# Convex Optimization

Homework # 5

**Instructions**

Please, submit your homeworks to alexey.barsukov@matfyz.cuni.cz not later than **January, 10,** 23:59 local time. The subject of your email should be of the following form: `Convex Optimization HW *#ofHW* *yourSurname* *yourName*`, for example: `Convex Optimization HW 5 Einstein Albert`

**Exercise 1 (25 points)**

Show that the following function is convex:

(a) $f(x,y) = x^{1/3}y^{2/3}$ on $\mathbb{R}^2_{++}$,

(b) $f(x,t) = \frac{|x_1|^3+\cdots+|x_n|^3}{t^2} = \frac{\|x\|_3^3}{t^2}$, on $\operatorname{dom} f = \{(x,t) \mid t > 0\}$,

(c) $f(x,t) = -(t^3 - \|x\|_3^3)^{1/3}$ on $\operatorname{dom} f = \{(x,t) \mid t > \|x\|_3\}$,

(d) $f(x,y) = -\sqrt{xy}$ on $\mathbb{R}^2_{++}$,

(e) $f(x,y) = -\sqrt{xy - z^2}$ on $\operatorname{dom} f = \{(x,y,z) \mid xy \geq z^2, \quad x,y > 0\}$.

**Exercise 2 (20 points)**

Suppose $A\colon \mathbb{R}^n \to \mathbb{S}^m$ is affine, i.e.,

$$A(x) = A_0 + x_1 A_1 + \cdots + x_n A_n$$

where $A_i \in \mathbb{S}^m$ are symmetric $m \times m$ matrices. Let $\lambda_1(x) \geq \lambda_2(x) \geq \cdots \geq \lambda_m(x)$ denote the eigenvalues of $A(x)$. Show how to pose the following problems as SDPs.

(a) Minimize the maximum eigenvalue $\lambda_1(x)$.

(b) Minimize the spread of eigenvalues $\lambda_1(x) - \lambda_m(x)$.

(c) Minimize the sum of the absolute values of the eigenvalues $|\lambda_1(x)| + \cdots + |\lambda_m(x)|$.
*Hint.* Express $A(x)$ as $A(x) = A_+ - A_-$, where $A_+ \succeq 0$ and $A_- \succeq 0$.

**Exercise 3 (20 points)**

Use `cvxpy` in order to solve the optimization problems (a), (b), and (c) from Exercise 2 for the following mapping $A(x)$:

$$A(x) = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} + x_1 \begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix} + x_2 \begin{pmatrix} 1 & 2 \\ 2 & 3 \end{pmatrix} + x_3 \begin{pmatrix} 3 & 1 \\ 1 & -1 \end{pmatrix}$$

**Exercise 4 (15 points)**

Prove (without using any linear programming code) that the optimal solution of the LP

$$\begin{array}{ll} \text{minimize} & 47x_1 + 93x_2 + 17x_3 - 93x_4 \\ \text{subject to} & \begin{pmatrix} -1 & -6 & 1 & 3 \\ -1 & -2 & 7 & 1 \\ 0 & 3 & -10 & -1 \\ -6 & -11 & -2 & 12 \\ 1 & 6 & -1 & -3 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} \preceq \begin{pmatrix} -3 \\ 5 \\ -8 \\ -7 \\ 4 \end{pmatrix} \end{array}$$

is unique and given by $x^* = (1,1,1,1)$.

**Exercise 5 (20 points)**

Let $C = \operatorname{conv}\{0, e_1, \ldots, e_n\}$ be the simplex, i.e., for $i = 1, \ldots, n$, $(e_i)_i = 1$ and $(e_i)_j = 0$ for all $i \neq j$. We want to find the Löwner-John ellipsoid $\mathcal{E}_{\text{lj}}$ of $C$. By symmetry, the center of $\mathcal{E}_{\text{lj}}$ lies on the direction

$\mathbf{1} = (1, \dots, 1)^T$ and the intersection of $\mathcal{E}_{\text{lj}}$ with every hyperplane orthogonal to $\mathbf{1}$ is a ball. Therefore, we can describe $\mathcal{E}_{\text{lj}}$ by a quadratic inequality

$$(x - \alpha\mathbf{1})^T(I + \beta\mathbf{1}\mathbf{1}^T)(x - \alpha\mathbf{1}) \leq \gamma,$$

where $\alpha, \beta, \gamma \in \mathbb{R}$ are three parameters.

(a) Knowing that $0, e_1, \dots, e_n$ must lie on the boundary of $\mathcal{E}_{\text{lj}}$, represent $\alpha$ and $\gamma$ as functions of $\beta$.

(b) The volume of $\mathcal{E}_{\text{lj}}$ is proportional to $\gamma^n \det(I + \beta\mathbf{1}\mathbf{1})^{-1} = \frac{\gamma^n}{1+\beta n}$. Find the values for $\alpha, \beta$, and $\gamma$, in which the derivative of the logarithm of the volume equals 0, and find the inequality describing $\mathcal{E}_{\text{lj}}$.

(c) The simplex $C$ is defined as the intersection of halfspaces

$$\left(\bigcap_{i=1}^{n} \{x \mid x_i \geq 0\}\right) \cap \{x \mid \mathbf{1}^T x \leq 1\}$$

For each hyperplane, find a point which belongs both to it and to the boundary of the shrunk ellipsoid:

$$(x - \alpha\mathbf{1})^T(I + \beta\mathbf{1}\mathbf{1}^T)(x - \alpha\mathbf{1}) = \frac{\gamma}{n^2}$$

This implies that $\mathcal{E}_{\text{lj}}$ must be shrunk by a factor $1/n$ to fit inside the simplex.