Convex Optimization

Practical session # 12

December 18, 2024

Exercise 1

Consider the function $f(x,y) = x^2 + y^2$ with domain $\mathbf{dom}\, f = \{(x,y) \mid x > 1\}$.

(a) What is $p^\star$ – the optimal value for the problem of minimizing f?

(b) Draw the sublevel set $S = \{(x,y) \mid f(x,y) \le f(2,2)\}$. Is S closed? Is f strongly convex on S?

(c) What happens if we apply the gradient method with backtracking line search, starting at $(2,2)$? Does $f(x^{(k)}, y^{(k)})$ converge to $p^\star$?

Exercise 2

Let $\Delta x_{\mathrm{nsd}} = \mathrm{argmin}_v \{\nabla f(x)^T v \mid \|v\| = 1\}$ and $\Delta x_{\mathrm{sd}} = \|\nabla f(x)\|_* \Delta x_{\mathrm{nsd}}$ be the normalized and unnormalized steepest descent directions at x, for the norm $\|\cdot\|$. Prove the following identities. Here, $\|\cdot\|_*$ is the dual norm: $\|z\|_* = \sup\{z^T x \mid \|x\| \le 1\}$.

(a) $\nabla f(x)^T \Delta x_{\mathrm{nsd}} = -\|\nabla f(x)\|_*$

(b) $\nabla f(x)^T \Delta x_{\mathrm{sd}} = -\|\nabla f(x)\|_*^2$

(c) $\Delta x_{\mathrm{sd}} = \mathrm{argmin}_v \big(\nabla f(x)^T v + (1/2)\|v\|^2\big)$

Exercise 3

Show that the Newton decrement $\lambda(x)$ satisfies

$$\lambda(x) = \sup_{v^T \nabla^2 f(x) v = 1} \big(-v^T \nabla f(x)\big) = \sup_{v \ne 0} \frac{-v^T \nabla f(x)}{\big(v^T \nabla^2 f(x) v\big)^2}$$

Exercise 4

Newton's method with fixed step size $t = 1$ can diverge if the initial point is not close to $x^\star$. In this problem we consider two examples.

(a) $f(x) = \log(e^x + e^{-x})$ has a unique minimizer $x^\star = 0$. Run Newton's method with fixed step size $t = 1$, starting at $x^{(0)} = 1$ and $x^{(0)} = 1.1$.

(b) $f(x) = -\log x + x$ has a unique minimizer $x^\star = 1$. Run Newton's method with fixed step size $t = 1$, starting at $x^{(0)} = 3$.

Plot f and f', and show the first few iterates.

Exercise 5

(a) Show that $f(x) = 1/x$ with domain $(0, 8/9)$ is self-concordant.

(b) Show that the function

$$f(x) = \alpha \sum_{i=1}^{m} \frac{1}{b_i - a_i^T x}$$

with $\mathbf{dom}\, f = \{x \in \mathbb{R}^n \mid a_i^T x < b_i,\ \ i = 1, \ldots, m\}$ is self-concordant if $\mathbf{dom}\, f$ is bounded and

$$\alpha > \frac{9}{8} \max_{i=1,\ldots,m} \sup_{x \in \mathbf{dom}\, f} (b_i - a_i^T x).$$