Convex Optimization

Practical session # 3

October 16, 2024

Exercise 1

1. Give an example of two closed convex sets that are disjoint but cannot be strictly separated.
2. Express the closed convex set $\{\mathbf{x} \in \mathbb{R}^2_+ \mid x_1 x_2 \geq 1\}$ as an intersection of halfspaces.
3. Let $C = \{\mathbf{x} \in \mathbb{R}^n \mid \|x\|_\infty \leq 1\}$, the ℓ_∞-norm unit ball in $\mathbb{R}^n$, and let $\hat{\mathbf{x}}$ be a point in the boundary of C. Identify the supporting hyperplanes of C at $\hat{\mathbf{x}}$ explicitly.

Exercise 2

Are the following functions convex, concave or neither? Here, $\mathbb{R}_{++} = \{x \in \mathbb{R} \mid x > 0\}$.

1. $f(x, y) = \frac{x^2}{y}$ on $\mathrm{dom}(f) = \mathbb{R} \times \mathbb{R}_{++}$
2. $f(x, y) = xy$ on $\mathrm{dom}(f) = \mathbb{R}^2$
3. $f(\mathbf{x}) = \sqrt[p]{x_1^p + \cdots + x_n^p}$ on $\mathrm{dom}(f) = \mathbb{R}^n_{++}$ and for $p > 0$
4. The negative entropy $f(\mathbf{x}) = \sum_{i=1}^n x_i \log(x_i)$ on $\mathrm{dom}(f) = \mathbb{R}^n_{++}$. This function appears in many optimization problems in physics, as thermodynamical systems tend to a state of maximal entropy.

Exercise 3

Let $A \in \mathbb{R}^n \times \mathbb{R}^n$ be a regular matrix, and $\mathbf{x}_c \in \mathbb{R}^n$. Show that the ellipsoid $E = \{\mathbf{x}_c + A\mathbf{u} \mid \|\mathbf{u}\| \leq 1\}$ is a convex set (for any norm $\|\cdot\|$).

Exercise 4

Show that, for any convex function $f \colon \mathbb{R}^n \to \mathbb{R}$, its *perspective function* $tf(\mathbf{x}/t)$ with domain $\{(\mathbf{x}, t) \mid \mathbf{x}/t \in \mathrm{dom}(f),\ t > 0\}$ is convex.

Exercise 5*

Let $f \colon \mathbb{R}^n \to \mathbb{R}$ be a convex function with $f(\mathbf{0}) \leq 0$.

1. Show that the perspective $tf(\mathbf{x}/t)$ is nonincreasing as a function of t.
2. Let g be concave and positive on its domain. Show that the function

$$h(\mathbf{x}) = g(\mathbf{x}) f\big(\mathbf{x}/g(\mathbf{x})\big), \qquad \mathrm{dom}(h) = \{\mathbf{x} \in \mathrm{dom}(g) \mid \mathbf{x}/g(\mathbf{x}) \in \mathrm{dom}(f)\}$$

   is convex.

3. As an example, show that

$$h(\mathbf{x}) = \frac{\mathbf{x}^T \mathbf{x}}{\left(\prod_{k=1}^n x_k\right)^{\frac{1}{n}}}, \qquad \mathrm{dom}(h) = \mathbb{R}^n_{++}$$

   is convex.