Convex Optimization

Practical session # 11

December 11, 2024

Exercise 1

Show that if $C \subseteq \mathbb{R}^n$ is nonempty, closed and convex, and the norm $\|\cdot\|$ is strictly convex, then for every x_0 there is exactly one $x \in C$ closest to x_0. In other words the projection of x_0 on C is unique.

Exercise 2

Let C, D be convex sets.

1. Show that $\text{dist}(C, x + D)$ is a convex function of x.
2. Show that $\text{dist}(tC, x + tD)$ is a convex function of (x, t) for $t > 0$.

Exercise 3

A set $C \subseteq \mathbb{R}^n$ is called a *Chebyshev set* if for every $x_0 \in \mathbb{R}^n$, there is a unique point in C closest (in Euclidean norm) to x_0. From the result in exercise 1, every nonempty, closed, convex set is a Chebyshev set. In this problem we show the converse, which is known as *Motzkin's theorem.* Let $C \subseteq \mathbb{R}^n$ be a Chebyshev set.

(a) Show that C is nonempty and closed.
(b) Show that P_C, the Euclidean projection on C, is continuous.
(c) Suppose $x_0 \notin C$. Show that $P_C(x) = P_C(x_0)$ for all $x = \theta x_0 + (1-\theta)P_C(x_0)$ with $0 \leq \theta \leq 1$.
(d) Suppose $x_0 \notin C$. Show that $P_C(x) = P_C(x_0)$ for all $x = \theta x_0 + (1-\theta)P_C(x_0)$ with $\theta \geq 1$.
Hint: Use the fact (following from Brower's fixed point theorem) that, for every continuous $g\colon \mathbb{R}^n \to \mathbb{R}^n$ such that $g(x) \neq 0$ for all $\|x\|_2 = 1$, there exists an x with $\|x\|_2 = 1$ and $g(x)/\|g(x)\|_2 = x$.
(e) Combining parts (c) and (d), we can conclude that all points on the ray with base $P_C(x_0)$ and direction $x_0 - P_C(x_0)$ have projection $P_C(x_0)$. Show that this implies that C is convex.

Exercise 4

Let $C = \text{conv}\{0, e_1, \ldots, e_n\}$ be the simplex, i.e., for $i = 1, \ldots, n$, $(e_i)_i = 1$ and $(e_i)_j = 0$ for all $i \neq j$. We want to find the Löwner-John ellipsoid $\mathcal{E}_{\text{lj}}$ of C. By symmetry, the center of $\mathcal{E}_{\text{lj}}$ lies on the direction $\mathbf{1} = (1, \ldots, 1)^T$ and the intersection of $\mathcal{E}_{\text{lj}}$ with every hyperplane orthogonal to $\mathbf{1}$ is a ball. Therefore, we can describe $\mathcal{E}_{\text{lj}}$ by a quadratic inequality

$$(x - \alpha\mathbf{1})^T(I + \beta\mathbf{1}\mathbf{1}^T)(x - \alpha\mathbf{1}) \leq \gamma,$$

where $\alpha, \beta, \gamma \in \mathbb{R}$ are three parameters.

(a) Knowing that $0, e_1, \ldots, e_n$ must lie on the boundary of $\mathcal{E}_{\text{lj}}$, represent α and γ as functions of β.
(b) The volume of $\mathcal{E}_{\text{lj}}$ is proportional to $\gamma^n \det(I + \beta\mathbf{1}\mathbf{1})^{-1} = \frac{\gamma^n}{1+\beta n}$. Find the values for α, β, and γ, in which the derivative of the logarithm of the volume equals 0, and find the inequality describing $\mathcal{E}_{\text{lj}}$.
(c) The simplex C is defined as the intersection of halfspaces

$$\left(\bigcap_{i=1}^{n} \{x \mid x_i \geq 0\}\right) \cap \{x \mid \mathbf{1}^T x \leq 1\}$$

For each hyperplane, find a point which belongs both to it and to the boundary of the shrunk ellipsoid:

$$(x - \alpha\mathbf{1})^T(I + \beta\mathbf{1}\mathbf{1}^T)(x - \alpha\mathbf{1}) = \frac{\gamma}{n^2}$$

This implies that $\mathcal{E}_{\text{lj}}$ must be shrunk by a factor $1/n$ to fit inside the simplex.