Clonoids between vector spaces

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AAA106 - Olomouc

Clones and clonoids

Clones

$\mathcal{A} \subseteq \bigcup_{n \geq 1} A^{A^n}$ is a clone on a set A if

- $\mathcal{A}$ contains all projections $\pi_i^n(x_1, \ldots, x_n) = x_i$
- $f, g_1, \ldots, g_k \in \mathcal{A} \Rightarrow f \circ (g_1, \ldots, g_k) \in \mathcal{A}$ $\qquad (\mathcal{A} \circ \mathcal{A} \subseteq \mathcal{A})$

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- $\mathcal{C} \circ \mathcal{A} \subseteq \mathcal{C}$,
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Background:
[Couceiro, Foldes '09], [Aichinger, Mayr '16], [Sparks '19], [Lehtonen],...

Clonoids between modules

For an $\mathbf{R}$-module $\mathbf{A} = (A, +, 0, -, (r)_{r \in \mathbf{R}})$
$\mathrm{Clo}(\mathbf{A})$ consists of all linear maps

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Applications for 2-nilpotent algebras

- Classification results:
  expansions of $\mathbb{Z}_p \times \mathbb{Z}_q$ [Aichinger, Mayr '07], [Fioravanti '21]
- finite basis results [Mayr, K. '24]
- complexity of computational problems
  CSAT / CEQV [Kawałek, K., Krzaczkowski '24], SMP [K. '24]

Generating sets

$\langle F\rangle_{\mathbf{A},\mathbf{B}} :=$ $(\mathbf{A},\mathbf{B})$-clonoid generated by $F \subseteq \bigcup_{n\geq 1} B^{A^n}$.

$$g(\mathbf{x}) \in \langle F\rangle_{\mathbf{A},\mathbf{B}} \Leftrightarrow g(\mathbf{x}) = \sum_{i=1}^{m} r_i\, f_i(M_i\mathbf{x}),$$

$$f_i \in F, r_i \in \mathbf{R}_{\mathbf{B}}, M_i \in \mathbf{R}_{\mathbf{A}}^{\mathrm{ar}(f_i)\times n}$$

Question: Which clonoids are finitely generated?

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Example: $\mathbf{A} = \mathbf{B} = \mathbb{Z}_p$, p prime

$f(x,y) = x^2y + 1$ $\qquad (\deg(f) = 3).$

$g(x,y,z) = f(x,2z) + f(0,y+z) = 2x^2z + 2$ $\qquad (\deg(g) = 3).$

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- full clonoid not not finitely generated
- full classification: [Kreinecker '20]

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"$\Leftarrow$" True for:

- $\mathbf{A} = \mathbb{F}$ field [Fioravanti '20]

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- $\mathbf{A} = \mathbb{F}$ field [Fioravanti '20]
- $\mathbf{A} = \mathbb{F}_1 \times \mathbb{F}_2 \times \cdots \times \mathbb{F}_m$ for coprime fields $\mathbb{F}_i$ [Fioravanti '21]

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- Con($\mathbf{A}$) is distributive [Mayr, Wynne '24]

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New result [Fioravanti, K., Rossi '25] (unpublished)

Clonoid from $\mathbf{A} = \mathbb{F}^k$ to a coprime module $\mathbf{B}$ are k-generated.

Clonoids from $\mathbb{F}$ to $\mathbf{B}$ [Fioravanti '20]

Lemma

For $f : \mathbb{F}^2 \to \mathbf{B}$ with $f(0,0) = 0$:

$$|F|^{-1} \left(\sum_{a \in \mathbb{F}} f(x + ay, 0) - f(ay, 0) \right) = \begin{cases} f(x,y) \text{ if } y = 0 \\ 0 \text{ else.} \end{cases}$$

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(similar for lines other than $y = 0$)

Consequence

There are $r_M \in \mathbf{R}_B$, for $M \in \mathbb{F}^{2\times 2}$ such that for every $f\colon \mathbb{F}^2 \to \mathbf{B}$:

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- f is generated by unaries
- (1) is uniform in f

Uniformly generated functions

Uniform generation [Mayr, Wynne 24]

B^{A^n} is uniformly generated by k-ary $(\mathbf{A}, \mathbf{B})$-minors if $\exists r_M$ for $M \in \mathbf{R}_{\mathbf{A}}^{n \times n}$:

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- definable for general clones $(\mathcal{A}, \mathcal{B})$
- u.g. for $n = k + 1 \Rightarrow$ u.g. for all $n > k$
- closed under products $\mathcal{A}_1 \times \mathcal{A}_2 \curvearrowright A_1 \times A_2$

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Goal for $\mathbf{A} = \mathbb{F}^k$

Find $r_M \in \mathbf{R_B}$, $M \in \mathbb{F}^{(k+1)\times(k+1)}$:

$$\forall f : \mathbb{F}^{(k+1)\times k} \to B : \; f(X) = \sum_{\mathrm{rk}(M) \leq k} r_M f(MX).$$

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4. transform terms from 3 and sum up. □

Upper bounds

Observation [Fioravanti, K., Rossi '25]

If $\langle B^{A^m} \rangle_{\mathbf{A},\mathbf{B}} = \bigcup_{n \in \mathbb{N}} B^{A^n}$, then $m \geq \frac{\log |A|}{\log |R_A|}$.

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Corollary [Fioravanti, K., Rossi '25]

$(\mathbb{F}^k, \mathbf{B})$-clonoids, for coprime $\mathbb{F}$, $\mathbf{B}$

- are generated by their k-ary functions
- in general not by their $(k-1)$-ary functions.

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Now confirmed for:

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- $\mathbf{A} = \mathbb{F}_1^{k_1} \times \mathbb{F}_2^{k_2} \times \cdots \times \mathbb{F}_n^{k_n} \times \mathbf{A}'$, as $(\mathbb{F}_1 \times \cdots \times \mathbb{F}_n \times \mathbf{R}_{\mathbf{A}'})$-module, with $\mathrm{Con}(\mathbf{A}')$ distributive.

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It is enough to prove:

Conjecture 2

$\mathbf{A}$... abelian p-group

$\mathbf{B}$... coprime abelian group

$\Rightarrow \exists k$: $B^{A^{k+1}}$ uniformly generated by k-ary $(\mathbf{A}, \mathbf{B})$-minors.

Thank you!

Questions? Remarks? Counterexamples?