Convex Optimization

Practical session # 2

October 9, 2024

Exercise 1

Consider the optimization problem

$$\text{minimize } x$$
$$\text{subject to } y \leq 5 \text{ and } x \geq 2$$

Suppose that there are four candidates: $A = (2, 5)$, $B = (1, 4)$, $C = (2, 4)$, and $D = (3, 3)$. Which of the following are true?

1. Candidate A is better than B.
2. Candidate A is better than C.
3. Candidate D cannot be optimal.

Exercise 2

You are given some samples of input-output pairs of an unknown real-valued function $f(x)$. The values are $(-1, 1), (0, 2), (1, 4)$, and $(2, 1)$. Your goal is to find the best affine function $g(x) = ax + b$ approximating such $f(x)$ on the given inputs, where we measure the approximation error as the sum of squared errors between outputs of $g(x)$ and correct outputs of $f(x)$.

1. Show that this problem is readily cast as an optimization problem.
2. Solve this problem.
3. Find the matrix M and the vector $\bar{c}$ such that the optimization problem can be written as

   $$\text{minimize } \sum_i \left(m_i^T (a\ b)^T - c_i\right)^2,$$

   where m_i^T is the ith row of M.
4. Solve the system $(M^T M)(a\ b)^T = M^T \bar{c}$ and compare the two solutions.

5.* Show that this system is solvable no matter the rank of $M^T M$ and also try to find conditions on M implying $M^T M$ being invertible.

Exercise 3

Consider a village consisting of four houses located in points: $(-1, 1), (-1, -2), (1, 2), (2, -2)$. Villagers want to dig a well in an optimal place. A place is optimal if the ℓ_∞-distance from the place to the furthest house is minimal. Recall that $\|\bar{x} - \bar{y}\|_\infty = \max_i |x_i - y_i|$.

1. Show that this problem is readily cast as an optimization problem.
2. Solve this problem.
3. Find the matrix M and the vector $\bar{c}$ such that the problem can be written as

   $$\text{minimize } \max_i \left| m_i^T x - c_i \right|$$

   and reduce it to a linear optimization problem.

4. Solve the resulting linear optimization using geometry (draw half-planes).

Exercise 4

Determine if each set from items 1-5 is convex.

1. $\{(x, y) \in \mathbb{R}^2_{>0} \mid x/y \leq 1\}$
2. $\{(x, y) \in \mathbb{R}^2_{>0} \mid x/y \geq 1\}$
3. $\{(x, y) \in \mathbb{R}^2_{>0} \mid xy \leq 1\}$
4. $\{(x, y) \in \mathbb{R}^2_{>0} \mid xy \geq 1\}$
5. $C^\circ := \{\bar{y} \in \mathbb{R}^n \mid \bar{y}^T \bar{x} \leq 1 \text{ for all } \bar{x} \in C\}$, where $C \subseteq \mathbb{R}^n$ is a set. The set C° is called the *polar* of C.

6.* Show that $C \subseteq \mathbb{R}^n$ is convex if and only if

$$(\alpha + \beta)C = \alpha C + \beta C \text{ for all } \alpha, \beta \geq 0.$$

Here, $\alpha C = \{\alpha c \mid c \in C\}$ and $A + B = \{a + b \mid a \in A, b \in B\}$.

Exercise 5* Let **XOR** be the boolean binary function whose values are determined by the table

x	y	**XOR**(x, y)
0	0	0
0	1	1
1	0	1
1	1	0

Find the best affine approximation to **XOR**, where again we measure effectiveness of our approximation as a sum of squared errors.

This exact problem played a major role in the early history of modern AI. In particular, inability of a linear model to adequately approximate XOR was a major critique to a that-time widely used perceptron algorithm. The solution to the problem presented itself only later in the form of multiple layered neural network.