Universal Algebra Exercises - Homework 1

Exercise 1.1. It is a well-known fact in logic, that every propositional formula $\phi(x_1, ..., x_n)$ using the symbols $\wedge, \vee, \neg, 0$ and 1 can be rewritten into a conjunctive normal form (CNF), i.e. a formula $C_1 \wedge C_2 \wedge \cdots \wedge C_k$, with $C_i = l_{i,1} \vee l_{i,2} \vee \cdots \vee l_{i,j_i}$, where every $l_{i,j}$ is either a variable or negation of a variable. Describe a term rewriting system E that rewrites every formula to a CNF. What is the variety $\mathrm{Mod}(E)$? You are allowed to be informal.

Exercise 1.2. Prove that a loop $(L, \cdot, /, \backslash, 1)$ is abelian if and only if $\cdot$ is a commutative group operation.

Exercise 1.3. Consider an algebra $\mathbb{A}$ and the congruence $\alpha := [1_{\mathbb{A}}, 1_{\mathbb{A}}]$.

- Show that α is the smallest congruence such that $\mathbb{A}/\alpha$ is abelian.
- Conclude that for any abelian algebra $\mathbb{B}$ (in the same signature as $\mathbb{A}$) there is a natural[1] bijection between the following sets of homomorphisms.

$$\mathrm{hom}(\mathbb{A}/\alpha, \mathbb{B}) \cong \mathrm{hom}(\mathbb{A}, \mathbb{B})$$

- (Bonus) Assume that $\mathbb{C}$ is an abelian algebra such that for any abelian algebra $\mathbb{B}$ there is a natural bijection $\mathrm{hom}(\mathbb{C}, \mathbb{B}) \cong \mathrm{hom}(\mathbb{A}, \mathbb{B})$. Show that $\mathbb{C} \cong \mathbb{A}/\alpha$.

[1] Ignore this word if you don't know what is means.