Convex Optimization

Practical session # 4

October 23, 2024

Exercise 1 (Ex. 2.3 of the last week)

Is the function $f(x_1, \dots, x_n) = \sqrt[p]{x_1^p + \cdots + x_n^p}$ convex, concave or neither on $\mathrm{dom}(f) = \mathbb{R}^n_{++}$? Here, $p > 0$ and $\mathbb{R}_{++} = \{x \in \mathbb{R} \mid x > 0\}$.

Exercise 2

For each of the following functions, determine whether it is quasiconvex or quasiconcave.

1. $\mathrm{card}\colon \mathbb{R}^n_+ \to \mathbb{R}$, where, for a vector $\mathbf{x} \in \mathbb{R}^n$, $\mathrm{card}(\mathbf{x})$ is the number of nonzero components of $\mathbf{x}$.
2. $g\colon \mathbb{R}^{n+1} \to \mathbb{R}$ such that $g(\mathbf{x}, t) = f(\mathbf{x}/t)$ with $\mathrm{dom}(g) = \{(\mathbf{x}, t) \mid \mathbf{x}/t \in \mathrm{dom}(f) \;\&\; t > 0\}$, where $f\colon \mathbb{R}^n \to \mathbb{R}$ is a convex function.
3. $\mathrm{rank}\colon \mathbb{S}^n_+ \to \mathbb{R}$ which assigns to a symmetric positive semidefinite matrix its rank.

Exercise 3

Show that every LP is equivalent to an LP of the form

$$\begin{aligned} &\text{minimize } \mathbf{c}^T\mathbf{y},\\ &\text{subject to } A\mathbf{y} = \mathbf{b},\\ &\qquad\qquad \mathbf{y} \succeq 0. \end{aligned}$$

(Hint: try to prove it for a concrete example. Find suitable substitutions for the variables, such that in the resulting equivalent problem $\mathbf{y} \succeq 0$).

Exercise 4

Find an analytic solution for the quadratically constraint quadratic program (QCQP)

$$\begin{aligned} &\text{minimize } 4x + 5y + 3z,\\ &\text{subject to } x^2 + 2y^2 + z^2 \leq 4. \end{aligned}$$

Can you similarly find an analytic solution for minimizing an affine function $\mathbf{c}^T\mathbf{x}$ over any ellipsoid given by $(\mathbf{x} - \mathbf{x}_c)^T A (\mathbf{x} - \mathbf{x}_c) \leq 1$? Here, $\mathbf{x}_c \in \mathbb{R}^n$ and $A \in \mathbb{S}^n_{++}$ is positive definite.

Exercise 5

Formulate the ℓ_4-norm approximation problem

$$\text{minimize } \|A\mathbf{x} - \mathbf{b}\|_4$$

as a QCQP. The matrix $A \in \mathbb{R}^{m\times n}$ and the vector $\mathbf{b} \in \mathbb{R}^m$ are fixed.