

Domáci úkol č. 3

Zadáno: 21. 10.

Deadline: 4. 11.

1. Najděte limitu

$$\lim_{x \rightarrow 0} \frac{\cotg(a + 2x) - 2\cotg(a + x) + \cotg a}{x^2},$$

kde $a \in \mathbb{R}$, $\sin a \neq 0$.

2. Najděte limitu

$$\lim_{x \rightarrow \frac{\pi}{4}} (\operatorname{tg} x)^{\operatorname{tg} 2x}.$$

Řešení

1.

$$\begin{aligned} & \lim_{x \rightarrow 0} \frac{\cotg(a + 2x) - 2\cotg(a + x) + \cotg a}{x^2} = \\ &= \lim_{x \rightarrow 0} \frac{\cotg(a + 2x) - \cotg(a + x) + \cotg a - \cotg(a + x)}{x^2} = \\ &= \text{použijeme} \quad \cotg A - \cotg B = \frac{\sin(B - A)}{\sin B \sin A} \\ &= \lim_{x \rightarrow 0} \frac{1}{x^2} \left(\frac{\sin(-x)}{\sin(a + 2x) \sin(a + x)} + \frac{\sin(x)}{\sin(a) \sin(a + x)} \right) = \\ &= \lim_{x \rightarrow 0} \frac{1 - \sin(x) \sin(a) + \sin(x) \sin(a + 2x)}{x^2 \sin(a + 2x) \sin(a + x) \sin(a)} = \\ &= \lim_{x \rightarrow 0} \frac{\sin(x)}{x} \lim_{x \rightarrow 0} \frac{1}{\sin(a + 2x) \sin(a + x) \sin(a)} \lim_{x \rightarrow 0} \frac{\sin(a + 2x) - \sin(a)}{x} = \\ &= \frac{1}{\sin^3(a)} \lim_{x \rightarrow 0} \frac{\sin(a + 2x) - \sin(a)}{x} = \\ &= \text{použijeme} \quad \sin A - \sin B = 2 \cos \frac{A + B}{2} \sin \frac{A - B}{2} \\ &= \frac{1}{\sin^3(a)} \lim_{x \rightarrow 0} \frac{2 \cos(a + x) \sin(x)}{x} = \frac{2 \cos(a)}{\sin^3(a)} \end{aligned}$$

2.

$$\begin{aligned} \lim_{x \rightarrow \frac{\pi}{4}} (\operatorname{tg} x)^{\operatorname{tg} 2x} &= \\ &= \lim_{x \rightarrow \frac{\pi}{4}} \exp(\operatorname{tg} 2x \ln \operatorname{tg} x) = \\ &= \exp \lim_{x \rightarrow \frac{\pi}{4}} (\operatorname{tg} 2x \ln \operatorname{tg} x) = \\ &= \exp \lim_{x \rightarrow \frac{\pi}{4}} \left(\operatorname{tg} 2x \frac{\ln \operatorname{tg} x}{\operatorname{tg} x - 1} (\operatorname{tg} x - 1) \right) = \\ &= \exp \lim_{x \rightarrow \frac{\pi}{4}} \frac{\ln \operatorname{tg} x}{\operatorname{tg} x - 1} \lim_{x \rightarrow \frac{\pi}{4}} (\operatorname{tg} 2x (\operatorname{tg} x - 1)) = \\ &= \exp \lim_{x \rightarrow \frac{\pi}{4}} \left(\frac{2 \sin x \cos x}{\cos^2 x - \sin^2 x} \frac{\sin x - \cos x}{\cos x} \right) = \\ &= \exp \lim_{x \rightarrow \frac{\pi}{4}} \left(\frac{-2 \sin x}{\cos x + \sin x} \right) = \exp(-1) = \frac{1}{e} \end{aligned}$$