Convex Optimization

Homework # 4

Instructions

Please, submit your homeworks to `alexey.barsukov@matfyz.cuni.cz` not later than **December, 22,** 23:59 local time. The subject of your email should be of the following form: `Convex Optimization HW *#ofHW* *yourSurname* *yourName*`, for example: `Convex Optimization HW 4 Einstein Albert`

Exercise 1 (10 points) Let $p(x, y)$ be the joint probability function of a parameter vector x and some observation vector y. Consider the MAP (maximum a posteriori probability) estimation $\hat{x}_{\text{map}}$ of the model parameter that, for fixed y maximizes

$$p_{y|x}(x, y)p_x(x).$$

Argue that if the prior density $p_x(x)$ describes a uniform distribution on some set C, then MAP estimation is equivalent to ML estimation subject to $x \in C$.

Exercise 2 (20 points)

Derive a Lagrange dual for the problem

$$\begin{array}{ll} \text{minimize} & \sum_{i=1}^{m} \phi(r_i) \\ \text{subject to} & r = Ax - b \end{array}$$

for the following penalty functions $\phi\colon \mathbb{R} \to \mathbb{R}$. The variables are $x \in \mathbb{R}^n$, $r \in \mathbb{R}^m$.

(a) *Huber penalty* (with $M = 1$),

$$\phi(u) = \begin{cases} u^2, & |u| \le 1 \\ 2|u| - 1, & |u| > 1 \end{cases}$$

(b) *Log-barrier* (with limit $a = 1$),

$$\phi(u) = -\log(1 - u^2), \qquad \operatorname{dom}\phi = (-1, 1).$$

Exercise 3 (20 points)

Consider the problem of minimizing a quadratic function:

$$\text{minimize} \qquad f(x) = (1/2)x^T P x + q^T x + r.$$

Here P, q, r are fixed and $P \in \mathbb{S}^n$ – symmetric. Notice we do not assume $P \succeq 0$.

(a) Suppose $P \not\succeq 0$, i.e. $f(x)$ is not convex. Show that the problem is unbounded from below.

(b) Suppose $P \succeq 0$, but the optimality condition $Px^* = -q$ does not have a solution. Show that the problem is unbounded below.

Exercise 4 (25 points)

Suppose $y \in \{0, 1\}$ is a random variable given by

$$y = \begin{cases} 1, & a^T u + b + v \le 0 \\ 0, & a^T u + b + v > 0 \end{cases}$$

where $u \in \mathbb{R}^n$ is a vector of explanatory variables, and v is a zero mean unit variance Gaussian variable. Formulate the ML estimation problem of estimating a and b, given data consisting of pairs (u_i, y_i), $i = 1, \ldots, n$, as a convex optimization problem.

Exercise 5 (25 points)

Suppose you would like to invest into some stocks A and B. Suppose that you are given two random variables X and Y which give the return on two investments (we quantize possible returns). In our case, both X and Y take value in $\{-1, -0.5, 0, 0.5, 1\}$. The joint distribution of X and Y is not known, but we are given their marginal distributions. The one corresponding to X is $(0.05, 0.25, 0.25, 0.25, 0.2)$ and the one corresponding to Y is $(0.2, 0.25, 0.25, 0.25, 0.05)$, i.e. the probability of X being -1 is 0.05, being -0.5 is 0.25, and so on. What is the expected return rate of each of the investments (denote the larger value among the two as α)? Now suppose you divided your budget into two equal parts and invested equally into X and Y. You would like to estimate the probability of getting return smaller than α. Formulate the problem of computing the worst case probability of the overall return less or equal to α as an LP. What if X and Y are independent? Solve the above LP using `cvxpy`.