{m^2,c}(t'_3) = U^{(2)}_{m^2,c}\left(-\frac{t_6^2t'_3}{2}\right)\,U^{(3)}_{m^2,c}\left[t'_3\left(1-\frac{t_6^2}{2}\right)\right]\,U^{(4)}_{m^2,c}(t'_3t_6)\,U^{(6)}_{m^2,c}(t_6), \quad \text{(A.12)}$$

$$U^{(6)}_{m^2,c}(t_6)\,U^{(4)}_{m^2,c}(t'_4) = U^{(2)}_{m^2,c}(-t'_4t_6)\,U^{(3)}_{m^2,c}(-t'_4t_6)\,U^{(4)}_{m^2,c}(t'_4)\,U^{(6)}_{m^2,c}(t_6). \quad \text{(A.13)}$$

Proof. Take any $\psi \in \mathcal{H}_2$.

(a) $U^{(1)}_{m^2,c}(t_1)\,U^{(5)}_{m^2,c}(t'_5)\psi(x) = e^{\frac{t_1}{2}}\psi(e^{t_1}x_1, x_2 + t'_5e^{t_1}x_1) = U^{(5)}_{m^2,c}(t'_5e^{t_1})\,U^{(1)}_{m^2,c}(t_1)\psi(x).$

(b) $U^{(6)}_{m^2,c}(t_6)\,U^{(1)}_{m^2,c}(t'_1)\psi(x)$

$$\begin{aligned}
&= e^{\frac{t'_1}{2}}\alpha^{(6)}(x;t_6)\,\psi\Big(e^{t'_1}X^{(6)}_1(x;t_6), X^{(6)}_2(x;t_6)\Big)\\
&= e^{\frac{t'_1}{2}}\alpha^{(6)}(e^{t'_1}x;t_6e^{t'_1})\,\psi\Big(X^{(6)}_1(e^{t'_1}x_1,x_2;t_6e^{t'_1}), X^{(6)}_2(e^{t'_1}x,x_2;t_6e^{t'_1})\Big)\\
&= U^{(1)}_{m^2,c}(t'_1)\,U^{(6)}_{m^2,c}(t_6e^{t'_1})\psi(x).
\end{aligned}$$

(c) $U^{(5)}_{m^2,c}\left(\frac{t'_5}{1-t'_5t_6}\right)\,U^{(1)}_{m^2,c}[-2\ln(1-t'_5t_6)]\,U^{(6)}_{m^2,c}\left(\frac{t_6}{1-t'_5t_6}\right)\psi(x)$

$$\begin{aligned}
&= U_{m^2,c}\left(-2\ln(1-t'_5t_6),0,0,0,\frac{t'_5}{1-t'_5t_6},\frac{t_6}{1-t'_5t_6}\right)\psi(x)\\
&= \alpha\left(x;-2\ln(1-t'_5t_6),0,0,0,\frac{t'_5}{1-t'_5t_6},\frac{t_6}{1-t'_5t_6}\right)\\
&\qquad\times\psi\left[X_1\left(x;-2\ln(1-t'_5t_6),\frac{t'_5}{1-t'_5t_6},\frac{t_6}{1-t'_5t_6}\right),\right.\\
&\qquad\qquad\left.X_2\left(x;-2\ln(1-t'_5t_6),\frac{t'_5}{1-t'_5t_6},\frac{t_6}{1-t'_5t_6}\right)\right]\\
&= \alpha^{(6)}(x;t_6)\,\psi\Big(X^{(6)}_1(x;t_6), X^{(6)}_2(x;t_6)+t'_5X^{(6)}_1(x;t_6)\Big)\\
&= U^{(6)}_{m^2,c}(t_6)\,U^{(5)}_{m^2,c}(t'_5)\psi(x),
\end{aligned}$$

since

$$\begin{aligned}
&X_1\left(x;-2\ln(1-t'_5t_6),\frac{t'_5}{1-t'_5t_6},\frac{t_6}{1-t'_5t_6}\right)\\
&= \frac{x_1}{(1-t'_5t_6)^2} - \frac{2t_6}{1-t'_5t_6}\left(x_2+\frac{t'_5x_1}{1-t'_5t_6}\right) + \frac{\left(x_2+\frac{t'_5x_1}{1-t'_5t_6}\right)^2+m^2}{\frac{x_1}{(1-t'_5t_6)^2}}\left(\frac{t_6}{1-t'_5t_6}\right)^2\\
&= \frac{1}{(1-t'_5t_6)^2}\left[x_1 - 2t_6x_2 + 2t'_5t_6^2x_2 - 2t'_5t_6x_1 + \frac{x_2^2t_6^2}{x_1} + \frac{(t'_5)^2t_6^4x_2^2}{x_1}\right.\\
&\qquad\left. +(t'_5)^2t_6^2x_1 - \frac{2t'_5t_6^3x_2^2}{x_1} + 2t'_5t_6^2x_2 - 2(t'_5)^2t_6^3x_2 + \frac{t_6^2m^2}{x_1}(1-t'_5t_6)^2\right]\\
&= x_1 - 2t_6x_2 + \frac{x_2^2+m^2}{x_1}t_6^2\\
&= X^{(6)}_1(x;t_6),
\end{aligned}$$

$$
\begin{aligned}
&X_2\left(x; -2\ln(1-t_5' t_6), \frac{t_5'}{1-t_5' t_6}, \frac{t_6}{1-t_5' t_6}\right) \\
&\quad = x_2 + \frac{t_5' x_1}{1-t_5' t_6} - \frac{\left(x_2 + \frac{t_5' x_1}{1-t_5' t_6}\right)^2 + m^2}{\frac{x_1}{(1-t_5' t_6)^2}} \cdot \frac{t_6}{1-t_5' t_6} \\
&\quad = \frac{1}{1-t_5' t_6}\left[x_2 - t_5' t_6 x_2 + t_5' x_1 - \frac{x_2^2 t_6}{x_1}(1-t_5' t_6)^2 - (t_5')^2 t_6 x_1\right. \\
&\qquad\qquad\qquad \left. -2x_2 t_5' t_6 (1-t_5' t_6) - \frac{m^2 t_6}{x_1}(1-t_5' t_6)^2\right] \\
&\quad = x_2 + t_5' x_1 - 2x_2 t_5' t_6 - \frac{x_2^2+m^2}{x_1} t_6 + t_5' \frac{x_2^2+m^2}{x_1} t_6^2 \\
&\quad = X_2^{(6)}(x; t_6) + t_5' X_1^{(6)}(x; t_6)
\end{aligned}
$$

and, if $m^2 \neq 0$,

$$
\begin{aligned}
&\alpha\left(x; -2\ln(1-t_5' t_6), 0, 0, 0, \frac{t_5'}{1-t_5' t_6}, \frac{t_6}{1-t_5' t_6}\right) \\
&\quad = \frac{1}{1-t_5' t_6} \cdot \frac{\sqrt{X_1\left(x; -2\ln(1-t_5' t_6), \frac{t_5'}{1-t_5' t_6}, \frac{t_6}{1-t_5' t_6}\right)}}{\sqrt{\frac{x_1}{(1-t_5' t_6)^2}}} \\
&\qquad \times \left[\frac{\frac{x_1}{(1-t_5' t_6)^2} - \frac{t_6 x_2}{1-t_5' t_6} - \frac{t_5' t_6 x_1}{(1-t_5' t_6)^2} + \frac{i t_6 \sqrt{m^2}}{1-t_5' t_6}}{\frac{x_1}{(1-t_5' t_6)^2} - \frac{t_6 x_2}{1-t_5' t_6} - \frac{t_5' t_6 x_1}{(1-t_5' t_6)^2} - \frac{i t_6 \sqrt{m^2}}{1-t_5' t_6}}\right]^{\frac{c}{\sqrt{m^2}}} \\
&\quad = \frac{\sqrt{X_1^{(6)}(x; t_6)}}{\sqrt{x_1}} \left[\frac{x_1 - (1-t_5' t_6) t_6 x_2 - t_5' t_6 x_1 + (1-t_5' t_6) i t_6 \sqrt{m^2}}{x_1 - (1-t_5' t_6) t_6 x_2 - t_5' t_6 x_1 - (1-t_5' t_6) i t_6 \sqrt{m^2}}\right]^{\frac{c}{\sqrt{m^2}}} \\
&\quad = \frac{\sqrt{X_1^{(6)}(x; t_6)}}{\sqrt{x_1}} \left[\frac{x_1 - t_6 x_2 + i t_6 \sqrt{m^2}}{x_1 - t_6 x_2 - i t_6 \sqrt{m^2}}\right]^{\frac{c}{\sqrt{m^2}}} \\
&\quad = \alpha^{(6)}(x; t_6),
\end{aligned}
$$

and for $m^2 = 0$ we also have

$$
\begin{aligned}
&\alpha\left(x; -2\ln(1-t_5' t_6), 0, 0, 0, \frac{t_5'}{1-t_5' t_6}, \frac{t_6}{1-t_5' t_6}\right) \\
&\quad = \frac{1}{1-t_5' t_6}\left[1 - \frac{t_6}{1-t_5' t_6} \cdot \frac{x_2 + \frac{t_5' x_1}{1-t_5' t_6}}{\frac{x_1}{(1-t_5' t_6)^2}}\right] \exp\left\{\frac{\frac{2ict_6}{1-t_5' t_6}}{\frac{x_1}{(1-t_5' t_6)^2} - \frac{t_6}{1-t_5' t_6}\left(x_2 + \frac{t_5' x_1}{1-t_5' t_6}\right)}\right\} \\
&\quad = \frac{1}{1-t_5' t_6}\left[1 - t_5' t_6 - (1-t_5' t_6)\frac{t_6 x_2}{x_1}\right] \exp\left\{\frac{(1-t_5' t_6) 2ict_6}{x_1 - t_5' t_6 x_1 - (1-t_5' t_6) t_6 x_2}\right\} \\
&\quad = \left(1 - \frac{t_6 x_2}{x_1}\right) \exp\left(\frac{2ict_6}{x_1 - t_6 x_2}\right) \\
&\quad = \alpha^{(6)}(x; t_6).
\end{aligned}
$$

(d) $U^{(2)}_{m^2,c}(t'_2\cosh t_1)\,U^{(3)}_{m^2,c}(-t'_2\sinh t_1)\,U^{(1)}_{m^2,c}(t_1)\psi(x)$

$$= e^{\frac{t_1}{2}+\frac{it'_2}{4}\left(e^{t_1}+e^{-t_1}\right)\left(x_1+\frac{x_2^2+m^2}{x_1}\right)+\frac{it'_2}{4}\left(e^{t_1}-e^{-t_1}\right)\left(x_1-\frac{x_2^2+m^2}{x_1}\right)}\psi(e^{t_1}x_1,x_2)$$

$$= e^{\frac{t_1}{2}+\frac{it'_2}{2}\left(e^{t_1}x_1+\frac{x_2^2+m^2}{e^{t_1}x_1}\right)}\psi(e^{t_1}x_1,x_2)$$

$$= U^{(1)}_{m^2,c}(t_1)\,U^{(2)}_{m^2,c}(t'_2)\psi(x).$$

(e) $U^{(2)}_{m^2,c}(-t'_3\sinh t_1)\,U^{(3)}_{m^2,c}(t'_3\cosh t_1)\,U^{(1)}_{m^2,c}(t_1)\psi(x)$

$$= e^{\frac{t_1}{2}-\frac{it'_3}{4}\left(e^{t_1}-e^{-t_1}\right)\left(x_1+\frac{x_2^2+m^2}{x_1}\right)-\frac{it'_3}{4}\left(e^{t_1}+e^{-t_1}\right)\left(x_1-\frac{x_2^2+m^2}{x_1}\right)}\psi(e^{t_1}x_1,x_2)$$

$$= e^{\frac{t_1}{2}-\frac{it'_3}{2}\left(e^{t_1}x_1-\frac{x_2^2+m^2}{e^{t_1}x_1}\right)}\psi(e^{t_1}x_1,x_2)$$

$$= U^{(1)}_{m^2,c}(t_1)\,U^{(3)}_{m^2,c}(t'_3)\psi(x).$$

(f) Commutation of $U^{(1)}_{m^2,c}(t_1)$ with $U^{(4)}_{m^2,c}(t'_4)$ is obvious.

(g) $U^{(2)}_{m^2,c}\left[t'_2\left(1+\frac{t_5^2}{2}\right)\right]\,U^{(3)}_{m^2,c}\left(-\frac{t'_2t_5^2}{2}\right)\,U^{(4)}_{m^2,c}(t'_2t_5)\,U^{(5)}_{m^2,c}(t_5)\psi(x)$

$$= e^{\frac{it'_2}{2}\left(1+\frac{t_5^2}{2}\right)\left(x_1+\frac{x_2^2+m^2}{x_1}\right)+\frac{it'_2t_5^2}{4}\left(x_1-\frac{x_2^2+m^2}{x_1}\right)+it'_2t_5x_2}\psi(x_1,x_2+t_5x_1)$$

$$= e^{\frac{it'_2}{2}\left(x_1+\frac{x_2^2+m^2}{x_1}+t_5^2x_1+2t_5x_2\right)}\psi(x_1,x_2+t_5x_1)$$

$$= e^{\frac{it'_2}{2}\left(x_1+\frac{(x_2+t_5x_1)^2+m^2}{x_1}\right)}\psi(x_1,x_2+t_5x_1)$$

$$= U^{(5)}_{m^2,c}(t_5)\,U^{(2)}_{m^2,c}(t'_2)\psi(x).$$

(h) $U^{(2)}_{m^2,c}\left(\frac{t'_3t_5^2}{2}\right)\,U^{(3)}_{m^2,c}\left[t'_3\left(1-\frac{t_5^2}{2}\right)\right]\,U^{(4)}_{m^2,c}(t'_3t_5)\,U^{(5)}_{m^2,c}(t_5)\psi(x)$

$$= e^{\frac{it'_3t_5^2}{4}\left(x_1+\frac{x_2^2+m^2}{x_1}\right)-\frac{it'_3}{2}\left(1-\frac{t_5^2}{2}\right)\left(x_1-\frac{x_2^2+m^2}{x_1}\right)+it'_3t_5x_2}\psi(x_1,x_2+t_5x_1)$$

$$= e^{-\frac{it'_3}{2}\left(-t_5^2x_1+x_1-\frac{x_2^2+m^2}{x_1}-2t_5x_2\right)}\psi(x_1,x_2+t_5x_1)$$

$$= e^{-\frac{it'_3}{2}\left(x_1-\frac{(x_2+t_5x_1)^2+m^2}{x_1}\right)}\psi(x_1,x_2+t_5x_1)$$

$$= U^{(5)}_{m^2,c}(t_5)\,U^{(3)}_{m^2,c}(t'_3)\psi(x).$$

(i) $U^{(2)}_{m^2,c}(t'_4t_5)\,U^{(3)}_{m^2,c}(-t'_4t_5)\,U^{(4)}_{m^2,c}(t'_4)\,U^{(5)}_{m^2,c}(t_5)\psi(x)$

$$= e^{\frac{it'_4t_5}{2}\left(x_1+\frac{x_2^2+m^2}{x_1}\right)+\frac{it'_4t_5}{2}\left(x_1-\frac{x_2^2+m^2}{x_1}\right)+it'_4x_2}\psi(x_1,x_2+t_5x_1)$$

$$= e^{it'_4(t_5x_1+x_2)}\psi(x_1,x_2+t_5x_1)$$

$$= U^{(5)}_{m^2,c}(t_5)\,U^{(4)}_{m^2,c}(t'_4)\psi(x).$$

(j) $U^{(2)}_{m^2,c}\left[t_2'\left(1+\frac{t_6^2}{2}\right)\right]\,U^{(3)}_{m^2,c}\left(\frac{t_2't_6^2}{2}\right)\,U^{(4)}_{m^2,c}(-t_2't_6)\,U^{(6)}_{m^2,c}(t_6)\psi(x)$

$$= e^{\frac{it_2'}{2}\left(1+\frac{t_6^2}{2}\right)\left(x_1+\frac{x_2^2+m^2}{x_1}\right)-\frac{it_2't_6^2}{4}\left(x_1-\frac{x_2^2+m^2}{x_1}\right)-it_2't_6x_2}\alpha(x;t_6)\psi\left(X_1^{(6)}(x;t_6),X_2^{(6)}(x;t_6)\right)$$

$$= e^{\frac{it_2'}{2}\left(x_1+\frac{x_2^2+m^2}{x_1}+\frac{x_2^2+m^2}{x_1}t_6^2-2t_6x_2\right)}\alpha(x;t_6)\psi\left(X_1^{(6)}(x;t_6),X_2^{(6)}(x;t_6)\right)$$

$$= e^{\frac{it_2'}{2}\left(X_1^{(6)}(x;t_6)+\frac{\left[X_2^{(6)}(x;t_6)\right]^2+m^2}{X_1^{(6)}(x;t_6)}\right)}\alpha(x;t_6)\psi\left(X_1^{(6)}(x;t_6),X_2^{(6)}(x;t_6)\right)$$

$$= U^{(6)}_{m^2,c}(t_6)\,U^{(2)}_{m^2,c}(t_2')\psi(x),$$

since

$$\frac{\left[X_2^{(6)}(x;t_6)\right]^2+m^2}{X_1^{(6)}(x;t_6)} = \frac{\left(x_2-\frac{x_2^2+m^2}{x_1}t_6\right)^2+m^2}{x_1-2x_2t_6+\frac{x_2^2+m^2}{x_1}t_6^2}$$

$$= \frac{x_2^2-2x_2\frac{x_2^2+m^2}{x_1}t_6+\left(\frac{x_2^2+m^2}{x_1}\right)^2t_6^2+m^2}{x_1-2x_2t_6+\frac{x_2^2+m^2}{x_1}t_6^2}$$

$$= \frac{\frac{x_2^2+m^2}{x_1}\left(x_1-2x_2t_6+\frac{x_2^2+m^2}{x_1}t_6^2\right)}{x_1-2x_2t_6+\frac{x_2^2+m^2}{x_1}t_6^2}$$

$$= \frac{x_2^2+m^2}{x_1}.$$

The same relation is used in the following as well.

(k) $U^{(2)}_{m^2,c}\left(-\frac{t_6^2t_3'}{2}\right)\,U^{(3)}_{m^2,c}\left[t_3'\left(1-\frac{t_6^2}{2}\right)\right]\,U^{(4)}_{m^2,c}(t_3't_6)\,U^{(6)}_{m^2,c}(t_6)\psi(x)$

$$= e^{-\frac{it_3't_6^2}{4}\left(x_1+\frac{x_2^2+m^2}{x_1}\right)-\frac{it_3'}{2}\left(1-\frac{t_6^2}{2}\right)\left(x_1-\frac{x_2^2+m^2}{x_1}\right)+it_3't_6x_2}\alpha(x;t_6)\psi\left(X_1^{(6)}(x;t_6),X_2^{(6)}(x;t_6)\right)$$

$$= e^{-\frac{it_3'}{2}\left(\frac{x_2^2+m^2}{x_1}t_6^2+x_1-\frac{x_2^2+m^2}{x_1}-2t_6x_2\right)}\alpha(x;t_6)\psi\left(X_1^{(6)}(x;t_6),X_2^{(6)}(x;t_6)\right)$$

$$= e^{-\frac{it_3'}{2}\left(X_1^{(6)}(x;t_6)-\frac{\left[X_2^{(6)}(x;t_6)\right]^2+m^2}{X_1^{(6)}(x;t_6)}\right)}\alpha(x;t_6)\psi\left(X_1^{(6)}(x;t_6),X_2^{(6)}(x;t_6)\right)$$

$$= U^{(6)}_{m^2,c}(t_6)\,U^{(3)}_{m^2,c}(t_3')\psi(x).$$

(l) $U^{(2)}_{m^2,c}(-t_4't_6)\,U^{(3)}_{m^2,c}(-t_4't_6)\,U^{(4)}_{m^2,c}(t_4')\,U^{(6)}_{m^2,c}(t_6)\psi(x)$

$$= e^{-\frac{it_4't_6}{2}\left(x_1+\frac{x_2^2+m^2}{x_1}\right)+\frac{it_4't_6}{2}\left(x_1-\frac{x_2^2+m^2}{x_1}\right)+it_4'x_2}\alpha(x;t_6)\psi\left(X_1^{(6)}(x;t_6),X_2^{(6)}(x;t_6)\right)$$

$$= e^{it_4'\left(x_2-\frac{x_2^2+m^2}{x_1}t_6\right)}\alpha(x;t_6)\psi\left(X_1^{(6)}(x;t_6),X_2^{(6)}(x;t_6)\right)$$

$$= e^{it_4'X_2^{(6)}(x;t_6)}\alpha(x;t_6)\psi\left(X_1^{(6)}(x;t_6),X_2^{(6)}(x;t_6)\right)$$

$$= U^{(6)}_{m^2,c}(t_6)\,U^{(4)}_{m^2,c}(t_4')\psi(x).$$

□

It is important for us that, with help of the previous lemma, the product of two operators $U_{m^2,c}(t)U_{m^2,c}(t')$ can be in reordered back to $U_{m^2,c}(t'')$ in finitely many steps. Recall that this can be done only locally, i.e. for t, t' taken from certain (unspecified) neighbourhood of $0 \in \mathbb{R}^6$. The procedure is outlined by the following scheme:

$$
\begin{gathered}
23451\underline{62}34516\\
2345\underline{12}34\underline{63}4516\\
234\underline{52}3\underline{13}4234\underline{64}516\\
234234\underline{53}23\underline{14}234234\underline{65}16\\
234234234\underline{52}34\underline{12}3423451\underline{61}6\\
234234234234\underline{53}423\underline{13}423451166\\
234234234234234\underline{54}2323\underline{14}23451166\\
234234234234234234\underline{52}3234\underline{12}3451166\\
234234234234234234234\underline{53}23423\underline{13}451166\\
234234234234234234234234\underline{52}342323\underline{14}51166\\
234234234234234234234234234\underline{53}423234\underline{15}1166\\
234234234234234234234234234234\underline{54}23234511166\\
234234234234234234234234234234234\underline{52}3234511166\\
234234234234234234234234234234234234\underline{53}234511166\\
234234234234234234234234234234234234234\underline{52}34511166\\
234234234234234234234234234234234234234234\underline{53}4511166\\
234234234234234234234234234234234234234234234\underline{54}511166\\
2342342342342342342342342342342342342342342342345511166
\end{gathered}
$$

Within each step, the underlined pairs were commuted. To complete the reordering, it is enough to realize that "2,3,4" obviously commute with each other.

A.3 Mackey Theory for $\mathcal{P}_3$

In the third part of the Appendix we show three auxiliary assertions used in "Mackey" construction of irreducible unitary representations of $\mathcal{P}_3$. Namely three Jacobian determinants of coordinate transformations are computed in order to establish Radon-Nikodym derivatives in §3.2.1, §3.2.2 and §3.2.3, respectively.

Lemma A.4. *For any* $\iota = \mathrm{I}, \mathrm{II}, \mathrm{III}$, *we have* $\left|\frac{\partial(X_1^\iota, X_2^\iota)}{\partial(x_1, x_2)}\right| = 1$.

Proof. First of all, it is useful to realize

$$
\begin{aligned}
X_1^\iota = x_1 + t_1 + \ln\Bigg[\left(1 - \frac{t_5 t_6}{e^{t_1}}\right)^2 &+ 2t_5 t_6^2 x_2 e^{-x_1-2t_1} \\
&+ t_6^2 x_2^2 e^{-2x_1-2t_1} - 2t_6 x_2 e^{-x_1-t_1} + \varepsilon^\iota t_6^2 e^{-2x_1-2t_1}\Bigg],
\end{aligned} \tag{A.14}
$$

$$
X_2^\iota = x_2 - e^{x_1-t_1} t_5^2 t_6 + e^{x_1} t_5 - 2e^{-t_1} t_5 t_6 x_2 - t_6 e^{-t_1-x_1} x_2^2 - \varepsilon^\iota t_6 e^{-t_1-x_1}, \tag{A.15}
$$

where $\varepsilon^{\mathrm{I}} = 0$, $\varepsilon^{\mathrm{II}} = 1$ and $\varepsilon^{\mathrm{III}} = -1$. Then, for any $\iota = \mathrm{I}, \mathrm{II}, \mathrm{III}$,

$$
\frac{\partial X_1^\iota}{\partial x_1} = 1 + \frac{-2t_5 t_6^2 x_2 e^{-x_1-2t_1} - 2t_6^2 x_2^2 e^{-2x_1-2t_1} + 2t_6 x_2 e^{-x_1-t_1} - 2\varepsilon^\iota t_6^2 e^{-2x_1-2t_1}}{\left(1 - \frac{t_5 t_6}{e^{t_1}}\right)^2 + 2t_5 t_6^2 x_2 e^{-x_1-2t_1} + t_6^2 x_2^2 e^{-2x_1-2t_1} - 2t_6 x_2 e^{-x_1-t_1} + \varepsilon^\iota t_6^2 e^{-2x_1-2t_1}}
$$

$$
\begin{aligned}
&= \frac{1 + t_5^2 t_6^2 e^{-2t_1} - 2t_5 t_6 e^{-t_1} - t_6^2 x_2^2 e^{-2x_1-2t_1} - \varepsilon^\iota t_6^2 e^{-2x_1-2t_1}}{\left(1 - \frac{t_5 t_6}{e^{t_1}}\right)^2 + 2t_5 t_6^2 x_2 e^{-x_1-2t_1} + t_6^2 x_2^2 e^{-2x_1-2t_1} - 2t_6 x_2 e^{-x_1-t_1} + \varepsilon^\iota t_6^2 e^{-2x_1-2t_1}} \\
&\equiv \frac{A_1^\iota}{B^\iota},
\end{aligned}
$$

$$
\begin{aligned}
\frac{\partial X_1^\iota}{\partial x_2} &= \frac{2t_5 t_6^2 e^{-x_1-2t_1} + 2t_6^2 x_2 e^{-2x_1-2t_1} - 2t_6 e^{-x_1-t_1}}{\left(1 - \frac{t_5 t_6}{e^{t_1}}\right)^2 + 2t_5 t_6^2 x_2 e^{-x_1-2t_1} + t_6^2 x_2^2 e^{-2x_1-2t_1} - 2t_6 x_2 e^{-x_1-t_1} + \varepsilon^\iota t_6^2 e^{-2x_1-2t_1}} \\
&\equiv \frac{A_2^\iota}{B^\iota},
\end{aligned}
$$

$$
\frac{\partial X_2^\iota}{\partial x_1} = -e^{x_1-t_1} t_5^2 t_6 + e^{x_1} t_5 + t_6 e^{-t_1-x_1} x_2^2 + \varepsilon^\iota t_6 e^{-t_1-x_1} \equiv A_3^\iota,
$$

$$
\frac{\partial X_2^\iota}{\partial x_2} = 1 - 2e^{-t_1} t_5 t_6 - 2t_6 e^{-t_1-x_1} x_2 \equiv A_4^\iota.
$$

Then

$$
\begin{aligned}
A_1^\iota A_4^\iota - A_2^\iota A_3^\iota &= \left(1 + t_5^2 t_6^2 e^{-2t_1} - 2t_5 t_6 e^{-t_1} - t_6^2 x_2^2 e^{-2x_1-2t_1} - \varepsilon^\iota t_6^2 e^{-2x_1-2t_1}\right) \\
&\quad \times \left(1 - 2e^{-t_1} t_5 t_6 - 2t_6 e^{-t_1-x_1} x_2\right) \\
&\quad - \left(2t_5 t_6^2 e^{-x_1-2t_1} + 2t_6^2 x_2 e^{-2x_1-2t_1} - 2t_6 e^{-x_1-t_1}\right) \\
&\quad \times \left(-e^{x_1-t_1} t_5^2 t_6 + e^{x_1} t_5 + t_6 e^{-t_1-x_1} x_2^2 + \varepsilon^\iota t_6 e^{-t_1-x_1}\right) \\
&= 1 + t_5^2 t_6^2 e^{-2t_1} - 2t_5 t_6 e^{-t_1} - t_6^2 x_2^2 e^{-2x_1-2t_1} - \varepsilon^\iota t_6^2 e^{-2x_1-2t_1} - 2e^{-t_1} t_5 t_6 \\
&\quad - 2t_5^3 t_6^3 e^{-3t_1} + 4t_5^2 t_6^2 e^{-2t_1} + 2t_5 t_6^3 x_2^2 e^{-2x_1-3t_1} + 2\varepsilon^\iota t_5 t_6^3 e^{-2x_1-3t_1} \\
&\quad - 2t_6 e^{-t_1-x_1} x_2 - 2t_5^2 t_6^3 x_2 e^{-3t_1-x_1} + 4t_5 t_6^2 x_2 e^{-2t_1-x_1} + 2t_6^3 x_2^3 e^{-3x_1-3t_1} \\
&\quad + 2\varepsilon^\iota t_6^3 x_2 e^{-3x_1-3t_1} + 2t_5^3 t_6^3 e^{-3t_1} - 2t_5^2 t_6^2 e^{-2t_1} - 2t_5 t_6^3 x_2^2 e^{-2x_1-3t_1} \\
&\quad - 2\varepsilon^\iota t_5 t_6^3 e^{-2x_1-3t_1} + 2t_5^2 t_6^3 x_2 e^{-x_1-3t_1} - 2t_5 t_6^2 x_2 e^{-x_1-2t_1} \\
&\quad - 2t_6^3 x_2^3 e^{-3x_1-3t_1} - 2\varepsilon^\iota t_6^3 x_2 e^{-3x_1-3t_1} - 2t_5^2 t_6^2 e^{-2t_1} + 2t_5 t_6 e^{-t_1} \\
&\quad + 2t_6^2 x_2^2 e^{-2x_1-2t_1} + 2\varepsilon^\iota t_6^2 e^{-2x_1-2t_1} \\
&= 1 + t_5^2 t_6^2 e^{-2t_1} - 2t_5 t_6 e^{-t_1} + t_6^2 x_2^2 e^{-2x_1-2t_1} + \varepsilon^\iota t_6^2 e^{-2x_1-2t_1} \\
&\quad - 2t_6 e^{-t_1-x_1} x_2 + 2t_5 t_6^2 x_2 e^{-x_1-2t_1} \\
&= B^\iota.
\end{aligned}
$$

Now it is easy to see

$$
\left| \frac{\partial(X_1^\iota, X_2^\iota)}{\partial(x_1, x_2)} \right| = \left| \frac{A_1^\iota}{B^\iota} A_4^\iota - \frac{A_2^\iota}{B^\iota} A_3^\iota \right| = |1| = 1. \qquad \square
$$

A.4 Relations in $\mathfrak{D}(\mathfrak{p}_4)$

Finally, we shall show that $\hat{s}_{23}, \hat{s}_{12}, \hat{s}_{13} \in \mathfrak{D}(\mathfrak{p}_4)$ defined by (4.10) - (4.12), satisfy the same relations as the respective elements of $\mathfrak{D}_{3,1;s}(\mathbb{R})$, i.e. (4.4) and (4.5).

Observe, first of all, that in $\mathfrak{D}(\mathfrak{p}_4)$ for $j = 2, 3$ we have

$$\begin{aligned}\left[(P_0 - P_1)^{-1}, L_{1j} + L_{0j}\right] &= (P_0 - P_1)^{-1}[L_{1j} + L_{0j}, P_0 - P_1](P_0 - P_1)^{-1} \\ &= -2P_j(P_0 - P_1)^{-2}\end{aligned}$$

and

$$[\hat{p}_j, \hat{q}_j^2] = [\hat{p}_j, \hat{q}_j]\hat{q}_j + \hat{q}_j[\hat{p}_j, \hat{q}_j] = 2\hat{q}_j.$$

Then,

Lemma A.5. *In $\mathfrak{D}(\mathfrak{p}_4)$, for any $j = 1, 2, 3$ and $1 \le k < l \le 3$ we have $[\hat{s}_{kl}, \hat{p}_j] = [\hat{s}_{kl}, \hat{q}_j] = 0$.*

Proof.
(a) First, for $\hat{s}_{23}$ we have

$$\begin{aligned}
[\hat{s}_{23}, \hat{q}_1] &= [L_{23}, P_0 - P_1] = 0, \\
[\hat{s}_{23}, \hat{p}_1] &= \left[L_{23}, (P_0 - P_1)^{-1}\left(L_{01} - \frac{1}{2}\right)\right] = 0, \\
[\hat{s}_{23}, \hat{q}_2] &= [L_{23}, P_2] - \hat{q}_3[\hat{p}_2, \hat{q}_2] = P_3 - \hat{q}_3 = 0, \\
[\hat{s}_{23}, \hat{p}_2] &= \left[L_{23}, (P_0 - P_1)^{-1}(L_{12} - L_{02})\right] + [\hat{q}_2, \hat{p}_2]\hat{p}_3 = (P_0 - P_1)^{-1}(L_{13} - L_{03}) - \hat{p}_3 \\
&= 0, \\
[\hat{s}_{23}, \hat{q}_3] &= [L_{23}, P_3] + \hat{q}_2[\hat{p}_3, \hat{q}_3] = -P_2 + \hat{q}_2 = 0, \\
[\hat{s}_{23}, \hat{p}_3] &= \left[L_{23}, (P_0 - P_1)^{-1}(L_{13} - L_{03})\right] - [\hat{q}_3, \hat{p}_3]\hat{p}_2 = -(P_0 - P_1)^{-1}(L_{12} - L_{02}) + \hat{p}_2 \\
&= 0.
\end{aligned}$$

(b) Second, with help of the already proven relations, for $\hat{s}_{12}$ we can write

$$\begin{aligned}
[\hat{s}_{12}, \hat{q}_1] &= \hat{q}_1[L_{12} + L_{02}, \hat{q}_1] + 2\hat{q}_1[\hat{p}_1, \hat{q}_1]\hat{q}_2 = \hat{q}_1[L_{12} + L_{02}, P_0 - P_1] + 2\hat{q}_1\hat{q}_2 \\
&= -2\hat{q}_1 P_2 + 2\hat{q}_1\hat{q}_2 = 0, \\
[\hat{s}_{12}, \hat{p}_1] &= [\hat{q}_1, \hat{p}_1](L_{12} + L_{02}) + \hat{q}_1[L_{12} + L_{02}, \hat{p}_1] + 2[\hat{q}_1, \hat{p}_1]\hat{p}_1\hat{q}_2 \\
&= -(L_{12} + L_{02}) + \hat{q}_1\left[L_{12} + L_{02}, (P_0 - P_1)^{-1}\right]\left(L_{01} - \frac{1}{2}\right) \\
&\quad + \hat{q}_1(P_0 - P_1)^{-1}[L_{12} + L_{02}, L_{01}] - 2\hat{q}_2\hat{p}_1 \\
&= -L_{12} - L_{02} + 2P_2(P_0 - P_1)^{-1}\left(L_{01} - \frac{1}{2}\right) + L_{12} + L_{02} - 2\hat{q}_2\hat{p}_1 \\
&= 0, \\
[\hat{s}_{12}, \hat{q}_2] &= \hat{q}_1[L_{12} + L_{02}, P_2] + (\hat{q}_2^2 + \hat{q}_3^2 - M^2)[\hat{p}_2, \hat{q}_2] = (P_0 - P_1)(-P_1 - P_0) + P_0^2 - P_1^2 \\
&= 0, \\
[\hat{s}_{12}, \hat{p}_2] &= \hat{q}_1[L_{12} + L_{02}, \hat{p}_2] + 2L_{01}[\hat{q}_2, \hat{p}_2] + [\hat{q}_2^2, \hat{p}_2]\hat{p}_2 \\
&= \hat{q}_1\left[L_{12} + L_{02}, (P_0 - P_1)^{-1}\right](L_{12} - L_{02}) + [L_{12} + L_{02}, L_{12} - L_{02}] \\
&\quad - 2L_{01} - 2\hat{q}_2\hat{p}_2 \\
&= 2P_2(P_0 - P_1)^{-1}(L_{12} - L_{02}) + 2L_{01} - 2L_{01} - 2\hat{q}_2\hat{p}_2 \\
&= 0,
\end{aligned}$$

$$
\begin{aligned}
[\hat{s}_{12}, \hat{q}_3] &= [L_{12} + L_{02}, P_3] = 0, \\
[\hat{s}_{12}, \hat{p}_3] &= \hat{q}_1 [L_{12} + L_{02}, \hat{p}_3] + [\hat{q}_3^2, \hat{p}_3]\hat{p}_2 + 2[\hat{q}_3, \hat{p}_3]\hat{s}_{23} \\
&= \hat{q}_1 \left[L_{12} + L_{02}, (P_0 - P_1)^{-1}\right] (L_{13} - L_{03}) + [L_{12} + L_{02}, L_{13} - L_{03}] - 2\hat{q}_3\hat{p}_2 - 2\hat{s}_{23} \\
&= 2P_2(P_0 - P_1)^{-1}(L_{13} - L_{03}) + 2L_{23} - 2\hat{q}_3\hat{p}_2 - 2\hat{s}_{23} \\
&= 0.
\end{aligned}
$$

(c) Finally, to show $\hat{s}_{13}$ commute with all $\hat{p}_j, \hat{q}_j$, $j = 1, 2, 3$, it is sufficient to realize that $\hat{s}_{13}$ could be obtained from $\hat{s}_{12}$ only by intertwining indices $2 \leftrightarrow 3$, with $\hat{s}_{23} := -\hat{s}_{32}$. Then the same modification of commutation relations for $\hat{s}_{12}$ leads to the desired result.

□

Lemma A.6. *In $\mathfrak{D}(\mathfrak{p}_4)$, the elements $\hat{s}_{23}$, $\hat{s}_{12}$ and $\hat{s}_{13}$ satisfy*

$$
[\hat{s}_{23}, \hat{s}_{12}] = \hat{s}_{13}, \quad [\hat{s}_{13}, \hat{s}_{23}] = \hat{s}_{12}, \quad [\hat{s}_{12}, \hat{s}_{13}] = -4M^2\hat{s}_{23}, \tag{A.16}
$$

$$
\hat{s}_{23}^* = -\hat{s}_{23}, \quad \hat{s}_{12}^* = \hat{s}_{12}, \quad \hat{s}_{13}^* = \hat{s}_{13}. \tag{A.17}
$$

Proof.
(a) Let us begin with the involution property. First,

$$
\hat{s}_{23}^* = L_{23}^* + \hat{p}_3^*\hat{q}_2^* - \hat{p}_2^*\hat{q}_3^* = -L_{23} - \hat{p}_3\hat{q}_2 - \hat{p}_2\hat{q}_3 = -\hat{s}_{23}.
$$

(b) Second,

$$
\begin{aligned}
\hat{s}_{12}^* &= (L_{12} + L_{02})\hat{q}_1 - 2\hat{q}_2 \left(\frac{1}{2} - \hat{p}_1\hat{q}_1\right) + \hat{p}_2(\hat{q}_2^2 + \hat{q}_3^2 - M^2) + 2\hat{s}_{23}\hat{q}_3 \\
&= [L_{12} + L_{02}, P_0 - P_1] + \hat{q}_1(L_{12} + L_{02}) + 2\left(\hat{q}_1\hat{p}_1 + \frac{1}{2}\right)\hat{q}_2 \\
&\qquad + [\hat{p}_2, \hat{q}_2^2] + (\hat{q}_2^2 + \hat{q}_3^2 - M^2)\hat{p}_2 + 2\hat{q}_3\hat{s}_{23} \\
&= -2P_2 + \hat{q}_1(L_{12} + L_{02}) + 2\left(\hat{q}_1\hat{p}_1 + \frac{1}{2}\right)\hat{q}_2 + 2\hat{q}_2 + (\hat{q}_2^2 + \hat{q}_3^2 - M^2)\hat{p}_2 + 2\hat{q}_3\hat{s}_{23} \\
&= \hat{q}_1(L_{12} + L_{02}) + 2\left(\hat{q}_1\hat{p}_1 + \frac{1}{2}\right)\hat{q}_2 + (\hat{q}_2^2 + \hat{q}_3^2 - M^2)\hat{p}_2 + 2\hat{q}_3\hat{s}_{23} \\
&= \hat{s}_{12}.
\end{aligned}
$$

(c) Third,

$$
\begin{aligned}
\hat{s}_{13}^* &= (L_{13} + L_{03})\hat{q}_1 - 2\hat{q}_3 \left(\frac{1}{2} - \hat{p}_1\hat{q}_1\right) + \hat{p}_3(\hat{q}_2^2 + \hat{q}_3^2 - M^2) - 2\hat{s}_{23}\hat{q}_2 \\
&= [L_{13} + L_{03}, P_0 - P_1] + \hat{q}_1(L_{13} + L_{03}) + 2\left(\hat{q}_1\hat{p}_1 + \frac{1}{2}\right)\hat{q}_3 \\
&\qquad + [\hat{p}_3, \hat{q}_3^2] + (\hat{q}_2^2 + \hat{q}_3^2 - M^2)\hat{p}_3 - 2\hat{q}_2\hat{s}_{23} \\
&= -2P_3 + \hat{q}_1(L_{13} + L_{03}) + 2\left(\hat{q}_1\hat{p}_1 + \frac{1}{2}\right)\hat{q}_3 + 2\hat{q}_3 + (\hat{q}_2^2 + \hat{q}_3^2 - M^2)\hat{p}_3 - 2\hat{q}_2\hat{s}_{23} \\
&= \hat{q}_1(L_{13} + L_{03}) + 2\left(\hat{q}_1\hat{p}_1 + \frac{1}{2}\right)\hat{q}_3 + (\hat{q}_2^2 + \hat{q}_3^2 - M^2)\hat{p}_3 - 2\hat{q}_2\hat{s}_{23} \\
&= \hat{s}_{13}.
\end{aligned}
$$

(d) As for the commutation relations, we make use of the previous lemma. First,

$$
\begin{aligned}
[\hat{s}_{23}, \hat{s}_{12}] &= [\hat{s}_{23}, \hat{q}_1(L_{12} + L_{02})] \\
&= \hat{q}_1[\hat{s}_{23}, L_{12} + L_{02}] \\
&= \hat{q}_1[L_{23}, L_{12} + L_{02}] + \hat{q}_1 \left[P_2(P_0 - P_1)^{-1}(L_{13} - L_{03}), L_{12} + L_{02}\right] \\
&\quad - \hat{q}_1 \left[P_3(P_0 - P_1)^{-1}(L_{12} - L_{02}), L_{12} + L_{02}\right] \\
&= \hat{q}_1[L_{23}, L_{12} + L_{02}] + \hat{q}_1 \left[P_2, L_{12} + L_{02}\right] (P_0 - P_1)^{-1}(L_{13} - L_{03}) \\
&\quad + \hat{q}_1 P_2 \left[(P_0 - P_1)^{-1}, L_{12} + L_{02}\right] (L_{13} - L_{03}) + P_2 \left[L_{13} - L_{03}, L_{12} + L_{02}\right] \\
&\quad - \hat{q}_1 P_3 \left[(P_0 - P_1)^{-1}, L_{12} + L_{02}\right] (L_{12} - L_{02}) - P_3 \left[L_{12} - L_{02}, L_{12} + L_{02}\right] \\
&= \hat{q}_1(L_{13} + L_{03}) + \hat{q}_1(P_0 + P_1)(P_0 - P_1)^{-1}(L_{13} - L_{03}) \\
&\quad - 2\hat{q}_1 P_2^2 (P_0 - P_1)^{-2}(L_{13} - L_{03}) - 2P_2 L_{23} \\
&\quad + 2\hat{q}_1 P_3 P_2 (P_0 - P_1)^{-2}(L_{12} - L_{02}) + 2L_{01}P_3 \\
&= \hat{q}_1(L_{13} + L_{03}) + (P_0^2 - P_1^2)\hat{p}_3 - 2\hat{q}_2^2\hat{p}_3 - 2\hat{q}_2 L_{23} + 2\hat{q}_2\hat{q}_3\hat{p}_2 + 2L_{01}\hat{q}_3 \\
&= \hat{s}_{13}.
\end{aligned}
$$

(e) Second, the proof of $[\hat{s}_{23}, \hat{s}_{13}] = -\hat{s}_{12}$ is essentially the same as in the previous case, up to interchange $2 \leftrightarrow 3$. Again, $\hat{s}_{32} \equiv -\hat{s}_{23}$ and $L_{32} \equiv -L_{23}$.

(f) Finally,

$$
\begin{aligned}
[\hat{s}_{13}, \hat{s}_{12}] &= \hat{q}_1[\hat{s}_{13}, L_{12} + L_{02}] + 2\hat{q}_3[\hat{s}_{13}, \hat{s}_{23}] \\
&= \hat{q}_1[(P_0 - P_1)(L_{13} + L_{03}), L_{12} + L_{02}] + 2\hat{q}_1[L_{01}P_3, L_{12} + L_{02}] \\
&\quad + \hat{q}_1\left[(P_0 + P_1)(L_{13} - L_{03}), L_{12} + L_{02}\right] - 2\hat{q}_1[\hat{q}_2\hat{s}_{23}, L_{12} + L_{02}] + 2\hat{q}_3\hat{s}_{12} \\
&= \hat{q}_1[P_0 - P_1, L_{12} + L_{02}](L_{13} + L_{03}) + \hat{q}_1(P_0 - P_1)[L_{13} + L_{03}, L_{12} + L_{02}] \\
&\quad + 2\hat{q}_1[L_{01}, L_{12} + L_{02}]P_3 + \hat{q}_1(P_0 + P_1)[L_{13} - L_{03}, L_{12} + L_{02}] \\
&\quad + \hat{q}_1\left[P_0 + P_1, L_{12} + L_{02}\right](L_{13} - L_{03}) - \hat{q}_1\hat{q}_2[\hat{s}_{23}, L_{12} + L_{02}] \\
&\quad - 2\hat{q}_1[P_2, L_{12} + L_{02}]\hat{s}_{23} + 2\hat{q}_3\hat{s}_{12} \\
&= 2\hat{q}_1\hat{q}_2(L_{13} + L_{03}) - 2\hat{q}_1\hat{q}_3(L_{12} + L_{02}) - 2\hat{q}_1(P_0 + P_1)L_{23} \\
&\quad - 2\hat{q}_2\hat{s}_{13} - 2\hat{q}_1(P_0 + P_1)\hat{s}_{23} + 2\hat{q}_3\hat{s}_{12} \\
&= -2(P_0^2 - P_1^2)(L_{23} + \hat{s}_{23}) + 4\hat{q}_2\hat{q}_3L_{01} + 2(\hat{q}_2^2 + \hat{q}_3^2 - M^2)\hat{q}_3\hat{p}_2 + 4\hat{q}_3^2\hat{s}_{23} \\
&\quad - 4L_{01}\hat{q}_2\hat{q}_3 - 2(\hat{q}_2^2 + \hat{q}_3^2 - M^2)\hat{q}_2\hat{p}_3 + 4\hat{q}_2^2\hat{s}_{23} \\
&= -2(P_0^2 - P_1^2)(L_{23} + \hat{s}_{23} - \hat{q}_3\hat{p}_2 + \hat{q}_2\hat{p}_3) + 4P_3^2\hat{s}_{23} + 4P_2^2\hat{s}_{23} \\
&= 4M^2\hat{s}_{23}.
\end{aligned}
$$

□

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