# Convex Optimization

Practical session # 13

January 8, 2025

**Exercise 1** Do you agree with the following statements?

(a) The $\ell_1$-norm of a vector can be expressed as

$$\|x\|_1 = \frac{1}{2} \inf_{y \succ 0} \left( \sum_{i=1}^{m} \frac{x_i^2}{y_i} + y_i \right)$$

(b) Therefore the $\|\cdot\|_1$-approximation problem (minimize $\sum_{i=1}^m |a_i^T x - b_i|$) is equivalent to the problem minimize $f(x,y) = \sum_{i=1}^m (a_i^T x - b_i)^2 / y_i + y_i$, with $\mathrm{dom}(f) = \{(x,y) \mid y \succ 0\}$.

(c) The objective function $f$ is twice differentiable and convex. Hence we can solve the $\|\cdot\|_1$-approximation problem by using Newton descent on $f$.

Recall other approaches we discussed to solve the $\|\cdot\|_1$-approximation problem.

**Exercise 2** Try to solve the LP

$$\begin{aligned} \text{minimize } & x_2 \\ \text{subject to } & x_1 \le x_2 \\ & 0 \le x_2, \end{aligned}$$

using the (log-)barrier method, i.e. try to compute the central path $x^*(t)$. What seems to be the problem?

**Exercise 3**

(a) Prove that for any general (strictly feasible, twice differentiable,...) convex problem

$$\begin{aligned} \text{minimize } & f_0(x) \\ \text{subject to } & f_i(x) \le 0 \text{ for } i = 1, \ldots, m \end{aligned}$$

we obtain a well-defined central path after adding the additional constraint $x^T x \le R$ for a big enough radius $R$. Hint: show that $tf_0(x) + \phi(x)$ is strongly convex, where $\phi(x)$ is the barrier function of the modified problem.

(b) Does this resolves the problem in Exercise 2?

**Exercise 4** We based the log barrier method on approximating the indicator function $I_-(u)$ by $-(1/t)\log(-u)$ for $t \to \infty$. We now discuss how this can also be achieved for different choices of barrier function.

For this let $h\colon \mathbf{R} \to \mathbf{R}$ twice differentiable, closed, increasing, convex, $\mathrm{dom}(h) = -\mathbf{R}_{++}$ (for example $h(u) = -1/u$). For a problem like in Exercise (3b) let us define the *h-barrier function*

$$\phi_h(x) = \sum_{i=1}^{m} h(f_i(x)).$$

(a) Explain why $tf_0(x) + \phi_h(x)$ is convex in $x$ for every $t > 0$.

(b) Show how to construct a feasible solution $\lambda$ from the central path $x^*(t)$, and discuss the corresponding duality gap (recall the proof for $h(u) = -\log(-u)$).

(c) For which functions $h$ does this gap only depend on $t$ and $m$, i.e. only the problem data?