Convex Optimization 2025/26

Homework # 3
November 13, 2025

Instructions

- Please, submit your homeworks to kompatscher@karlin.mff.cuni.cz. The subject of your email should start with `[Convex Optimization]`.
- The written solutions are expected to be submitted in a single `.pdf` file and include your name. The use of LaTeX is encouraged - if you use handwriting, make sure it's legible. Additionally attach any Python code you used.
- There will be 4 homework assignments, on each of which you need to score at least 26 out of 40 points to obtain the credit (zápočet) for the course.
- Please, send your submissions no later than November 27, 15:40.

Exercise 1 (4 points) Find two symmetric matrices $A, B \in \mathbf{R}^{2\times 2}$ that are incomparable with respect to $\preceq$, i.e. neither $A \preceq B$ nor $B \preceq A$ holds.

Exercise 2 (10 points) Consider the problem

$$\begin{aligned} &\text{minimize } e^{-x} \\ &\text{subject to } x^2/y \leq 0, \end{aligned}$$

with domain $\mathcal{D} = \{(x, y) \mid y > 0\}$.

What is its optimal value p^*? Determine also its dual problem and its optimal value d^*. Why does strong duality not hold?

Exercise 3 (13 points) Recall that the Max-Cut problem of a graph $G = (V, E)$ with vertices $V = \{1, 2, \dots, n\}$ can be phrased as the problem

$$\text{maximize } \sum_{(i,j)\in E} \frac{1}{4}(1 - v_i v_j) \text{ subject to } v_i^2 = 1 \text{ for } i = 1, \dots, n \tag{1}$$

Let A be the adjacency matrix A of the graph G (so $A_{ij} = 1$ if $(i, j) \in E$ and 0 else). Show that the SDP relaxation of (1) can be computed via

$$\begin{aligned} &\text{maximize } \frac{1}{4}(tr(A^T A) - tr(AX)) \\ &\text{subject to } X_{ii} = 1 \text{ for all } i = 1, \dots, n \\ &X \succeq 0. \end{aligned}$$

Let us consider the graph on 28 vertices, given by the adjacency matrix A you can find on https://tinyurl.com/ypy4jrw8.

Using CVXPY, solve this SDP. Based on the result, give an upper bound on the maximum number of cuts of A. (An example on how to solve SDPs in CVXPY can e.g. be found in the documentation https://www.cvxpy.org/examples/basic/sdp.html).

Exercise 4 (13 points) Note that (1) is also equivalent to the problem

$$\text{minimize } v^T A v \text{ subject to } v_i^2 = 1, \text{ for } i = 1, \ldots, n, \tag{2}$$

where A is the adjacency matrix. Show that the dual of (2) can be expressed as the following SDP:

$$\begin{aligned} &\text{maximize } -\sum_{i=1}^{n} \nu_i \\ &\text{subject to } A + \text{diag}(\nu_1, \nu_2, \ldots, \nu_\text{n}) \succeq 0, \end{aligned}$$

Using CVXPY, solve also this dual for the adjacency matrix from https://tinyurl.com/ypy4jrw8.

Use the result to get another bound to the original Max-Cut problem (1). How does it compare with the bound from Exercise 2?