Convex Optimization

Practical session # 6

November 6, 2024

Exercise 1

Consider the optimization problem

$$\begin{array}{ll} \text{minimize} & x^2+1 \\ \text{subject to} & (x-2)(x-4) \leq 0, \end{array}$$

with variable $x \in \mathbb{R}$.

1. Give the feasible set, the optimal value, and the optimal solution.
2. Plot the objective x^2+1 versus x. On the same plot, show the feasible set, optimal point and value, and plot the Lagrangian $L(x,\lambda)$ versus x for a few positive values of λ. Verify the lower bound property

$$p^\star \geq \inf_x L(x,\lambda), \quad \text{for } \lambda \geq 0$$

   Derive and sketch the Lagrange dual function $g(\lambda)$.
3. State the dual problem, and verify that it is a concave maximization problem. Find the dual optimal value and dual optimal solution $\lambda^\star$. Does strong duality hold?
4. Let $p^\star(u)$ denote the optimal value of the problem

$$\begin{array}{ll} \text{minimize} & x^2+1 \\ \text{subject to} & (x-2)(x-4) \leq u, \end{array}$$

   as a function of the parameter u. Plot $p^\star(u)$. Verify that $(p^\star)'(0) = -\lambda^\star$.

Exercise 2

Consider the inequality form LP

$$\begin{array}{ll} \text{minimize} & c^T x \\ \text{subject to} & Ax \preceq b, \end{array}$$

and its dual

$$\begin{array}{ll} \text{maximize} & -b^T\lambda \\ \text{subject to} & A^T\lambda + c = 0, \\ & \lambda \succeq 0, \end{array}$$

with $A \in \mathbb{R}^{m\times n}$ and $b \in \mathbb{R}^m$. If x is feasible for the LP, i.e., satisfies $Ax \preceq b$, then it also satisfies the inequality

$$w^T Ax \leq w^T b$$

for all $w \in \mathbb{R}^m_+$. Geometrically, for every $w \succeq 0$, the halfspace $H_w = \{x \mid w^T Ax \leq w^T b\}$ contains the feasible set for the LP. So, if we minimize the objective $c^T x$ over H_w, we get a lower bound on $p^\star$.

1. Derive an expression for the minimum value of $c^T x$ over the halfspace H_w (which will depend on the choice of $w \succeq 0$).

2. Formulate the problem of finding the best such bound, by maximizing the lower bound over $w \succeq 0$.

3. Relate the results of 1. and 2. to the Lagrange dual problem.

Exercise 3

Recall from that the conjugate f^* of a function $f\colon \mathbb{R}^n \to \mathbb{R}$ is given by

$$f^*(y) = \sup_{x \in \mathrm{dom}\, f} (y^T x - f(x))$$

1. Consider an optimization problem with linear constraints

$$\begin{aligned} \text{minimize} \quad & f_0(x) \\ \text{subject to} \quad & Ax \preceq b \\ & Cx = d \end{aligned} \tag{1}$$

Show that $g(\lambda, \nu) = -b^T\lambda - d^T\nu - f_0^*(-A^T\lambda - C^T\nu)$.

2. Consider the problem with variable $X \in \mathbb{S}^n_{++}$ that determines the minimum volume ellipsoid, centered at the origin, that includes the points $a_1, \dots, a_m \in \mathbb{R}^n$.

$$\begin{aligned} \text{minimize} \quad & f_0(X) = \log \det X^{-1} \\ \text{subject to} \quad & a_i^T X a_i \leq 1, \quad i = 1, \dots, m, \end{aligned} \tag{2}$$

where dom $f_0 = \mathbb{S}^n_{++}$. For $i \in \{1, \dots, m\}$, fin a matrix A_i such that $\mathrm{tr}(A_i X) = a_i^T X a_i$, so that the constraints (2) can be rewritten in the form of constraints (1) from Ex 3.1.

3.* Show that $f_0^*(Y) = \log\det(-Y)^{-1} - n$.

4. Apply this result in order to find the dual function $g(\lambda)$ for the problem from Ex. 3.2.

Exercise 4

For the minimum volume ellipsoid problem from Ex. 3.2, assume that the vectors $a_1, \dots, a_m$ span $\mathbb{R}^n$. This implies that the problem is bounded below.

1. Show that the matrix

$$X_{\text{sim}} = \left(\sum_{k=1}^m a_k a_k^T \right)^{-1}$$

is a feasible point of the problem. *Hint:* Show that

$$\begin{pmatrix} \sum_{k=1}^m a_k a_k^T & a_i \\ a_i & 1 \end{pmatrix} \succeq 0,$$

and use Schur complements to prove that $a_i^T X a_i \leq 1$ for $i = 1, \dots, m$.

2. Now we establish a bound on how suboptimal the feasible point X_{sim} is, via the dual problem

$$\begin{aligned} \text{maximize} \quad & g(\lambda) \\ \text{subject to} \quad & \lambda \succeq 0, \end{aligned}$$

with the implicit constraint $\sum_{i=1}^m \lambda_i a_i a_i^T \succ 0$. Consider λ's of the form $(t,\ t, \dots, t)$. Find (analytically) the optimal value of t, and evaluate the dual objective at this $\lambda(t)$. Use this to prove that the volume of the ellipsoid $\{u \mid u^T X_{\text{sim}} u \leq 1\}$ is no more than a factor $(m/n)^{n/2}$ more than the volume of the minimum volume ellipsoid. The volume, for a matrix X, is equal to $\exp\bigl(\frac{1}{2}\log\det(X)\bigr)$.