Clonoids and uniform generation by minors

Stefano Fioravanti, **Michael Kompatscher**, Bernardo Rossi

Charles University

AAA 107
June 20th 2025, BF Bern

Clones and clonoids

Clones

$\mathcal{A} \subseteq \bigcup_{n \geq 1} A^{A^n}$ is a clone on A if

- all $\pi_i^n \in \mathcal{A}$ with $\pi_i^n(x_1, \ldots, x_n) = x_i$
- $f, g_1, \ldots, g_k \in \mathcal{A} \Rightarrow f \circ (g_1, \ldots, g_k) \in \mathcal{A}$ $\quad (\mathcal{A} \circ \mathcal{A} \subseteq \mathcal{A})$

Clones and clonoids

Clones

$\mathcal{A} \subseteq \bigcup_{n\geq 1} A^{A^n}$ is a clone on A if

- all $\pi_i^n \in \mathcal{A}$ with $\pi_i^n(x_1, \dots, x_n) = x_i$
- $f, g_1, \dots, g_k \in \mathcal{A} \Rightarrow f \circ (g_1, \dots, g_k) \in \mathcal{A}$ $\qquad (\mathcal{A} \circ \mathcal{A} \subseteq \mathcal{A})$

Clonoids

For clones $\mathcal{A}, \mathcal{B}$ (on A, B), $\mathcal{C} \subseteq \bigcup_{n\geq 1} B^{A^n}$ is a $(\mathcal{A}, \mathcal{B})$-clonoid if

- $\mathcal{C} \circ \mathcal{A} \subseteq \mathcal{C}$,
- $\mathcal{B} \circ \mathcal{C} \subseteq \mathcal{C}$.

Clones and clonoids

Clones

$\mathcal{A} \subseteq \bigcup_{n \geq 1} A^{A^n}$ is a clone on A if

- all $\pi_i^n \in \mathcal{A}$ with $\pi_i^n(x_1, \dots, x_n) = x_i$
- $f, g_1, \dots, g_k \in \mathcal{A} \Rightarrow f \circ (g_1, \dots, g_k) \in \mathcal{A}$ $\qquad (\mathcal{A} \circ \mathcal{A} \subseteq \mathcal{A})$

Clonoids

For clones $\mathcal{A}, \mathcal{B}$ (on A, B), $\mathcal{C} \subseteq \bigcup_{n \geq 1} B^{A^n}$ is a $(\mathcal{A}, \mathcal{B})$-clonoid if

- $\mathcal{C} \circ \mathcal{A} \subseteq \mathcal{C}$,
- $\mathcal{B} \circ \mathcal{C} \subseteq \mathcal{C}$.

An $(\mathbf{A}, \mathbf{B})$-clonoid is a $(\mathrm{Clo}(\mathbf{A}), \mathrm{Clo}(\mathbf{B}))$-clonoid.

Clones and clonoids

Clones

$\mathcal{A} \subseteq \bigcup_{n\geq 1} A^{A^n}$ is a clone on A if

- all $\pi_i^n \in \mathcal{A}$ with $\pi_i^n(x_1, \ldots, x_n) = x_i$
- $f, g_1, \ldots, g_k \in \mathcal{A} \Rightarrow f \circ (g_1, \ldots, g_k) \in \mathcal{A}$ $\qquad (\mathcal{A} \circ \mathcal{A} \subseteq \mathcal{A})$

Clonoids

For clones $\mathcal{A}, \mathcal{B}$ (on A, B), $\mathcal{C} \subseteq \bigcup_{n\geq 1} B^{A^n}$ is a $(\mathcal{A}, \mathcal{B})$-clonoid if

- $\mathcal{C} \circ \mathcal{A} \subseteq \mathcal{C}$,
- $\mathcal{B} \circ \mathcal{C} \subseteq \mathcal{C}$.

An $(\mathbf{A}, \mathbf{B})$-clonoid is a $(\text{Clo}(\mathbf{A}), \text{Clo}(\mathbf{B}))$-clonoid.

Goal: For given clones $\mathcal{A}$, $\mathcal{B}$, describe the $(\mathcal{A}, \mathcal{B})$-clonoids.

Some known results

Pippenger '02 : (A, B)-clonoid $=$ minor closed set/minion.
Minions are equal to $\mathrm{Pol}(\mathbb{A}, \mathbb{B}) = \{h\colon \mathbb{A}^n \to \mathbb{B}, n \geq 1\}$ for relational structures $\mathbb{A}$,$\mathbb{B}$.

Couceiro, Foldes '09 : $(\mathcal{A}, \mathcal{B})$-clonoid $=$ left/right stable under $\mathcal{A}/\mathcal{B}$
$(\mathcal{A}, \mathcal{B})$-clonoids $= \mathrm{Pol}(\mathbb{A}, \mathbb{B})$ for $\mathbb{A}$,$\mathbb{B}$ invariant under $\mathcal{A}$,$\mathcal{B}$.

Lehtonen, Szendrei '11 : $(\mathcal{A}, A)$-clonoids
study of clones $\mathcal{A}$ with finitely many $\mathcal{A}$-equivalence classes
$(f \equiv g \Leftrightarrow f \circ \mathcal{A} = g \circ \mathcal{A})$

Aichinger, Mayr '16 : $(A, \mathrm{Clo}(\mathbf{B}))$-clonoid $=$ clonoid with target $\mathbf{B}$

Sparks '19 : The number of $(A, \mathcal{B})$-clonoid, for $|A|, |B| \geq 2$ is

1. finite if $\mathcal{B}$ has NU operation,
2. ω if $\mathcal{B}$ has few subpowers, no NU-term,
3. 2^ω else. (*)

Lehtonen '25 : classification of $(\mathcal{A}, \mathcal{B})$-clonoid, for Boolean $\mathcal{A}, \mathcal{B}$

Erkko's results on Boolean clonoids

Cardinalities of (C_1, C_2)-clonoid lattices

	$[J, I]$	$[I^*, \Omega(1)]$	$[V_{01}, V]$ $[\Lambda_{01}, \Lambda]$	$[MU_{01}^{\infty}, MU^{\infty}]$ $[MW_{01}^{\infty}, MW^{\infty}]$	U_{01}^{∞} W_{01}^{∞}	U^{∞} W^{∞}	$[L_{01}, L]$	$[\{SM, MU_{01}^{k}, MW_{01}^{k}\}, \Omega]$
J	U	U	U	U	U	U	C	F
I_0, I_1	U	U	U	U	U	U	C	F
I	U	U	U	F	F	F	C	F
I^*	U	U	U	U	U	U	C	F
$\Omega(1)$	U	U	U	F	F	F	C	F
V_{01}, Λ_{01}	U	U	U	U	U	U	F	F
V_{0*}, Λ_{*1}	U	U	U	F	F	F	F	F
V_{*1}, Λ_{0*}	U	U	U	U	U	U	F	F
V, Λ	U	U	U	F	F	F	F	F
MU_{01}^{k}, MW_{01}^{k}	U	U	U	U	U	U	F	F
MU^{k}, MW^{k}	U	U	U	U	U	U	F	F
U_{01}^{k}, W_{01}^{k}	U	U	U	U	U	U	F	F
U^{k}, W^{k}	U	U	U	U	U	U	F	F
L_{01}	U	U	U	U	U	U	C	F
L_{0*}, L_{*1}	U	U	U	U	U	U	C	F
LS	U	U	U	U	U	U	C	F
L	U	U	U	F	F	F	C	F
SM	U	U	U	U	U	U	F	F
$[M_{01}, M]$	C	C	F	F	F	F	F	F
$[S_{01}, \Omega]$	F	F	F	F	F	F	F	F

Generating sets of clonoids

For clones $\mathcal{A}$, $\mathcal{B}$ on finite sets A, B; $F \subseteq \bigcup_{n \in \mathbb{N}} B^{A^n}$

$\langle F \rangle := \mathcal{B} \circ F \circ \mathcal{A}$ is the $(\mathcal{A}, \mathcal{B})$-clonoid generated by F.

Generating sets of clonoids

For clones $\mathcal{A}$, $\mathcal{B}$ on finite sets A, B; $F \subseteq \bigcup_{n\in\mathbb{N}} B^{A^n}$

$\langle F\rangle := \mathcal{B} \circ F \circ \mathcal{A}$ is the $(\mathcal{A},\mathcal{B})$-clonoid generated by F.

Goal revisited

- Describe the *lattice* of $(\mathcal{A},\mathcal{B})$-clonoids.
- For which $\mathcal{A}$, $\mathcal{B}$ is it finite?
- Are there finite generating sets?

Generating sets of clonoids

For clones $\mathcal{A}$, $\mathcal{B}$ on finite sets A, B; $F \subseteq \bigcup_{n\in\mathbb{N}} B^{A^n}$

$\langle F \rangle := \mathcal{B} \circ F \circ \mathcal{A}$ is the $(\mathcal{A}, \mathcal{B})$-clonoid generated by F.

Goal revisited

- Describe the *lattice* of $(\mathcal{A}, \mathcal{B})$-clonoids.
- For which $\mathcal{A}$, $\mathcal{B}$ is it finite?
- Are there finite generating sets?

Observation, for fixed $\mathcal{A}$, $\mathcal{B}$:

$$\text{Finite lattice} \Leftrightarrow \exists k \in \mathbb{N}\colon \mathcal{C} = \langle \mathcal{C}^{(k)} \rangle \text{ for every clonoid } \mathcal{C}.$$
$$\Leftrightarrow \exists k \in \mathbb{N}\colon \mathcal{C} = \langle \mathcal{C}^{(k)} \rangle \text{ for every } \mathcal{C} = \langle f \rangle$$

Our motivation: Clonoids between affine clones

Clonoids between *affine* clones $\leftrightsquigarrow$ 2-nilpotent algebras.

Our motivation: Clonoids between affine clones

Clonoids between *affine* clones $\leftrightsquigarrow$ 2-nilpotent algebras.

Conjecture for affine $\mathcal{A}$, $\mathcal{B}$:

$$\exists \text{ finitely many } (\mathcal{A}, \mathcal{B})\text{-clonoids} \Leftrightarrow \gcd(|A|, |B|) = 1.$$

Our motivation: Clonoids between affine clones

Clonoids between *affine* clones $\leftrightsquigarrow$ 2-nilpotent algebras.

Conjecture for affine $\mathcal{A}$, $\mathcal{B}$:

$$\exists \text{ finitely many } (\mathcal{A},\mathcal{B})\text{-clonoids} \Leftrightarrow \gcd(|A|,|B|) = 1.$$

wlog $\mathcal{A} = \mathrm{Clo}(\mathbf{A})$, $\mathcal{B} = \mathrm{Clo}(\mathbf{B})$ for modules $\mathbf{A}, \mathbf{B}$.

Our motivation: Clonoids between affine clones

Clonoids between *affine* clones $\leftrightsquigarrow$ 2-nilpotent algebras.

Conjecture for affine $\mathcal{A}$, $\mathcal{B}$:

$$\exists \text{ finitely many } (\mathcal{A},\mathcal{B})\text{-clonoids} \Leftrightarrow \gcd(|A|,|B|) = 1.$$

wlog $\mathcal{A} = \mathrm{Clo}(\mathbf{A})$, $\mathcal{B} = \mathrm{Clo}(\mathbf{B})$ for modules $\mathbf{A}, \mathbf{B}$.

Conjecture confirmed for $\mathbf{A}$:

- $\mathbb{F}$ [Fioravanti '20]
- $\mathbb{F}_1 \times \mathbb{F}_2 \times \cdots \times \mathbb{F}_m$ (as regular module) [Fioravanti '21]
- distributive module [Mayr, Wynne '24]
- $\mathbb{F}^k$ [Fioravanti, MK, Rossi '25] (in preparation)

Our motivation: Clonoids between affine clones

Clonoids between *affine* clones $\leftrightsquigarrow$ 2-nilpotent algebras.

Conjecture for affine $\mathcal{A}$, $\mathcal{B}$:

$$\exists \text{ finitely many } (\mathcal{A}, \mathcal{B})\text{-clonoids} \Leftrightarrow \gcd(|A|, |B|) = 1.$$

wlog $\mathcal{A} = \mathrm{Clo}(\mathbf{A})$, $\mathcal{B} = \mathrm{Clo}(\mathbf{B})$ for modules $\mathbf{A}, \mathbf{B}$.

Conjecture confirmed for $\mathbf{A}$:

- $\mathbb{F}$ [Fioravanti '20]
- $\mathbb{F}_1 \times \mathbb{F}_2 \times \cdots \times \mathbb{F}_m$ (as regular module) [Fioravanti '21]
- distributive module [Mayr, Wynne '24]
- $\mathbb{F}^k$ [Fioravanti, MK, Rossi '25] (in preparation)

All these results were proved by uniform generation by minors.

Generation by n-ary minors

Let $\mathcal{A}$, $\mathcal{B}$ be clones, $f\colon A^k \to B$, $\mathcal{C} = \langle f \rangle$. When is $\mathcal{C} = \langle \mathcal{C}^{(n)} \rangle$?

Generation by n-ary minors

Let $\mathcal{A}$, $\mathcal{B}$ be clones, $f\colon A^k \to B$, $\mathcal{C} = \langle f \rangle$. When is $\mathcal{C} = \langle \mathcal{C}^{(n)} \rangle$?

$$\mathcal{C}^{(n)} = \mathcal{B} \circ f \circ \mathcal{A}^{(n)}$$
$$\langle \mathcal{C}^{(n)} \rangle = \mathcal{B} \circ f \circ \mathcal{A}^{(n)} \circ \mathcal{A}$$

Generation by n-ary minors

Let $\mathcal{A}$, $\mathcal{B}$ be clones, $f\colon A^k \to B$, $\mathcal{C} = \langle f \rangle$. When is $\mathcal{C} = \langle \mathcal{C}^{(n)} \rangle$?

$$\begin{aligned} \mathcal{C}^{(n)} &= \mathcal{B} \circ f \circ \mathcal{A}^{(n)} \\ \langle \mathcal{C}^{(n)} \rangle &= \mathcal{B} \circ f \circ \mathcal{A}^{(n)} \circ \mathcal{A} \\ &= \mathcal{B} \circ \{\mathbf{x} \mapsto f(r(\mathbf{x})) : r \in R_n(\mathcal{A})\} \end{aligned}$$

$$r \in R_n(\mathcal{A}) :\Leftrightarrow r(\mathbf{x}) = \begin{bmatrix} u_1 \circ (v_1, \dots, v_n)(\mathbf{x}) \\ \vdots \\ u_k \circ (v_1, \dots, v_n)(\mathbf{x}) \end{bmatrix}, u_i \in \mathcal{A}^{(n)}, v_j \in \mathcal{A}$$

Generation by n-ary minors

Let $\mathcal{A}$, $\mathcal{B}$ be clones, $f\colon A^k \to B$, $\mathcal{C} = \langle f \rangle$. When is $\mathcal{C} = \langle \mathcal{C}^{(n)} \rangle$?

$$\begin{aligned} \mathcal{C}^{(n)} &= \mathcal{B} \circ f \circ \mathcal{A}^{(n)} \\ \langle \mathcal{C}^{(n)} \rangle &= \mathcal{B} \circ f \circ \mathcal{A}^{(n)} \circ \mathcal{A} \\ &= \mathcal{B} \circ \{\mathbf{x} \mapsto f(r(\mathbf{x})) : r \in R_n(\mathcal{A})\} \end{aligned}$$

$$r \in R_n(\mathcal{A}) :\Leftrightarrow r(\mathbf{x}) = \begin{bmatrix} u_1 \circ (v_1, \dots, v_n)(\mathbf{x}) \\ \vdots \\ u_k \circ (v_1, \dots, v_n)(\mathbf{x}) \end{bmatrix}, u_i \in \mathcal{A}^{(n)}, v_j \in \mathcal{A}$$

Thus $\mathcal{C} = \langle \mathcal{C}^{(n)} \rangle$ iff $\exists t \in \mathcal{B}, r_1, \dots, r_s \in R_n(\mathcal{A})$:

$$f = t \circ (f \circ r_1, f \circ r_2, \dots, f \circ r_s).$$

Uniform generation by n-ary minors

Definition

For clones $\mathcal{A}$,$\mathcal{B}$, $U \subseteq B^{A^k}$ is uniformly generated (**ug**) by n-ary $(\mathcal{A}, \mathcal{B})$-minors if $\exists t \in \mathcal{B}, r_1, \ldots, r_s \in R_n(\mathcal{A})$:

$$\forall f \in U \colon f = t \circ (f \circ r_1, f \circ r_2, \ldots, f \circ r_s).$$

Uniform generation by n-ary minors

Definition

For clones $\mathcal{A}$,$\mathcal{B}$, $U \subseteq B^{A^k}$ is uniformly generated (**ug**) by n-ary $(\mathcal{A}, \mathcal{B})$-minors if $\exists t \in \mathcal{B}, r_1, \dots, r_s \in R_n(\mathcal{A})$:

$$\forall f \in U\colon f = t \circ (f \circ r_1, f \circ r_2, \dots, f \circ r_s).$$

Example: for term clones of modules $\mathbf{A}$, $\mathbf{B}$:

$U \subseteq B^{A^k}$ is uniformly generated by n-ary minors if $\exists r_M \in \mathbf{R_B}$.

$$\forall f \in U\colon f = \sum_{rk(M) \leq n} r_M f(M\mathbf{x}).$$

Uniform finite generation

Observation [Rossi, MK, Fioravanti '25]

For clones $\mathcal{A}$, $\mathcal{B}$, the following are equivalent:

1. $B^{A^{n+1}}$ is **ug** by n-ary $(\mathcal{A}, \mathcal{B})$-minors,

Uniform finite generation

Observation [Rossi, MK, Fioravanti '25]

For clones $\mathcal{A}$, $\mathcal{B}$, the following are equivalent:

1. $B^{A^{n+1}}$ is **ug** by n-ary $(\mathcal{A},\mathcal{B})$-minors,
2. $\forall k : B^{A^k}$ is **ug** by n-ary $(\mathcal{A},\mathcal{B})$-minors,

Uniform finite generation

Observation [Rossi, MK, Fioravanti '25]

For clones $\mathcal{A}$, $\mathcal{B}$, the following are equivalent:

1. $B^{A^{n+1}}$ is **ug** by n-ary $(\mathcal{A}, \mathcal{B})$-minors,
2. $\forall k : B^{A^k}$ is **ug** by n-ary $(\mathcal{A}, \mathcal{B})$-minors,
3. (All **ur** $I : B^{A^k} \to B^{A^l}$ are **ur** by n-ary $(\mathcal{A}, \mathcal{B})$-minors.)

Uniform finite generation

Observation [Rossi, MK, Fioravanti '25]

For clones $\mathcal{A}$, $\mathcal{B}$, the following are equivalent:

1. $B^{A^{n+1}}$ is **ug** by n-ary $(\mathcal{A}, \mathcal{B})$-minors,
2. $\forall k : B^{A^k}$ is **ug** by n-ary $(\mathcal{A}, \mathcal{B})$-minors,
3. (All **ur** $I : B^{A^k} \to B^{A^l}$ are **ur** by n-ary $(\mathcal{A}, \mathcal{B})$-minors.)

Consequence

$B^{A^{n+1}}$ is **ug** by n-ary $(\mathcal{A}, \mathcal{B})$-minors $\Rightarrow \forall (\mathcal{A}, \mathcal{B})$-clonoid: $\mathcal{C} = \langle \mathcal{C}^{(n)} \rangle$

Uniform finite generation

Observation [Rossi, MK, Fioravanti '25]

For clones $\mathcal{A}$, $\mathcal{B}$, the following are equivalent:

1. $B^{A^{n+1}}$ is **ug** by n-ary $(\mathcal{A}, \mathcal{B})$-minors,
2. $\forall k : B^{A^k}$ is **ug** by n-ary $(\mathcal{A}, \mathcal{B})$-minors,
3. (All **ur** $I\colon B^{A^k} \to B^{A^l}$ are **ur** by n-ary $(\mathcal{A}, \mathcal{B})$-minors.)

Consequence

$B^{A^{n+1}}$ is **ug** by n-ary $(\mathcal{A}, \mathcal{B})$-minors $\Rightarrow \forall(\mathcal{A}, \mathcal{B})$-clonoid: $\mathcal{C} = \langle \mathcal{C}^{(n)} \rangle$

Example [Fioravanti '20]

For a field $\mathbb{F}$, coprime module $\mathbf{B}$:
$\{f : \mathbb{F}^2 \to \mathbf{B}\}$ is **ug** by 1-minors $\Rightarrow \mathcal{C} = \langle \mathcal{C}^{(1)} \rangle$ for $(\mathbb{F}, \mathbf{B})$-clonoids

Example: Uniform generation by binary minors

Let $\mathcal{A} = \mathrm{Clo}(\{0, 1\}, \cdot, 0)$, $\mathcal{B} = \mathrm{Clo}(\mathbb{Z}_2)$.

Example: Uniform generation by binary minors

Let $\mathcal{A} = \text{Clo}(\{0,1\}, \cdot, 0)$, $\mathcal{B} = \text{Clo}(\mathbb{Z}_2)$.

Represent $f\colon \{0,1\}^3 \to \{0,1\}$ by polynomials over $\mathbb{Z}_2$:

$$f(x,y,z) = a_1xyz + a_2xy + \cdots + a_7z + a_8$$

Example: Uniform generation by binary minors

Let $\mathcal{A} = \text{Clo}(\{0,1\}, \cdot, 0)$, $\mathcal{B} = \text{Clo}(\mathbb{Z}_2)$.

Represent $f\colon \{0,1\}^3 \to \{0,1\}$ by polynomials over $\mathbb{Z}_2$:

$$f(x,y,z) = a_1xyz + a_2xy + \cdots + a_7z + a_8$$
$$I(f)(x,y,z) := f(x,y,0) + f(x,0,z) + \cdots + f(0,0,z) + f(0,0,0)$$

Then $(f - I(f))(xyz) = a_1xyz$.

Example: Uniform generation by binary minors

Let $\mathcal{A} = \mathrm{Clo}(\{0,1\}, \cdot, 0)$, $\mathcal{B} = \mathrm{Clo}(\mathbb{Z}_2)$.

Represent $f\colon \{0,1\}^3 \to \{0,1\}$ by polynomials over $\mathbb{Z}_2$:

$$
\begin{aligned}
f(x,y,z) &= a_1 xyz + a_2 xy + \cdots + a_7 z + a_8 \\
I(f)(x,y,z) &:= f(x,y,0) + f(x,0,z) + \cdots + f(0,0,z) + f(0,0,0)
\end{aligned}
$$

Then $(f - I(f))(xyz) = a_1 xyz$. Moreover

$$
\begin{aligned}
J(f)(xyz) := (f - I(f))(xy, xy, z) &= f(xy, xy, z) - I(f)(xy, xy, z) \\
&= a_1 xyz.
\end{aligned}
$$

By $f = I(f) + J(f)$, 3-ary functions are **ug** by 2-ary $(\mathcal{A}, \mathcal{B})$-minors

Example: Uniform generation by binary minors

Let $\mathcal{A} = \text{Clo}(\{0,1\}, \cdot, 0)$, $\mathcal{B} = \text{Clo}(\mathbb{Z}_2)$.

Represent $f\colon \{0,1\}^3 \to \{0,1\}$ by polynomials over $\mathbb{Z}_2$:

$$\begin{aligned} f(x,y,z) &= a_1xyz + a_2xy + \cdots + a_7z + a_8 \\ I(f)(x,y,z) &:= f(x,y,0) + f(x,0,z) + \cdots + f(0,0,z) + f(0,0,0) \end{aligned}$$

Then $(f - I(f))(xyz) = a_1xyz$. Moreover

$$\begin{aligned} J(f)(xyz) &:= (f - I(f))(xy,xy,z) = f(xy,xy,z) - I(f)(xy,xy,z) \\ &= a_1xyz. \end{aligned}$$

By $f = I(f) + J(f)$, 3-ary functions are **ug** by 2-ary $(\mathcal{A},\mathcal{B})$-minors

$\Rightarrow \mathcal{C} = \langle \mathcal{C}^{(2)} \rangle$ for clonoids from $\mathcal{A}$ to $\mathcal{B}$. [Couceiro, Lehtonen '24]

Example: Uniform generation by binary minors

Let $\mathcal{A} = \text{Clo}(\{0,1\}, \cdot, 0)$, $\mathcal{B} = \text{Clo}(\mathbb{Z}_2)$.

Represent $f\colon \{0,1\}^3 \to \{0,1\}$ by polynomials over $\mathbb{Z}_2$:

$$f(x,y,z) = a_1 xyz + a_2 xy + \cdots + a_7 z + a_8$$
$$I(f)(x,y,z) := f(x,y,0) + f(x,0,z) + \cdots + f(0,0,z) + f(0,0,0)$$

Then $(f - I(f))(xyz) = a_1 xyz$. Moreover

$$\begin{aligned} J(f)(xyz) &:= (f - I(f))(xy,xy,z) = f(xy,xy,z) - I(f)(xy,xy,z) \\ &= a_1 xyz. \end{aligned}$$

By $f = I(f) + J(f)$, 3-ary functions are **ug** by 2-ary $(\mathcal{A},\mathcal{B})$-minors

$\Rightarrow \mathcal{C} = \langle \mathcal{C}^{(2)} \rangle$ for clonoids from $\mathcal{A}$ to $\mathcal{B}$. [Couceiro, Lehtonen '24]

(I, J are *uniformly representable* (**ur**) by 2-ary $(\mathcal{A},\mathcal{B})$-minors.)

Products

Observation 2 [Rossi, MK, Fioravanti '25]

- $B^{A_1^k}$ **ug** by n-ary $(\mathcal{A}_1, \mathcal{B})$-minors
- $B^{A_2^k}$ **ug** by n-ary $(\mathcal{A}_2, \mathcal{B})$-minors

$\Rightarrow B^{(A_1 \times A_2)^k}$ **ug** by n-ary $(\mathcal{A}_1 \times \mathcal{A}_2, \mathcal{B})$-minors!

Products

Observation 2 [Rossi, MK, Fioravanti '25]

- $B^{A_1^k}$ **ug** by n-ary $(\mathcal{A}_1, \mathcal{B})$-minors
- $B^{A_2^k}$ **ug** by n-ary $(\mathcal{A}_2, \mathcal{B})$-minors

$\Rightarrow B^{(A_1 \times A_2)^k}$ **ug** by n-ary $(\mathcal{A}_1 \times \mathcal{A}_2, \mathcal{B})$-minors!

Example [Fioravanti '21]

For $\mathbf{A} = \mathbb{F}_1 \times \mathbb{F}_2 \times \cdots \times \mathbb{F}_m$, $\mathbf{B}$ coprime: $\mathcal{C} = \langle \mathcal{C}^{(1)} \rangle$.

Products

Observation 2 [Rossi, MK, Fioravanti '25]

- $B^{A_1^k}$ **ug** by n-ary $(\mathcal{A}_1, \mathcal{B})$-minors
- $B^{A_2^k}$ **ug** by n-ary $(\mathcal{A}_2, \mathcal{B})$-minors

$\Rightarrow B^{(A_1 \times A_2)^k}$ **ug** by n-ary $(\mathcal{A}_1 \times \mathcal{A}_2, \mathcal{B})$-minors!

Example [Fioravanti '21]

For $\mathbf{A} = \mathbb{F}_1 \times \mathbb{F}_2 \times \cdots \times \mathbb{F}_m$, $\mathbf{B}$ coprime: $\mathcal{C} = \langle \mathcal{C}^{(1)} \rangle$.

Example [Mayr, Wynne '24]

Conjecture true for uniserial modules $\mathbf{A} \Rightarrow$ true for distributive modules.

Polynomial equivalence

$\mathcal{A}$, $\mathcal{B}$... clones
$\mathcal{A}$... *polynomially equivalent* to a module $\mathbf{A}$. Then

Observation 3

- B^{A^k} **ug** by n-ary $(\mathrm{Clo}(\mathbf{A}), \mathcal{B})$-minors $\Rightarrow$ B^{A^k} **ug** by $(n+1)$-ary $(\mathcal{A}, \mathcal{B})$-minors.
- A^{B^k} **ug** by n-ary $(\mathcal{B}, \mathrm{Clo}(\mathbf{A}))$-minors $\Rightarrow$ A^{B^k} **ug** by n-ary $(\mathcal{B}, \mathcal{A})$-minors.

Polynomial equivalence

$\mathcal{A}$, $\mathcal{B}$... clones
$\mathcal{A}$... *polynomially equivalent* to a module **A**. Then

Observation 3

- B^{A^k} **ug** by n-ary $(\mathrm{Clo}(\mathbf{A}), \mathcal{B})$-minors $\Rightarrow$
  B^{A^k} **ug** by $(n+1)$-ary $(\mathcal{A}, \mathcal{B})$-minors.
- A^{B^k} **ug** by n-ary $(\mathcal{B}, \mathrm{Clo}(\mathbf{A}))$-minors $\Rightarrow$
  A^{B^k} **ug** by n-ary $(\mathcal{B}, \mathcal{A})$-minors.

Example [Couceiro, Lehtonen '24]

$\Rightarrow \mathcal{C} = \langle \mathcal{C}^{(2)} \rangle$ for $(\mathcal{A}, \mathcal{B})$-clonoids

$\mathcal{A} = \mathrm{Clo}(\{0,1\}, \cdot, 0)$
$\mathcal{B} = \mathrm{Clo}(\{0,1\}, x - y + z)$.

Useful in the affine case:

Lemma [Wynne, Mayr '24]

Lemma 2.14. *Let* $\mathbf{A}$ *be an* $\mathbf{R}$*-module,* $\mathbf{B}$ *an* $\mathbf{S}$*-module,* $k, n \in \mathbb{N}$, $U \subseteq F(A,B)^{(k)}$ *and* $d : U \to F(A,B)^{(k)}$, $f \mapsto f'$.

(1) *Then* d *can be uniformly represented by* n*-ary* $\mathbf{A}, \mathbf{B}$*-minors on* U *if and only if there exists* $s \colon \{r \in R^{k \times k} : \mathrm{rk}(r) \leq n\} \to S$ *with finite support such that for all* $f \in U$ *and all* $x \in A^k$,

$$f'(x) = \sum_{r \in R^{k\times k}, \mathrm{rk}(r) \leq n} s(r) f(rx).$$

(2) *Assume that* d *can be uniformly represented by* n*-ary* $\mathbf{A}, \mathbf{B}$*-minors on* U *and that* $\{f - d(f) \colon f \in U\}$ *is uniformly generated by* n*-ary* $\mathbf{A}, \mathbf{B}$*-minors. Then* U *is uniformly generated by* n*-ary* $\mathbf{A}, \mathbf{B}$*-minors.*

(3) *Let* $\ell \in \mathbb{N}$ *and let* $\mathbf{M}_1, \ldots, \mathbf{M}_\ell$ *be submodules of* $\mathbf{A}$*. Assume that* $F(M_i, B)^{(k)}$ *is uniformly generated by* n*-ary* $\mathbf{M}_i, \mathbf{B}$*-minors for each* $i \in [\ell]$ *and that*

$$U_0 := \{f \in F(A,B)^{(k)} : f(M_i^k) = 0 \text{ for all } i \in [\ell]\}$$

is uniformly generated by n*-ary* $\mathbf{A}, \mathbf{B}$*-minors.*

Then $F(A,B)^{(k)}$ *is uniformly generated by* n*-ary* $\mathbf{A}, \mathbf{B}$*-minors.*

Beyond affine clones

Example [Sparks '19]

For a clone $\mathcal{B}$ with n-ary NU-operation:
$\forall k : \{f : A^k \to B\}$ is **ug** by $|A|^n$-ary $(A, \mathcal{B})$-minors.

$\Rightarrow$ only finitely many $(A, \mathcal{B})$-clonoids.

Beyond affine clones

Example [Sparks '19]

For a clone $\mathcal{B}$ with n-ary NU-operation:
$\forall k : \{f : A^k \to B\}$ is **ug** by $|A|^n$-ary $(A, \mathcal{B})$-minors.

$\Rightarrow$ only finitely many $(A, \mathcal{B})$-clonoids.

Caution

Not all finiteness results are covered by **ug**!

Beyond affine clones

Example [Sparks '19]

For a clone $\mathcal{B}$ with n-ary NU-operation:
$\forall k : \{f : A^k \to B\}$ is **ug** by $|A|^n$-ary $(A, \mathcal{B})$-minors.

$\Rightarrow$ only finitely many $(A, \mathcal{B})$-clonoids.

Caution

Not all finiteness results are covered by **ug**!

Example [Lehtonen, Szendrei '11]

There are only finitely many (Ω_A, A)-clonoids, for $\Omega_A = \bigcup_{n\in\mathbb{N}} A^{A^n}$.

But $\forall n : A^{A^{n+1}}$ is **not ug** by n-ary (Ω_A, A)-minors.

Thank you!