Clonoids and uniform generation by minors

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Outline

Clones and clonoids

Clones

$\mathcal{A} \subseteq \bigcup_{n\geq 1} A^{A^n}$ is a clone on A if

- all $\pi_i^n \in \mathcal{A}$ with $\pi_i^n(x_1, \ldots, x_n) = x_i$
- $f, g_1, \ldots, g_k \in \mathcal{A} \Rightarrow f \circ (g_1, \ldots, g_k) \in \mathcal{A}$ $\qquad (\mathcal{A} \circ \mathcal{A} \subseteq \mathcal{A})$

$\mathrm{Clo}(\mathbf{A}) :=$ term clone of algebra $\mathbf{A}$

Clonoids

For clones $\mathcal{A}, \mathcal{B}$ (on A, B), $\mathcal{C} \subseteq \bigcup_{n\geq 1} B^{A^n}$ is a $(\mathcal{A}, \mathcal{B})$-clonoid if

- $\mathcal{C} \circ \mathcal{A} \subseteq \mathcal{C}$,
- $\mathcal{B} \circ \mathcal{C} \subseteq \mathcal{C}$.

An $(\mathbf{A}, \mathbf{B})$-clonoid is a $(\mathrm{Clo}(\mathbf{A}), \mathrm{Clo}(\mathbf{B}))$-clonoid.

Goal: For given algebras $\mathbf{A}$, $\mathbf{B}$, describe the $(\mathbf{A}, \mathbf{B})$-clonoids.

Example: $\mathbf{A} = \mathbf{B} = (\mathbb{Z}_2, +, -, 0)$

$\mathcal{C} \subseteq \bigcup_{n\geq 1} \mathbb{Z}_2^{\mathbb{Z}_2^n}$ is $(\mathbf{A}, \mathbf{B})$-clonoid $\Leftrightarrow$ $\circ$-closed under linear functions

Represent $f\colon \mathbb{Z}_2^n \to \mathbb{Z}_2$ by reduced polynomials, e.g.

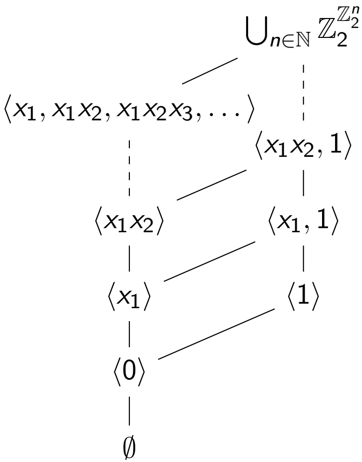
$$f(x, y, z, u) = 1 + xz + xyz.$$

$$\begin{aligned} \text{Then } f \in \mathcal{C} &\Rightarrow f(0,0,0,0) = 1 \in \mathcal{C}, \\ &\Rightarrow f(x,0,z,0) - 1 = xz \in \mathcal{C} \\ &\Rightarrow f(x,y,z,0) - 1 - xz = xyz \in \mathcal{C} \end{aligned}$$

Conversely $f(x, y, z, u) = 1 + xxz + xyz$.

In general: $f \in \mathcal{C} \Leftrightarrow f(0, \ldots, 0), x_1 x_2 \cdots x_{\deg(f)} \in \mathcal{C}$

Example: $\mathbf{A} = \mathbf{B} = (\mathbb{Z}_2, +, -, 0)$



$\langle F\rangle_{\mathbf{A},\mathbf{B}}$ = smallest $(\mathbf{A}, \mathbf{B})$-clonoid containing F.

Observations

- lattice ordered by $\subseteq$
- ω-many $(\mathbf{A}, \mathbf{B})$-clonoids
- all but top two clonoid have finite generating set F

[Kreinecker '19]

Complete classification for all $\mathbf{A} = \mathbf{B} = \mathbb{Z}_p$.

Goal revisited

Goal

For given finite algebras **A**, **B**:

- Describe the *lattice* of $(\mathbf{A}, \mathbf{B})$-clonoids.
- What is its cardinality?
- Find nice generating sets.

Observation

Finite lattice $\Leftrightarrow \exists k \in \mathbb{N}\colon \mathcal{C} = \langle \mathcal{C}^{(k)} \rangle$ for every $(\mathbf{A}, \mathbf{B})$-clonoid $\mathcal{C}$.

$\mathcal{C}^{(k)} = \mathcal{C} \cap B^{A^k}$

Some known results

Pippenger '02 : (A, B)-clonoid = minor closed set/minion.
Minions are equal to $\mathrm{Pol}(\mathbb{A}, \mathbb{B}) = \{h\colon \mathbb{A}^n \to \mathbb{B}, n \geq 1\}$ for relational structures $\mathbb{A}$,$\mathbb{B}$.

Couceiro, Foldes '09 : $(\mathcal{A}, \mathcal{B})$-clonoid = left/right stable under $\mathcal{A}/\mathcal{B}$
$(\mathcal{A}, \mathcal{B})$-clonoids = $\mathrm{Pol}(\mathbb{A}, \mathbb{B})$ for $\mathbb{A}$,$\mathbb{B}$ invariant under $\mathcal{A}$,$\mathcal{B}$.

Lehtonen, Szendrei '11 : $(\mathcal{A}, A)$-clonoids
study of clones $\mathcal{A}$ with finitely many $\mathcal{A}$-equivalence classes
$(f \equiv g \Leftrightarrow \langle f \rangle = f \circ \mathcal{A} = g \circ \mathcal{A} = \langle g \rangle)$

Aichinger, Mayr '16 : $(A, \mathbf{B})$-clonoid = **B**-clonoid

Sparks '19 : The number of $(A, \mathbf{B})$-clonoid is

1. finite if **B** has NU-term,
2. ω if **B** has few subpowers, no NU-term,
3. 2^ω else.

Lehtonen '25 : classification of $(\mathcal{A}, \mathcal{B})$-clonoid, for Boolean $\mathcal{A}, \mathcal{B}$

Erkko's results on Boolean clonoids

Cardinalities of (C_1, C_2)-clonoid lattices

	$[\mathsf{J}, \mathsf{I}]$	$[\mathsf{I}^*, \Omega(1)]$	$[\mathsf{V}_{01}, \mathsf{V}]$ $[\Lambda_{01}, \Lambda]$	$[\mathsf{MU}^\infty_{01}, \mathsf{MU}^\infty]$ $[\mathsf{MW}^\infty_{01}, \mathsf{MW}^\infty]$	U^∞_{01} W^∞_{01}	U^∞ W^∞	$[\mathsf{L}_{01}, \mathsf{L}]$	$[\{\mathsf{SM}, \mathsf{MU}^k_{01}, \mathsf{MW}^k_{01}\}, \Omega]$
J	U	U	U	U	U	U	C	F
$\mathsf{I}_0, \mathsf{I}_1$	U	U	U	U	U	U	C	F
I	U	U	U	F	F	F	C	F
I^*	U	U	U	U	U	U	C	F
$\Omega(1)$	U	U	U	F	F	F	C	F
$\mathsf{V}_{01}, \Lambda_{01}$	U	U	U	U	U	U	F	F
$\mathsf{V}_{0*}, \Lambda_{*1}$	U	U	U	F	F	F	F	F
$\mathsf{V}_{*1}, \Lambda_{0*}$	U	U	U	U	U	U	F	F
V, Λ	U	U	U	F	F	F	F	F
$\mathsf{MU}^k_{01}, \mathsf{MW}^k_{01}$	U	U	U	U	U	U	F	F
$\mathsf{MU}^k, \mathsf{MW}^k$	U	U	U	U	U	U	F	F
$\mathsf{U}^k_{01}, \mathsf{W}^k_{01}$	U	U	U	U	U	U	F	F
$\mathsf{U}^k, \mathsf{W}^k$	U	U	U	U	U	U	F	F
L_{01}	U	U	U	U	U	U	C	F
$\mathsf{L}_{0*}, \mathsf{L}_{*1}$	U	U	U	U	U	U	C	F
LS	U	U	U	U	U	U	C	F
L	U	U	U	F	F	F	C	F
SM	U	U	U	U	U	U	F	F
$[\mathsf{M}_{01}, \mathsf{M}]$	C	C	F	F	F	F	F	F
$[\mathsf{S}_{01}, \Omega]$	F	F	F	F	F	F	F	F

Clonoids between modules

R-modules are algebras $\mathbf{A} = (A, +, -, 0, (r)_{r \in \mathbf{R}})$, with $r(x) = r \cdot x$.

$$\mathrm{Clo}(\mathbf{A}) = \{\mathbf{x} \mapsto \mathbf{r}^T \mathbf{x} = \sum_{i=1}^{n} r_i \cdot x_i \text{ with } r_i \in \mathbf{R}.\}$$

Goal

Describe the $(\mathbf{A}, \mathbf{B})$-clonoids for finite modules $\mathbf{A}$, $\mathbf{B}$.

Motivation: 2-nilpotent algebras

- classification results [Aichinger, Mayr '07], [Fioravanti '21]
- equational bases [Mayr, K. '24]
- complexity of computational problems
  CEQV [Kawałek, K., Krzaczkowski '24], SMP [K. '24] [Patrick Wynne's talk]

Finitely many clonoids

Conjecture for $\mathbf{A}$, $\mathbf{B}$ finite modules

There are finitely many $(\mathbf{A}, \mathbf{B})$-clonoids $\Leftrightarrow$ $\gcd(|A|, |B|) = 1$.

"$\Rightarrow$" [Mayr, Wynne '24]

as for $\mathbf{A} = \mathbf{B} = \mathbb{Z}_p$

"$\Leftarrow$" True for:

- $\mathbf{A} = \mathbb{F}$ (1-dim. vector space) [Fioravanti '20]
- $\mathbf{A} = \mathbb{F}_1 \times \mathbb{F}_2 \times \cdots \times \mathbb{F}_m$ (as regular module) [Fioravanti '21]
- $\mathrm{Con}(\mathbf{A})$ is distributive [Mayr, Wynne '24]
  (k-generated, k = nilpotence-degree of Jacobson radical of $\mathbf{R}_\mathbf{A}$)
- $\mathbf{A} = \mathbb{F}^k$ (k-dim. vector space) [Fioravanti, MK, Rossi '25]
  (k-generated) (in preparation)

Outline

Generation from minors

Let **A**, **B** be modules, $f\colon A^k \to B$, $\mathcal{C} = \langle f \rangle$.
Question: When is $\mathcal{C} = \langle \mathcal{C}^{(n)} \rangle$?

$$\begin{aligned} \mathcal{C}^{(n)} &= \mathcal{B} \circ f \circ \mathcal{A}^{(n)} \\ &= \mathcal{B} \circ \{\mathbf{x} \mapsto f(U\mathbf{x}) : U \in \mathbf{R}_{\mathbf{A}}^{k\times n}\} \\ \langle \mathcal{C}^{(n)} \rangle &= \mathcal{B} \circ \{\mathbf{x} \mapsto f(M\mathbf{x}) : M \in \mathbf{R}_{\mathbf{A}}^{k\times m}, rk(M) \leq n\} \\ rk(M) \leq n &\Leftrightarrow M = UV, U \in \mathbf{R}_{\mathbf{A}}^{k\times n}, V \in \mathbf{R}_{\mathbf{A}}^{n\times m} \end{aligned}$$

So $\mathcal{C} = \langle \mathcal{C}^{(n)} \rangle \Leftrightarrow f(\mathbf{x}) = \sum_{rk(M)\leq n} r_M f(M\mathbf{x})$ for $r_M \in \mathbf{R_B}$

Generation from minors

Let $\mathcal{A}$, $\mathcal{B}$ be clones, $f\colon A^k \to B$, $\mathcal{C} = \langle f \rangle$.
Question: When is $\mathcal{C} = \langle \mathcal{C}^{(n)} \rangle$?

$$\begin{aligned}
\mathcal{C}^{(n)} &= \mathcal{B} \circ f \circ \mathcal{A}^{(n)} \\
&= \mathcal{B} \circ \{\mathbf{x} \mapsto f(U\mathbf{x}) : U \in \mathbf{R}_{\mathbf{A}}^{k \times n}\} \\
\langle \mathcal{C}^{(n)} \rangle &= \mathcal{B} \circ \{\mathbf{x} \mapsto f(r(\mathbf{x})) : rk_{\mathcal{A}}(r) \leq n\} \\
rk_{\mathcal{A}}(r) \leq n &\Leftrightarrow \exists u_i \in \mathcal{A}^{(n)}, v_j \in \mathcal{A} : r_i = u_i \circ (v_1, \ldots, v_n)
\end{aligned}$$

So $\mathcal{C} = \langle \mathcal{C}^{(n)} \rangle \Leftrightarrow f = t \circ (f \circ r_1, \ldots, f \circ r_s)$,
for $t \in \mathcal{B}, rk_{\mathcal{A}}(r_i) \leq n$

Uniform generation from minors

Definition

For clones $\mathcal{A}$,$\mathcal{B}$, $U \subseteq B^{A^k}$ is uniformly generated by n-ary $(\mathcal{A}, \mathcal{B})$-minors if $\exists t \in \mathcal{B}, r_1, \ldots, r_s$ with $rk_{\mathcal{A}}(r_i) \leq n$

$$\forall f \in U \colon f = t \circ (f \circ r_1, f \circ r_2, \ldots, f \circ r_s).$$

For modules $\mathbf{A}, \mathbf{B}$:

$U \subseteq B^{A^k}$ is uniformly generated by n-ary minors if $\exists r_M \in \mathbf{R_B}$.

$$\forall f \in U \colon f = \sum_{rk(M) \leq n} r_M f(M\mathbf{x}).$$

Example [Fioravanti '20]

$\mathbf{A} = \mathbb{F}$, $\mathbf{B}$ coprime module

For all $f\colon \mathbb{F}^2 \to B$ with $f(0,0) = 0$:

$$I(f)(x,y) := |F|^{-1}\left(\sum_{a\in\mathbb{F}} f(x+ay,0) - f(ay,0)\right) = \begin{cases} f(x,y) \text{ if } y = 0 \\ 0 \text{ else.} \end{cases}$$

(similar for lines other than $y = 0$)

$$\Rightarrow \exists r_M \in \mathbf{R_B} \forall f \in B^{\mathbb{F}^2}\colon f(\mathbf{x}) = f(\mathbf{0}) + \sum_{\text{rk}(M)=1} r_M f(M\mathbf{x})$$

- $\{f\colon \mathbb{F}^2 \to B\}$ is uniformly generated by unary minors.
- I is *uniformly representable* by unary minors.

Uniform finite generation

Observation [Fioravanti, MK, Rossi '25]

For clones $\mathcal{A}$, $\mathcal{B}$, the following are equivalent:

1. $B^{A^{n+1}}$ is **ug** by n-ary $(\mathcal{A}, \mathcal{B})$-minors,
2. $\forall k : B^{A^k}$ is **ug** by n-ary $(\mathcal{A}, \mathcal{B})$-minors,
3. All **ur** $I : B^{A^k} \to B^{A^l}$ are **ur** by n-ary $(\mathcal{A}, \mathcal{B})$-minors.

Consequence

$B^{A^{n+1}}$ is **ug** by n-ary $(\mathcal{A}, \mathcal{B})$-minors $\Rightarrow \forall (\mathcal{A}, \mathcal{B})$-clonoid: $\mathcal{C} = \langle \mathcal{C}^{(n)} \rangle$

Example [Fioravanti '20]

$\{f : \mathbb{F}^2 \to \mathbf{B}\}$ is **ug** by 1-minors $\Rightarrow \mathcal{C} = \langle \mathcal{C}^{(1)} \rangle$ for $(\mathbb{F}, \mathbf{B})$-clonoids

Products

Observation 2 [Fioravanti, MK, Rossi '25]

- $B^{A_1^k}$ **ug** by n-ary $(\mathcal{A}_1, \mathcal{B})$-minors
- $B^{A_2^k}$ **ug** by n-ary $(\mathcal{A}_2, \mathcal{B})$-minors

$\Rightarrow B^{(A_1 \times A_2)^k}$ **ug** by n-ary $(\mathcal{A}_1 \times \mathcal{A}_2, \mathcal{B})$-minors.

Example [Fioravanti '21]

For $\mathbf{A} = \mathbb{F}_1 \times \mathbb{F}_2 \times \cdots \times \mathbb{F}_m$, $\mathbf{B}$ coprime: $\mathcal{C} = \langle \mathcal{C}^{(1)} \rangle$.

Example [Mayr, Wynne '24]

Conjecture true for uniserial $\mathbf{A} \Rightarrow$ true for $\mathrm{Con}(\mathbf{A})$ distributive.

Beyond modules

Example [Sparks '19]

- $\mathcal{P}_A$ projection clone on A
- $\mathcal{B}$ contains a NU-operation of arity n

$\Rightarrow B^{A^k}$ is **ug** by $|A|^n$-ary $(\mathcal{P}_A, \mathcal{B})$-minors.

Caution

Not all finiteness results are covered by **ug**!

Example [Lehtonen, Szendrei '11]

For $\mathcal{A} = \bigcup_{n \in \mathbb{N}} A^{A^n}$ $\exists^{<\omega}$ $(\mathcal{A}, \mathcal{P}_A)$-clonoids, but
$\forall n : A^{A^{n+1}}$ is **not ug** by n-ary $(\mathcal{A}, \mathcal{P}_A)$-minors.

Additional consequences

Relational bases

Assume $\mathcal{C} = \langle \mathcal{C}^{(n)} \rangle$, for *every* $(\mathcal{A}, \mathcal{B})$-clonoid. Then

$$\mathcal{C} = \mathrm{Pol}(\mathbb{A}, \mathbb{B}), \text{ for } \mathbb{A} = (A, \mathcal{A}^{(n)}), \mathbb{B} = (B, \mathcal{C} \circ \mathcal{A}^{(n)}).$$

Equational basis

Assume $\mathcal{C} = \langle \mathcal{C}^{(n)} \rangle$, for *every* $(\mathbf{A}, \mathbf{B})$-clonoid. Then

$$f = g \Leftrightarrow f \circ (t_1, \dots, t_k) = g \circ (t_1, \dots, t_k) \text{ for all } t_1, \dots, t_k \in \mathrm{Clo}(\mathbf{A})^{(n)}$$

$\rightsquigarrow$ **ug** helpful for equational basis of $\mathcal{C}$ (as many-sorted algebra $\mathbf{A}, \mathbf{B}, \mathcal{C}^{(k)}$).

Example [Mayr, K.'24]

Nilpotent extensions of $\mathbb{Z}_p \times \mathbb{Z}_q$ for primes $p \neq q$ are finitely based.

Outline

Clonoids from $\mathbb{F}^k$ to $\mathbf{B}$

Goal for $\mathbf{A} = \mathbb{F}^k$:

Find $r_M \in \mathbf{R}_\mathbf{B}$ for all $M \in \mathbb{F}^{(k+1)\times(k+1)}$:

$$\forall f\colon \mathbb{F}^{(k+1)\times k} \to B\colon\ f(X) = \sum_{\mathrm{rk}(M)\leq k} r_M f(MX).$$

Proof outline (induction step $k-1 \to k$):

1. By induction hypothesis: $\exists r'_M$:

$$f(X) = \sum_{\mathrm{rk}(M)\leq k-1} r'_M f(MX) \text{ for } \mathrm{rk}(X) \leq k-1.$$

2. $\Rightarrow$ wlog $f(X) = 0$ if $\mathrm{rk}(X) \leq k-1$.
3. Find coefficients r_M such that

$$I(f)(X) = \sum_{\mathrm{rk}(M)=k} r_M f(MX) = \begin{cases} f(X) \text{ if } \mathbf{e}_{k+1}^T X = 0 \\ 0 \text{ else.} \end{cases}$$

4. use operations as in (3) to cover all subspaces $\mathbf{a}^T X = 0$ and sum up. □

Proof step 3

Find coefficients r_M such that

$$I(f)(X) = \sum_{\text{rk}(M)=k} r_M f(MX) = \begin{cases} f(X) \text{ if } \mathbf{e}_{k+1}^T X = 0 \\ 0 \text{ else.} \end{cases}$$

wlog $\mathbf{B} = \mathbb{K}$ is a field.

1. Let
$$\chi_N(X) := \begin{cases} 1 \text{ if } X = N \\ 0 \text{ else.} \end{cases}$$
2. It is enough to find r_M with
$$\chi_{(Id,0)^T}(X) = \sum_{\text{rk}(M)=k, M \in \mathbb{K}^{k \times (k+1)}} r_M \chi_{Id}(MX)$$
3. $MX = Id$ has solution space of form
$$\theta_{X_0,\mathbf{u}} = \{X_0 + \mathbf{u}\mathbf{v}^T | \mathbf{v} \in \mathbb{F}^k\}, \mathbf{u} \notin R(X_0)$$
4. find formula $\chi_{(Id,0)^T}(X) = \sum_{V \text{ is } \theta-\text{space}} \chi_V(X)$ □

Upper bounds

Observation [Fioravanti, MK, Rossi '25]

For modules $\mathbf{A}, \mathbf{B}$:

$$\langle B^{A^m} \rangle_{\mathbf{A},\mathbf{B}} = \bigcup_{n \in \mathbb{N}} B^{A^n} \Rightarrow m \geq \frac{\log |A|}{\log |R_A|}.$$

Corollary [Fioravanti, MK, Rossi '25]

$(\mathbb{F}^k, \mathbf{B})$-clonoids, for coprime $\mathbb{F}$, $\mathbf{B}$

- are generated by their k-ary functions
- in general not by their $(k-1)$-ary functions.

The lattice of $(\mathbb{F}^k, \mathbf{B})$-clonoids

Task

Describe the lattice of $(\mathbb{F}^k, \mathbf{B})$-clonoids $\mathcal{C}$.

- $\mathcal{C}^{(0)} \leq \mathbf{B}$,
- $\{f \in \mathcal{C}^{(1)} \mid f(0) = 0\} \leq \mathbf{B}^{\mathbb{F}^k \setminus \{0\}}$ as $\mathbf{R}_\mathbf{B}[\mathbb{F}^*]$-module
- $\{f \in \mathcal{C}^{(m)} \mid f(X) = 0 \text{ for } X \notin R_m\} \leq \mathbf{B}^{R_m}$ as $\mathbf{R}_\mathbf{B}[GL(\mathbb{F}, m)]$-module
  $R_m = \{X \in \mathbb{F}^{k \times m} \mid rk(X) = m\}$

Corollary [Fioravanti, MK, Rossi '25]

The lattice of $(\mathbb{F}^k, \mathbf{B})$-clonoids $\cong \Pi_{m=0}^k L_m$, with

$L_m =$ lattice of $\mathbf{R}_\mathbf{B}[GL(\mathbb{F}, m)]$-submodules of $\mathbf{B}^{R_m}$.

Back to the conjecture

Conjecture

Every $(\mathbf{A}, \mathbf{B})$-clonoid is finitely generated $\Leftrightarrow \gcd(|A|, |B|) = 1$.

Now confirmed for:

- $\mathbf{A} = \mathbb{F}^k$ vector spaces
- $\mathbf{A} = \mathbb{F}_1^{k_1} \times \mathbb{F}_2^{k_2} \times \cdots \times \mathbb{F}_n^{k_n} \times \mathbf{A}'$, as $(\mathbb{F}_1 \times \cdots \times \mathbb{F}_n \times \mathbf{R}_{\mathbf{A}'})$-module, with $\mathrm{Con}(\mathbf{A}')$ distributive.

It would be enough to prove:

Conjecture (*)

For $\mathbf{A}$ abelian p-group, $\mathbf{B}$ coprime abelian group.
$\exists k$: $B^{A^{k+1}}$ uniformly generated by k-ary $(\mathbf{A}, \mathbf{B})$-minors.

Thank you!

Questions? Remarks? Counterexamples?