Convex Optimization 2025/26

Practical session # 7

November 13, 2025

1. Consider the optimization problem

$$\begin{aligned} &\text{minimize } x^2 + 1 \\ &\text{subject to } (x-2)(x-4) \leq 0. \end{aligned}$$

   - Find the optimal solution x^* and optimal value p^*
   - Determine the Lagrangian $L(x, \lambda)$ and the Lagrange dual function $g(\lambda)$
   - Solve the dual problem

$$\begin{aligned} &\text{maximize } g(\lambda) \\ &\text{subject to } \lambda \geq 0 \end{aligned}$$

     Compare its optimal value d^* with p^*. What do you observe?

2. Take the vector-optimization problem

$$\text{minimize } (x^2, y^2 + 1)$$

   - Determine its (Pareto) optimal solutions. Discuss the scalarizations: In particular, what happens if we allow parameters $\lambda \succeq 0$ instead of just $\lambda \succ 0$?
   - Conversely, find a (convex) vector-optimization problem there exists a Pareto-optimal solution that is not a solution of any scalarization.

3. Consider the problem

$$\begin{aligned} &\text{minimize } \|x\|_2 \\ &\text{subject to } Ax = b, \end{aligned}$$

   for $A \in \mathbf{R}^{m \times n}, b \in \mathbf{R}^m$, and determine its dual function.
   (Hint: $\inf_x(\|x\|_2 - c^T x) = 0$ if $\|c\|_2 \leq 1$, and $-\infty$ else).
   Compare the result with the dual we obtained for the objective function $\|x\|_2^2 = x^T x$ (last lecture).

4. For any function $f\colon \mathbf{R}^n \to \mathbf{R}$, the function $f^*\colon \mathbf{R}^n \to \mathbf{R} \cup \{\infty\}$ defined by $f^*(y) = \sup_{x \in \mathrm{dom}(f)}(y^T x - f(x))$ is called the *conjugate* of f. Note that f^* is convex.

   - Determine the conjugate for $f(x) = a^T x + b$, $f(x) = -\log(x)$ and $f(x) = e^x$.
   - Describe the dual problem of

$$\begin{aligned} &\text{minimize } c^T x \\ &\text{subject to } f(x) \leq 0, \end{aligned}$$

     in terms of f^*.