Convex Optimization

Practical session # 7

November 13, 2024

Exercise 1

Consider the problem of minimizing a nonconvex quadratic function over the unit ball

$$\begin{array}{ll} \text{minimize} & x^2 - 2y^2 \\ \text{subject to} & x^2 + y^2 \leq 1, \end{array}$$

Give the dual function and solve the primal and dual problems. Does strong duality hold?

Exercise 2

Consider the optimization problem

$$\begin{array}{ll} \text{minimize} & e^{-x} \\ \text{subject to} & x^2/y \leq 0 \end{array}$$

with variables x and y, and domain $\{(x, y) \mid y > 0\}$.

What is the optimal value $p^\star(u)$ of the perturbed problem

$$\begin{array}{ll} \text{minimize} & e^{-x} \\ \text{subject to} & x^2/y \leq u \end{array}$$

as a function of u? Verify that the global sensitivity inequality does not hold:

$$p^\star(u) \geq p^\star(0) - \lambda^\star u$$

Exercise 3

For each of the following optimization problems, draw a sketch of the sets

$$\mathcal{G} = \{(u, t) \mid \exists x \in \mathbb{R},\ f_0(x) = t,\ f_1(x) = u\},$$
$$\mathcal{A} = \{(u, t) \mid \exists x \in \mathbb{R},\ f_0(x) \leq t,\ f_1(x) \leq u\},$$

give the dual problem, and solve the primal and dual problems. Is the problem convex? Is Slater's condition satisfied? Does strong duality hold?

(a) Minimize x subject to $x^2 \leq 1$.

(b) Minimize x subject to $x^2 \leq 0$.

(c) Minimize x subject to $|x| \leq 0$.

(d) Minimize x subject to $f_1(x) \leq 0$, where

$$f_1(x) = \begin{cases} -x + 2 & x \geq 1, \\ x & -1 \leq x \leq 1 \\ -x - 2 & x \leq -1. \end{cases}$$

(e) Minimize x^3 subject to $-x + 1 \leq 0$.

Exercise 4

Show that the weak max-min inequality

$$\sup_{z \in Z} \inf_{w \in W} f(w, z) \leq \inf_{w \in W} \sup_{z \in Z} f(w, z)$$

always holds, with no assumptions on $f \colon \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}$, $W \subseteq \mathbb{R}^n$, or $Z \subseteq \mathbb{R}^m$.