INTERNATIONAL JOURNAL OF
NUMERICAL ANALYSIS AND MODELING
Volume 2, Number 4, Pages 439–358

ON KORN'S FIRST INEQUALITY FOR QUADRILATERAL NONCONFORMING FINITE ELEMENTS OF FIRST ORDER APPROXIMATION PROPERTIES

PETR KNOBLOCH AND LUTZ TOBISKA

(Communicated by Roger Temam)

Abstract. We investigate the Korn first inequality for quadrilateral nonconforming finite elements of first order approximation properties and clarify the dependence of the constant in this inequality on the discretization parameter h. Then we use the nonconforming elements for approximating the velocity in a discretization of the Stokes equations with boundary conditions involving surface forces and, using the result on the Korn inequality, we prove error estimates which are optimal for the pressure and suboptimal for the velocity.

Key Words. nonconforming finite elements, Korn's inequality, Stokes equations, error estimates.

1. Introduction

Let $\Omega \subset \mathbb{R}^2$ be a bounded domain with a Lipschitz–continuous boundary $\partial\Omega$ and let Γ^D be a measurable subset of $\partial\Omega$ with a positive one–dimensional measure. The Korn first inequality (cf. e.g. [16]) states that there exists a positive constant C such that

$$|\boldsymbol{v}|_{1,\Omega} \leq C\,\|\nabla\boldsymbol{v} + (\nabla\boldsymbol{v})^{\mathsf{T}}\|_{0,\Omega} \qquad \forall\ \boldsymbol{v} \in V\,, \tag{1}$$

where

$$V = \{\boldsymbol{v} \in H^1(\Omega)^2;\ \boldsymbol{v} = \boldsymbol{0}\ \text{ on } \Gamma^D\}\,.$$

This inequality guarantees the coerciveness of the bilinear form which is related to weak formulations of problems from linear elasticity and fluid dynamics in which forces are prescribed on a part of the boundary of the computational domain Ω.

Let $\mathcal{T}_h$ be a triangulation of Ω consisting of shape–regular elements K satisfying the usual compatibility conditions and let V_h be a finite element space build up over $\mathcal{T}_h$ and approximating the space V. Then a discrete analogue of (1) is the inequality

$$\sum_{K\in\mathcal{T}_h} |\boldsymbol{v}_h|^2_{1,K} \leq C_h \sum_{K\in\mathcal{T}_h} \|\nabla\boldsymbol{v}_h + (\nabla\boldsymbol{v}_h)^{\mathsf{T}}\|^2_{0,K} \qquad \forall\ \boldsymbol{v}_h \in V_h\,. \tag{2}$$

For clarity we shall assume that C_h denotes the smallest constant for which the inequality (2) holds. In order to derive optimal convergence results, one usually needs C_h to be bounded from above by a constant C_0 independent of h. Such a constant C_0 always exists in the conforming finite element method where $V_h \subset V$

Received by the editors November 26, 2004.

2000 *Mathematics Subject Classification.* 65N12, 65N30, 65N15.

This research was supported by the Czech Grant Agency under the grant No. 201/05/0005, by the grant MSM 0021620839 and by the Otto–von–Guericke–Universität Magdeburg.

but if V_h is a nonconforming finite element space, then the dependence of C_h on h is not clear. It may even happen that, for some $\boldsymbol{v}_h \in V_h$, the right–hand side of (2) vanishes whereas the left–hand side does not and hence the inequality (2) does not hold for any constant C_h (cf. [10], [1]). Recently it was shown in [2] and [13] that $C_h \le C_0$ holds if, for any $\boldsymbol{v}_h \in V_h$ and for any edge E of $\mathcal{T}_h$ which does not lie on $\partial\Omega \setminus \Gamma^D$, the jump $[|\boldsymbol{v}_h|]_E$ of $\boldsymbol{v}_h$ across E satisfies

$$\int_E [|\boldsymbol{v}_h|]_E \, q \, \mathrm{d}\gamma = \mathbf{0} \qquad \forall\, q \in P_1(E)\,. \tag{3}$$

Unfortunately, this condition does not hold for the most nonconforming finite elements of first order approximation properties. In fact it is known that for many of these elements the inequality $C_h \le C_0$ fails but a detailed analysis of the dependence of C_h on h is not available.

In [9], the case of the nonconforming linear triangular Crouzeix–Raviart element was studied and a modification of the discrete bilinear form a_h from Section 5 was proposed which is uniformly coercive with respect to the discrete H^1 norm. This modification can be used also for the nonconforming quadrilateral finite elements considered below and leads to optimal error estimates. However, it has been observed in numerical experiments with quadrilateral elements of first order approximation properties that the standard discretization, which is simpler to implement, already shows optimal order of convergence. Up to now there is no theoretical explanation for such a behaviour. Note that also the technique of [18] for proving convergence without using (2) cannot be applied in this case since this technique uses the property (3) which is – in general – not satisfied for first order elements.

Therefore, in this paper, we will focus our attention on a theoretical support of the unexpected behaviour of nonconforming quadrilateral finite elements of first order approximation properties observed in numerical calculations. In doing this we shall concentrate on two questions:

a) characterization of the asymptotic behaviour of the constant C_h if $h \to \infty$;
b) convergence properties of discrete solutions in the case $C_h \to \infty$.

In order to cover most of the used first order nonconforming finite element spaces on quadrilaterals, we consider a class of finite element spaces V_h constructed using spaces of the type

$$\operatorname{span}\{1, \widehat{x}, \widehat{y}, \theta(\widehat{x}) - \theta(\widehat{y})\} \tag{4}$$

defined on the reference square. A precise definition of V_h will be given in Section 2. The function θ is usually an even polynomial, e.g.,

$$\theta(x) = x^2\,, \tag{5}$$

$$\theta(x) = x^2 - \frac{5}{3}\,x^4\,, \tag{6}$$

$$\theta(x) = x^2 - \frac{25}{6}\,x^4 + \frac{7}{2}\,x^6\,. \tag{7}$$

The function (5) leads to the rotated bilinear element of [17] and the functions (6), (7) were proposed in [8] in order to improve the properties of the element of [17]. In the literature one can also find nonconforming spaces which contain the above–described space as a subspace, cf. [5], [15].

In what follows we shall consider the model case

$$\Omega = (0,1)^2\,, \qquad \Gamma^D = ([0,1] \times \{0\}) \cup (\{0\} \times [0,1])\,,$$

i.e., Ω is a unit square and Γ^D is formed by the two sides of Ω lying on the coordinate axes. Moreover, for simplicity, we consider uniform triangulations $\mathcal{T}_h$ of Ω consisting of equal squares although rectangles could also be used.

First we show that for nonconforming spaces defined using the reference space (4) and approximating the space V one has $C_h = \mathcal{O}(h^{-2})$. One step in establishing this result will be the proof of the inequality

$$\|\boldsymbol{v}_h\|_{0,\Omega}^2 \leq \bar{C} \sum_{K\in\mathcal{T}_h} \|\nabla \boldsymbol{v}_h + (\nabla \boldsymbol{v}_h)^{\mathsf{T}}\|_{0,K}^2 \qquad \forall\, \boldsymbol{v}_h \in V_h\,,$$

which holds with a constant $\bar{C}$ independent of h. This result excludes the case that the right–hand side of (2) does not define a norm on V_h as it can happen for the nonconforming piecewise linear element (cf. [10], [1]). In particular, the corresponding discrete problems can be solved uniquely for each fixed meshsize.

Then we apply this result to investigate the convergence of finite element solutions of the Stokes equations

$$-\Delta\, \boldsymbol{u} + \nabla\, p = \boldsymbol{f} \quad \text{in } \Omega, \tag{8}$$

$$\operatorname{div} \boldsymbol{u} = 0 \quad \text{in } \Omega, \tag{9}$$

$$\boldsymbol{u} = \boldsymbol{u}_b \quad \text{on } \Gamma^D, \tag{10}$$

$$\boldsymbol{t}\cdot \sigma(\boldsymbol{u},p)\,\boldsymbol{n} = 0 \quad \text{on } \Gamma^N \equiv \partial\Omega \setminus \Gamma^D, \tag{11}$$

$$\boldsymbol{u}\cdot\boldsymbol{n} = 0 \quad \text{on } \Gamma^N. \tag{12}$$

Here $\boldsymbol{u}$ and p are the unknown velocity and pressure, respectively, $\boldsymbol{f}$ is an outer volume force, $\boldsymbol{u}_b$ is a prescribed velocity on Γ^D, $\boldsymbol{n}$ is the outer unit normal vector to $\partial\Omega$, $\boldsymbol{t}$ is a tangent vector to $\partial\Omega$ and $\sigma(\boldsymbol{u},p)$ is the stress tensor defined by

$$\sigma(\boldsymbol{u},p) = -p\, I + 2\, D(\boldsymbol{u})\,, \qquad D(\boldsymbol{u}) = \frac{1}{2}\left(\nabla\boldsymbol{u} + (\nabla\boldsymbol{u})^{\mathsf{T}}\right)$$

with I being the identity tensor. Boundary conditions of the form (10)–(12) appear when a part of the boundary of Ω represents a free surface.

The discretization of the problem (8)–(12) investigated in this paper will be obtained directly from the standard weak formulation of (8)–(12) by replacing the test function spaces by finite element spaces. We shall approximate the velocity $\boldsymbol{u}$ using the space V_h and the pressure p using piecewise constant functions. Since the ellipticity condition for the bilinear form in the discrete problem corresponding to the Laplace operator is equivalent to the inequality (2) and, as we mentioned, the constant C_h in this inequality behaves like h^{-2}, there is apparently little hope to prove any convergence results. Despite these negative expectations we succeeded in proving an optimal convergence of the discrete pressure p_h, namely, $\|p - p_h\|_{0,\Omega} = \mathcal{O}(h)$ and a suboptimal convergence for the discrete velocity $\boldsymbol{u}_h$ in the L^2 norm, namely, $\|\boldsymbol{u} - \boldsymbol{u}_h\|_{0,\Omega} = \mathcal{O}(h)$. Moreover, we were able to prove an optimal L^2 convergence of the derivatives of the non–rotational component of the velocity. To our knowledge, these are first convergence results for discretizations of the type described above. It is surprising that in the literature no deterioration of the convergence rates is reported when using such discretizations. Indeed, in the numerical experiments in [19], the velocity error convergences as $\mathcal{O}(h^2)$ in the L^2 norm and as $\mathcal{O}(h)$ in the H^1 norm.

The paper is organized as follows. In Section 2, we introduce the class of first order nonconforming finite elements on quadrilaterals considered in this paper. In Section 3, we recall an example showing that C_h tends to infinity at least as fast as $\mathcal{O}(h^{-1})$. Moreover, we give a new example improving this result to $\mathcal{O}(h^{-2})$.

Further, in Section 4, we prove an estimate of C_h from above with the conclusion that C_h behaves asymptotically as h^{-2} indeed. Finally, in Section 5, we establish the above–mentioned convergence results for a discretization of the Stokes equations (8)–(12).

Throughout the paper we use a standard notation (cf. e.g. [6]). Particularly, we denote by $\|\cdot\|_{0,G}$ the norm in the Lebesque space $L^2(G)$ and by $\|\cdot\|_{k,G}$ and $|\cdot|_{k,G}$ the norm and seminorm, respectively, in the Sobolev space $H^k(G) \equiv W^{k,2}(G)$, $k \geq 1$. For definitions of this notation in case of fractional Sobolev spaces (non–entire k) we refer to [11]. The notation $L^2_0(G)$ is used for the space of those functions from $L^2(G)$ which have zero mean value over G. As usual, we shall denote by C a generic positive constant independent of h.

2. Definition of the nonconforming space V_h

Let $\widehat{K} = [-1,1]^2$ be the reference square and let $\widehat{E}_1, \ldots, \widehat{E}_4$ be its edges. We denote by $C_{\widehat{E}_i}$ the midpoint of $\widehat{E}_i$, $i = 1, \ldots, 4$, and assume that $C_{\widehat{E}_1} = (1,0)$, $C_{\widehat{E}_2} = (0,1)$, $C_{\widehat{E}_3} = (-1,0)$, and $C_{\widehat{E}_4} = (0,-1)$.

Let $\theta \in C^1(\mathbb{R})$ be an even function satisfying $\theta(0) = 0$ and set

$$\widehat{Q} = \operatorname{span}\{1, \widehat{x}, \widehat{y}, \theta(\widehat{x}) - \theta(\widehat{y})\}\,. \tag{13}$$

We introduce nodal functionals $J_{\widehat{E}_1}, \ldots, J_{\widehat{E}_4}$ on $\widehat{Q}$ defined either by

$$J_{\widehat{E}_i}(\widehat{v}) = \widehat{v}(C_{\widehat{E}_i})\,, \qquad i = 1, \ldots, 4\,,$$

or by

$$J_{\widehat{E}_i}(\widehat{v}) = \frac{1}{2}\int_{\widehat{E}_i} \widehat{v}\,\mathrm{d}\widehat{\gamma}\,, \qquad i = 1, \ldots, 4\,.$$

We denote $\varphi(\widehat{x},\widehat{y}) = \theta(\widehat{x}) - \theta(\widehat{y})$, find

$$J_{\widehat{E}_1}(\varphi) = -J_{\widehat{E}_2}(\varphi) = J_{\widehat{E}_3}(\varphi) = -J_{\widehat{E}_4}(\varphi),$$

and set

$$\kappa = 4\,J_{\widehat{E}_1}(\varphi)\,.$$

We assume that θ is chosen in such a way that $\kappa \neq 0$ which is indeed the case for the three examples mentioned in Section 1. Then the both sets of the nodal functionals are unisolvent with the space $\widehat{Q}$ and we can introduce a basis $\widehat{q}_1, \ldots, \widehat{q}_4$ in $\widehat{Q}$ which is dual to the nodal functionals, i.e.,

$$J_{\widehat{E}_i}(\widehat{q}_j) = \delta_{ij}\,, \qquad i,j = 1, \ldots, 4\,,$$

where δ_{ij} is the Kronecker symbol. The basis functions are given by the following formulas:

$$\begin{aligned}
\widehat{q}_1(\widehat{x},\widehat{y}) &= \frac{1}{4} + \frac{1}{2}\,\widehat{x} + \frac{1}{\kappa}\,(\theta(\widehat{x}) - \theta(\widehat{y}))\,,\\
\widehat{q}_2(\widehat{x},\widehat{y}) &= \frac{1}{4} + \frac{1}{2}\,\widehat{y} - \frac{1}{\kappa}\,(\theta(\widehat{x}) - \theta(\widehat{y}))\,,\\
\widehat{q}_3(\widehat{x},\widehat{y}) &= \frac{1}{4} - \frac{1}{2}\,\widehat{x} + \frac{1}{\kappa}\,(\theta(\widehat{x}) - \theta(\widehat{y}))\,,\\
\widehat{q}_4(\widehat{x},\widehat{y}) &= \frac{1}{4} - \frac{1}{2}\,\widehat{y} - \frac{1}{\kappa}\,(\theta(\widehat{x}) - \theta(\widehat{y}))\,.
\end{aligned}$$

We have

$$|\widehat{q}_1|_{1,\widehat{K}} = |\widehat{q}_2|_{1,\widehat{K}} = |\widehat{q}_3|_{1,\widehat{K}} = |\widehat{q}_4|_{1,\widehat{K}}$$

and define

$$M = |\widehat{q}_i|_{1,\widehat{K}}\,, \qquad i = 1, \dots, 4\,. \tag{14}$$

Let $\mathcal{T}_h$ be a triangulation of Ω consisting of $n \times n$ equal square elements K of edge length $h = 1/n$, where $n \geq 3$. We denote by $\mathcal{E}_h$ the set of all edges E of $\mathcal{T}_h$, by $\mathcal{E}_h^{in}$ the set of the inner edges (i.e., $E \in \mathcal{E}_h^{in} \Leftrightarrow E \not\subset \partial\Omega$), by $\mathcal{E}_h^D$ the set of the edges lying on Γ^D, by $\mathcal{E}_h^N$ the set of the edges lying on Γ^N, by C_E the midpoint of the edge E, and by $\boldsymbol{n}_E$ a fixed unit normal vector to E which corresponds to the outer normal vector $\boldsymbol{n}$ for $E \subset \partial\Omega$. Further, for any inner edge $E \in \mathcal{E}_h^{in}$, we define the jump $[|v|]_E$ of a function v across E by

$$[|v|]_E = (v|_K)|_E - (v|_{\widetilde{K}})|_E\,,$$

where K, $\widetilde{K}$ are the two elements adjacent to E denoted in such a way that $\boldsymbol{n}_E$ points into $\widetilde{K}$. For boundary edges, we simply set

$$[|v|]_E = v|_E\,.$$

For any element K we denote by x_K, y_K the coordinates of its barycentre and we define

$$F_K(\widehat{x}, \widehat{y}) = \left(x_K + \frac{\widehat{x}\,h}{2}, y_K + \frac{\widehat{y}\,h}{2} \right)\,, \qquad (\widehat{x}, \widehat{y}) \in \widehat{K}\,.$$

Then F_K is a one–to–one mapping which maps the reference element $\widehat{K}$ onto K. We denote

$$Q(K) = \{\widehat{v} \circ F_K^{-1};\ \widehat{v} \in \widehat{Q}\}\,,$$

where $\widehat{Q}$ is the space defined in (13). Note that

$$\nabla(\widehat{v} \circ F_K^{-1}) = \frac{2}{h}\,(\widehat{\nabla}\widehat{v}) \circ F_K^{-1} \qquad \forall\, \widehat{v} \in H^1(\widehat{K})\,. \tag{15}$$

This particularly implies that

$$|\widehat{v} \circ F_K^{-1}|_{1,K} = |\widehat{v}|_{1,\widehat{K}} \qquad \forall\, \widehat{v} \in H^1(\widehat{K})\,. \tag{16}$$

Finally, for any edge $E \in \mathcal{E}_h$, we define a nodal functional J_E by

$$J_E(v) = v(C_E) \qquad \text{or} \qquad J_E(v) = \frac{1}{|E|} \int_E v \,\mathrm{d}\gamma$$

in correspondence to the definition of the nodal functionals $J_{\widehat{E}_i}$.

Now, the nonconforming space V_h approximating the space V is defined by

$$V_h = \{\boldsymbol{v}_h \in L^2(\Omega)^2;\ \boldsymbol{v}_h|_K \in Q(K)^2\ \ \forall\, K \in \mathcal{T}_h,\ \ J_E([|\boldsymbol{v}_h|]_E) = \boldsymbol{0}\ \ \forall\, E \in \mathcal{E}_h^{in} \cup \mathcal{E}_h^D\}\,.$$

The condition $J_E([|\boldsymbol{v}_h|]_E) = \boldsymbol{0}$ implies that $J_E(\boldsymbol{v}_h|_K) = J_E(\boldsymbol{v}_h|_{\widetilde{K}})$ for any inner edge E, where K, $\widetilde{K}$ are the two elements adjacent to E. We shall denote this value simply by $J_E(\boldsymbol{v}_h)$, which is the vector–valued degree of freedom of $\boldsymbol{v}_h$ associated with E. Note that the degrees of freedom associated with boundary edges lying on Γ^D vanish for any function from the space V_h.

3. Counterexamples to the uniform validity of (2)

Since the square roots of both sides of the inequality (2) represent norms on the space V_h, it follows from the equivalence of norms on finite–dimensional spaces that, for any h, there exists a positive constant C_h such that the inequality (2) holds. However, the constants C_h cannot be bounded from above by a constant independent of h, which will be shown in this section by constructing suitable counterexamples.

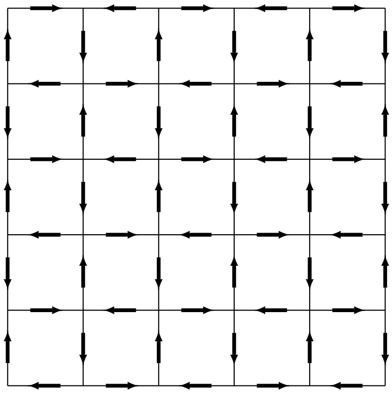

FIGURE 1. Degrees of freedom of $\widetilde{\boldsymbol{v}}_h$.

For any $K \in \mathcal{T}_h$, we define a function

$$\boldsymbol{v}_K(x,y) = \frac{2}{h}\,(y - y_K, x_K - x)$$

which represents a vortex in the clockwise direction around the barycentre of K. Note that

$$\nabla \boldsymbol{v}_K + (\nabla \boldsymbol{v}_K)^\mathsf{T} = 0 \qquad \text{and} \qquad |\boldsymbol{v}_K|_{1,K} = 2\sqrt{2}\,. \tag{17}$$

Let $\mathcal{T}_h^1$ and $\mathcal{T}_h^2$ be disjoint subsets of $\mathcal{T}_h$ representing a checkerboard decomposition of $\mathcal{T}_h$ and let $\mathcal{T}_h^1$ contain the lower left element of $\mathcal{T}_h$. We define a piecewise linear function $\widetilde{\boldsymbol{v}}_h$ by

$$\widetilde{\boldsymbol{v}}_h|_K = \boldsymbol{v}_K \qquad \forall\ K \in \mathcal{T}_h^1\,, \qquad\qquad \widetilde{\boldsymbol{v}}_h|_K = -\boldsymbol{v}_K \qquad \forall\ K \in \mathcal{T}_h^2\,.$$

Then $\widetilde{\boldsymbol{v}}_h$ has jumps across all inner edges but it is continuous at the midpoints of inner edges. Thus,

$$J_E([|\widetilde{\boldsymbol{v}}_h|]_E) = \mathbf{0} \qquad \forall\ E \in \mathcal{E}_h^{in}$$

for both definitions of J_E. The values of $\widetilde{\boldsymbol{v}}_h$ at the midpoints of edges are depicted in Fig. 1 for a triangulation consisting of 5×5 elements. For each edge E, the value $\widetilde{\boldsymbol{v}}_h(C_E) = J_E(\widetilde{\boldsymbol{v}}_h)$ is parallel to E and of magnitude 1.

Let us define a function $\boldsymbol{v}_h \in V_h$ by

$$J_E(\boldsymbol{v}_h) = J_E(\widetilde{\boldsymbol{v}}_h) \quad \forall\ E \in \mathcal{E}_h^{in}\,, \qquad\qquad J_E(\boldsymbol{v}_h) = \mathbf{0} \quad \forall\ E \in \mathcal{E}_h\,,\ E \subset \partial\Omega\,.$$

Then $\boldsymbol{v}_h = \widetilde{\boldsymbol{v}}_h$ on all elements which do not intersect the boundary of Ω and hence it follows from (17), (16) and (14) that

$$\sum_{K \in \mathcal{T}_h} \|\nabla \boldsymbol{v}_h + (\nabla \boldsymbol{v}_h)^\mathsf{T}\|_{0,K}^2 = \sum_{K \in \mathcal{T}_h,\, K \cap \partial\Omega \neq \emptyset} \|\nabla \boldsymbol{v}_h + (\nabla \boldsymbol{v}_h)^\mathsf{T}\|_{0,K}^2 \le 16\, M^2\, n\,,$$

$$\sum_{K \in \mathcal{T}_h} |\boldsymbol{v}_h|_{1,K}^2 \ge \sum_{K \in \mathcal{T}_h,\, K \cap \partial\Omega = \emptyset} |\boldsymbol{v}_h|_{1,K}^2 = 8\,(n-2)^2\,.$$

This shows that the constant C_h from (2) has to satisfy $C_h \ge (n-2)^2/(2\,M^2\,n) \ge h^{-1}/(18\,M^2)$. The above function $\boldsymbol{v}_h$ is basically the counterexample from [13]. It is also similar to the counterexample of [10] used for the Crouzeix–Raviart element. In the following, we construct another counterexample which even shows that $C_h \ge C\,h^{-2}$.

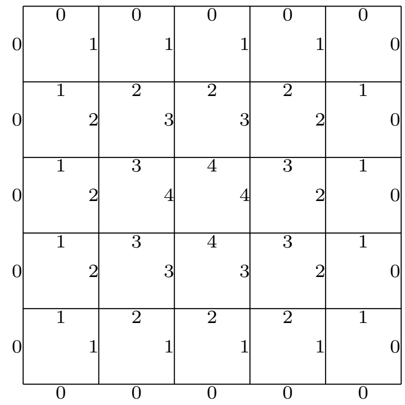

FIGURE 2. Decomposition of the set of the edges.

We decompose the set $\mathcal{E}_h$ of the edges of $\mathcal{T}_h$ into the sets $\mathcal{E}_h^0, \dots, \mathcal{E}_h^{n-1}$ defined in the following way:

$$\mathcal{E}_h^0 = \{E \in \mathcal{E}_h;\ E \subset \partial\Omega\}\,,$$

$$\mathcal{E}_h^i = \{E \in \mathcal{E}_h \setminus \bigcup_{j=0}^{i-1} \mathcal{E}_h^j;\ \exists\, E' \in \mathcal{E}_h^{i-1} : \ E \cap E' \neq \emptyset \wedge E \perp E'\},\ i = 1, \dots, n-1\,.$$

Then

$$\operatorname{card} \mathcal{E}_h^i = 4\,(n-i)\,, \qquad i = 0, \dots, n-1\,.$$

The decomposition of $\mathcal{E}_h$ is depicted in Fig. 2 for a triangulation consisting of 5×5 elements.

Now we introduce a function $\overline{\boldsymbol{v}}_h \in V_h$ satisfying

$$J_E(\overline{\boldsymbol{v}}_h) = i\, J_E(\widetilde{\boldsymbol{v}}_h) \qquad \forall\ E \in \mathcal{E}_h^i,\ i = 0, \dots, n-1\,.$$

Consider any $K \in \mathcal{T}_h$. Then there exists $i = i_K \in \{1, \dots, n-2\}$ such that all four edges of K belong to $\mathcal{E}_h^{i-1} \cup \mathcal{E}_h^i \cup \mathcal{E}_h^{i+1}$. Moreover, either two edges of K belong to $\mathcal{E}_h^i$ or all edges of K belong to $\mathcal{E}_h^{i+1} = \mathcal{E}_h^{n-1}$. Set

$$\boldsymbol{u}_K = \overline{\boldsymbol{v}}_h|_K - i_K\, \widetilde{\boldsymbol{v}}_h|_K\,.$$

Then $|J_E(\boldsymbol{u}_K)| \leq 1$ for any edge $E \subset \partial K$. In addition, either $J_E(\boldsymbol{u}_K) = \boldsymbol{0}$ for two edges of K or $\boldsymbol{u}_K = \widetilde{\boldsymbol{v}}_h|_K$. Thus, it follows from (16), (14) and (17) that

$$|\boldsymbol{u}_K|_{1,K} \leq 2\,M \qquad \text{or} \qquad \nabla \boldsymbol{u}_K + (\nabla \boldsymbol{u}_K)^{\mathsf{T}} = 0\,.$$

Consequently, in view of (17), we get

$$\sum_{K \in \mathcal{T}_h} \|\nabla \overline{\boldsymbol{v}}_h + (\nabla \overline{\boldsymbol{v}}_h)^{\mathsf{T}}\|_{0,K}^2 = \sum_{K \in \mathcal{T}_h} \|\nabla \boldsymbol{u}_K + (\nabla \boldsymbol{u}_K)^{\mathsf{T}}\|_{0,K}^2 \leq 16\, M^2\, n^2\,. \tag{18}$$

For any $K \in \mathcal{T}_h$, the function $\overline{\boldsymbol{v}}_h \circ F_K$ has the form

$$\overline{\boldsymbol{v}}_h \circ F_K = \alpha_1\,(0,-1)\,\widehat{q}_1 + \alpha_2\,(1,0)\,\widehat{q}_2 + \alpha_3\,(0,1)\,\widehat{q}_3 + \alpha_4\,(-1,0)\,\widehat{q}_4$$

with some real numbers $\alpha_1, \dots, \alpha_4$. Since $|\cdot|_{1,\widehat{K}}$ is a norm on the spaces $\operatorname{span}\{\widehat{q}_1, \widehat{q}_3\}$ and $\operatorname{span}\{\widehat{q}_2, \widehat{q}_4\}$, it follows from the equivalence of norms on finite–dimensional spaces that there exists a positive constant L such that

$$|\overline{\boldsymbol{v}}_h \circ F_K|_{1,\widehat{K}}^2 = |\alpha_1\,\widehat{q}_1 - \alpha_3\,\widehat{q}_3|_{1,\widehat{K}}^2 + |\alpha_2\,\widehat{q}_2 - \alpha_4\,\widehat{q}_4|_{1,\widehat{K}}^2 \geq L \sum_{i=1}^{4} \alpha_i^2\,.$$

Thus, we deduce using (16) that

$$\begin{aligned}\sum_{K\in\mathcal{T}_h} |\overline{\boldsymbol{v}}_h|^2_{1,K} \quad &\ge \quad 2\,L \sum_{E\in\mathcal{E}_h} |J_E(\overline{\boldsymbol{v}}_h)|^2 = 2\,L \sum_{i=1}^{n-1} \sum_{E\in\mathcal{E}_h^i} |J_E(\overline{\boldsymbol{v}}_h)|^2 \\ &= \quad 8\,L \sum_{i=1}^{n-1} (n-i)\,i^2 = \frac{2}{3}\,L\,(n^4-n^2) \ge \frac{1}{2}\,L\,n^4\,.\end{aligned}$$

This implies that the constant C_h in (2) has to satisfy $C_h \ge h^{-2}\,L/(32\,M^2)$.

4. Estimate of the constant C_h from above

In this section we prove that there exists a constant C_0 independent of h such that the constant C_h from the inequality (2) satisfies $C_h \le C_0\,h^{-2}$. The proof will be carried out by rewriting the right–hand side of (2) in terms of the degrees of freedom of $\boldsymbol{v}_h$ and by estimating the resulting sum from below by the sum of the squares of the degrees of freedom multiplied by h^2. This expression is equivalent to $\|\boldsymbol{v}_h\|^2_{0,\Omega}$ and hence also larger than $C\,h^2 \sum_{K\in\mathcal{T}_h} |\boldsymbol{v}_h|^2_{1,K}$ due to an inverse inequality.

First let us investigate the terms from the right–hand side of (2) on the reference element. Let $\boldsymbol{w}_i \equiv (u_i, v_i) \in \mathbb{R}^2$, $i = 1,\dots,4$, be arbitrary and set

$$\widehat{\boldsymbol{v}} = \sum_{i=1}^{4} \boldsymbol{w}_i\,\widehat{q}_i\,, \qquad u = \widehat{\boldsymbol{v}}\cdot(1,0)\,, \qquad v = \widehat{\boldsymbol{v}}\cdot(0,1)\,.$$

Then

$$\|\widehat{\nabla}\widehat{\boldsymbol{v}} + (\widehat{\nabla}\widehat{\boldsymbol{v}})^{\mathsf{T}}\|^2_{0,\widehat{K}} = \int_{\widehat{K}} 4\left(\frac{\partial u}{\partial\widehat{x}}\right)^2 + 2\left(\frac{\partial u}{\partial\widehat{y}} + \frac{\partial v}{\partial\widehat{x}}\right)^2 + 4\left(\frac{\partial v}{\partial\widehat{y}}\right)^2 \mathrm{d}\widehat{x}\,\mathrm{d}\widehat{y}\,.$$

In view of the definition of the functions $\widehat{q}_1,\dots,\widehat{q}_4$, we have

$$\begin{aligned}\frac{\partial u}{\partial\widehat{x}} &= \frac{1}{2}\,(u_1-u_3) + \frac{1}{\kappa}\,(u_1-u_2+u_3-u_4)\,\theta'(\widehat{x})\,,\\ \frac{\partial v}{\partial\widehat{y}} &= \frac{1}{2}\,(v_2-v_4) - \frac{1}{\kappa}\,(v_1-v_2+v_3-v_4)\,\theta'(\widehat{y})\,,\\ \frac{\partial u}{\partial\widehat{y}} + \frac{\partial v}{\partial\widehat{x}} &= \frac{1}{2}\,(u_2-u_4+v_1-v_3) - \frac{1}{\kappa}\,(u_1-u_2+u_3-u_4)\,\theta'(\widehat{y})\\ &\qquad + \frac{1}{\kappa}\,(v_1-v_2+v_3-v_4)\,\theta'(\widehat{x})\,.\end{aligned}$$

Using the fact that θ' is odd and denoting

$$\varepsilon = \frac{2}{\kappa^2}\int_{-1}^{1} [\theta'(x)]^2\,\mathrm{d}x\,,$$

we easily derive that

$$\begin{aligned}\text{(19)}\qquad \frac{1}{2}\,\|\widehat{\nabla}\widehat{\boldsymbol{v}} + (\widehat{\nabla}\widehat{\boldsymbol{v}})^{\mathsf{T}}\|^2_{0,\widehat{K}} &= 2\,(u_1-u_3)^2 + 2\,(v_2-v_4)^2 + (u_2-u_4+v_1-v_3)^2\\ &\quad + 3\,\varepsilon\,(u_2+u_4-u_1-u_3)^2\\ &\quad + 3\,\varepsilon\,(v_2+v_4-v_1-v_3)^2\,.\end{aligned}$$

To apply the identity (19) to the right–hand side of (2), we have to introduce a suitable numbering of the edges of the triangulation $\mathcal{T}_h$. First, we number the elements $K \in \mathcal{T}_h$ by indices $i, j = 1,\dots,n$, where i corresponds to the horizontal

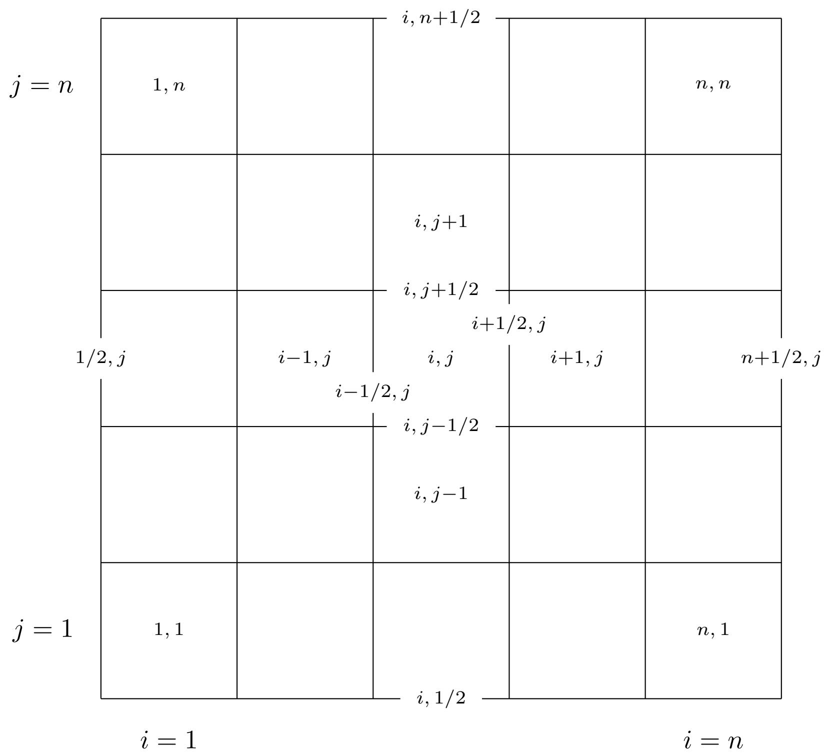

FIGURE 3. Numbering of the elements and edges.

direction and j corresponds to the vertical direction, see Fig. 3. The indices of the edges of the triangulation are defined as the averages of the indices of the two elements adjacent to the respective edge. For edges lying on the boundary of Ω, the indices are also defined in this way imagining that the triangulation continues outside Ω, cf. Fig. 3.

Now, for a given function $\boldsymbol{v}_h \in V_h$, we denote

$$(u_{i,j+1/2}, v_{i,j+1/2}) = J_{E_{i,j+1/2}}(\boldsymbol{v}_h)\,, \qquad i = 1, \dots, n\,, \quad j = 0, \dots, n\,,$$
$$(u_{i+1/2,j}, v_{i+1/2,j}) = J_{E_{i+1/2,j}}(\boldsymbol{v}_h)\,, \qquad i = 0, \dots, n\,, \quad j = 1, \dots, n\,.$$

Recall that the degrees of freedom associated with boundary edges lying on Γ^D vanish, i.e.,

$$u_{i,1/2} = v_{i,1/2} = 0\,, \quad i = 1, \dots, n\,, \qquad u_{1/2,j} = v_{1/2,j} = 0\,, \quad j = 1, \dots, n\,. \tag{20}$$

Applying (15) and (19), we obtain for any $i, j \in \{1, \dots, n\}$

$$\begin{aligned} \frac{1}{2}\,\|\nabla \boldsymbol{v}_h + (\nabla \boldsymbol{v}_h)^{\mathsf{T}}\|^2_{0,K_{ij}} = \;& 2\,(u_{i+1/2,j} - u_{i-1/2,j})^2 + 2\,(v_{i,j+1/2} - v_{i,j-1/2})^2 \\ & + (u_{i,j+1/2} - u_{i,j-1/2} + v_{i+1/2,j} - v_{i-1/2,j})^2 \\ & + 3\,\varepsilon\,(u_{i,j+1/2} + u_{i,j-1/2} - u_{i+1/2,j} - u_{i-1/2,j})^2 \\ & + 3\,\varepsilon\,(v_{i,j+1/2} + v_{i,j-1/2} - v_{i+1/2,j} - v_{i-1/2,j})^2\,. \end{aligned} \tag{21}$$

For investigating the sum of the right–hand sides of (21) over all indices $i, j \in \{1, \dots, n\}$, we shall need the following simple lemma.

Lemma 1. *Consider any $n \in \mathbb{N}$ and let $\alpha_1, \dots, \alpha_n$ be arbitrary real numbers. Set $\alpha_0 = 0$. Then*

$$\sum_{i=1}^{n} (\alpha_i - \alpha_{i-1})^2 \geq \frac{1}{n^2} \sum_{i=1}^{n} \alpha_i^2 \,. \tag{22}$$

Proof. For any $j \in \{1, \dots, n\}$, we have

$$\alpha_j = \sum_{i=1}^{j} (\alpha_i - \alpha_{i-1})$$

and hence

$$\alpha_j^2 \leq j \sum_{i=1}^{j} (\alpha_i - \alpha_{i-1})^2 \,.$$

Thus,

$$\sum_{j=1}^{n} \alpha_j^2 \leq \frac{n(n+1)}{2} \sum_{i=1}^{n} (\alpha_i - \alpha_{i-1})^2 \,,$$

which implies (22). □

Remark 1. Setting $\alpha_i = i$ for $i = 1, \dots, n$, we realize that the factor $1/n^2$ in (22) is optimal.

First let us analyze the sum of the right–hand sides of (21) for $\varepsilon = 1/3$ (this value of ε corresponds to θ defined by (5) if midpoint–oriented degrees of freedom are used). The following result is crucial for proving the desired Korn inequality.

Lemma 2. *Let $u_{i,j+1/2}$, $v_{i,j+1/2}$ with $i = 1, \dots, n$, $j = 0, \dots, n$, and $u_{i+1/2,j}$, $v_{i+1/2,j}$ with $i = 0, \dots, n$, $j = 1, \dots, n$ be arbitrary real numbers satisfying (20). Then*

$$\begin{aligned} \sum_{i,j=1}^{n} \{ & 2\,(u_{i+1/2,j} - u_{i-1/2,j})^2 + 2\,(v_{i,j+1/2} - v_{i,j-1/2})^2 \\ & + (u_{i,j+1/2} - u_{i,j-1/2} + v_{i+1/2,j} - v_{i-1/2,j})^2 \\ & + (u_{i,j+1/2} + u_{i,j-1/2} - u_{i+1/2,j} - u_{i-1/2,j})^2 \\ & + (v_{i,j+1/2} + v_{i,j-1/2} - v_{i+1/2,j} - v_{i-1/2,j})^2 \} \\ \geq \frac{1}{18\,n^2} \sum_{i,j=1}^{n} & \{ u_{i,j+1/2}^2 + v_{i,j+1/2}^2 + u_{i+1/2,j}^2 + v_{i+1/2,j}^2 \} \,. \end{aligned} \tag{23}$$

Proof. Let us denote for $i, j = 1, \dots, n$

$$\begin{aligned} a_{ij} &= u_{i+1/2,j} - u_{i-1/2,j} \,, \\ b_{ij} &= v_{i,j+1/2} - v_{i,j-1/2} \,, \\ c_{ij} &= u_{i,j+1/2} - u_{i,j-1/2} + v_{i+1/2,j} - v_{i-1/2,j} \,, \\ d_{ij} &= u_{i,j+1/2} + u_{i,j-1/2} - u_{i+1/2,j} - u_{i-1/2,j} \,, \\ e_{ij} &= v_{i,j+1/2} + v_{i,j-1/2} - v_{i+1/2,j} - v_{i-1/2,j} \,, \\ s_{ij} &= 2\,a_{ij}^2 + 2\,b_{ij}^2 + c_{ij}^2 + d_{ij}^2 + e_{ij}^2 \,. \end{aligned}$$

Then the left–hand side of (23) is $\sum_{i,j=1}^{n} s_{ij}$.

For any real numbers a, b, we have

$$a^2 + b^2 \geq \frac{1}{2}\,(a+b)^2$$

and hence, for any real numbers a, b, c, d, e, we derive

$$2\,a^2+2\,b^2+c^2+d^2+e^2 \ge \frac{1}{2}\,(a+b+c)^2+\frac{1}{2}\,(d+e)^2 \ge \frac{1}{4}\,(a+b+c+d+e)^2\,.$$

Consequently,

$$s_{ij} \ge \frac{1}{4}\,(a_{ij}+b_{ij}+c_{ij}+d_{ij}+e_{ij})^2 = (u_{i,j+1/2}+v_{i,j+1/2}-u_{i-1/2,j}-v_{i-1/2,j})^2 \tag{24}$$

and

$$s_{ij} \ge \frac{1}{4}\,(a_{ij}+b_{ij}+c_{ij}-d_{ij}-e_{ij})^2 = (u_{i,j-1/2}+v_{i,j-1/2}-u_{i+1/2,j}-v_{i+1/2,j})^2\,. \tag{25}$$

We denote

$$\begin{aligned} f_{i,j+1/2} &= u_{i,j+1/2}+v_{i,j+1/2}\,, \qquad i=1,\dots,n\,, \quad j=0,\dots,n\,,\\ f_{i+1/2,j} &= u_{i+1/2,j}+v_{i+1/2,j}\,, \qquad i=0,\dots,n\,, \quad j=1,\dots,n\,. \end{aligned}$$

Then, according to (20),

$$f_{i,1/2}=0\,, \quad i=1,\dots,n\,, \qquad f_{1/2,j}=0\,, \quad j=1,\dots,n\,. \tag{26}$$

For convenience we further set

$$f_{0,j+1/2}=0\,, \quad j=0,\dots,n\,, \qquad f_{i+1/2,0}=0\,, \quad i=0,\dots,n\,. \tag{27}$$

It follows from (24) and (25) that

$$2\,s_{ij} \ge (f_{i,j+1/2}-f_{i-1/2,j})^2+(f_{i,j-1/2}-f_{i+1/2,j})^2\,, \qquad i,j=1,\dots,n\,.$$

In view of (26) and (27) we have

$$\begin{aligned} \sum_{i,j=1}^{n}(f_{i,j+1/2}-f_{i-1/2,j})^2 &= \sum_{\substack{i,j=1\\ i-j\le 0}}^{n}(f_{i,j+1/2}-f_{i-1/2,j})^2+\sum_{\substack{i,j=1\\ i-j\ge 0}}^{n}(f_{i,j-1/2}-f_{i-1/2,j-1})^2\\ &= \sum_{k=1-n}^{0}\sum_{i=1}^{n+k}(f_{i,i-k+1/2}-f_{i-1/2,i-k})^2+\sum_{k=0}^{n-1}\sum_{i=1+k}^{n}(f_{i,i-k-1/2}-f_{i-1/2,i-k-1})^2\,, \end{aligned}$$

where now the summation runs over diagonals. Similarly,

$$\begin{aligned} \sum_{i,j=1}^{n}(f_{i,j-1/2}-f_{i+1/2,j})^2 &= \sum_{\substack{i,j=1\\ i-j\le 0}}^{n}(f_{i-1,j-1/2}-f_{i-1/2,j})^2+\sum_{\substack{i,j=1\\ i-j\ge 0}}^{n}(f_{i,j-1/2}-f_{i+1/2,j})^2\\ &= \sum_{k=1-n}^{0}\sum_{i=1}^{n+k}(f_{i-1,i-k-1/2}-f_{i-1/2,i-k})^2+\sum_{k=0}^{n-1}\sum_{i=1+k}^{n}(f_{i,i-k-1/2}-f_{i+1/2,i-k})^2\,. \end{aligned}$$

Thus,

$$\begin{aligned} 2\sum_{i,j=1}^{n}s_{ij} \ge{}& \sum_{k=1-n}^{0}\sum_{i=1}^{n+k}\{(f_{i,i-k+1/2}-f_{i-1/2,i-k})^2+(f_{i-1/2,i-k}-f_{i-1,i-k-1/2})^2\}\\ &+\sum_{k=0}^{n-1}\sum_{i=1+k}^{n}\{(f_{i+1/2,i-k}-f_{i,i-k-1/2})^2+(f_{i,i-k-1/2}-f_{i-1/2,i-k-1})^2\}\,. \end{aligned}$$

Applying Lemma 1 to each inner sum on the right–hand side of this inequality, we obtain

$$\begin{aligned} 2\sum_{i,j=1}^{n} s_{ij} \quad &\ge \quad \sum_{k=1-n}^{0} \frac{1}{4\,(n+k)^2} \sum_{i=1}^{n+k} \{f_{i,i-k+1/2}^2 + f_{i-1/2,i-k}^2\} \\ &\quad + \sum_{k=0}^{n-1} \frac{1}{4\,(n-k)^2} \sum_{i=1+k}^{n} \{f_{i+1/2,i-k}^2 + f_{i,i-k-1/2}^2\} \\ &\ge \quad \frac{1}{4\,n^2} \sum_{\substack{i,j=1 \\ i-j\le 0}}^{n} \{f_{i,j+1/2}^2 + f_{i-1/2,j}^2\} + \frac{1}{4\,n^2} \sum_{\substack{i,j=1 \\ i-j\ge 0}}^{n} \{f_{i+1/2,j}^2 + f_{i,j-1/2}^2\}\,. \end{aligned}$$

Due to (26) we have

$$\sum_{\substack{i,j=1 \\ i-j\le 0}}^{n} f_{i-1/2,j}^2 = \sum_{\substack{i,j=1 \\ i-j< 0}}^{n} f_{i+1/2,j}^2\,, \qquad \sum_{\substack{i,j=1 \\ i-j\ge 0}}^{n} f_{i,j-1/2}^2 = \sum_{\substack{i,j=1 \\ i-j> 0}}^{n} f_{i,j+1/2}^2\,.$$

This implies that

$$\sum_{i,j=1}^{n} s_{ij} \ge \frac{1}{8\,n^2} \sum_{i,j=1}^{n} \{f_{i,j+1/2}^2 + f_{i+1/2,j}^2\}\,. \tag{28}$$

To obtain another lower bound of the left–hand side of (23), we observe that (20) and Lemma 1 imply

$$\sum_{i=1}^{n} (u_{i+1/2,j} - u_{i-1/2,j})^2 \ge \frac{1}{n^2} \sum_{i=1}^{n} u_{i+1/2,j}^2\,, \qquad j = 1, \dots, n\,.$$

Similarly, we derive

$$\sum_{j=1}^{n} (v_{i,j+1/2} - v_{i,j-1/2})^2 \ge \frac{1}{n^2} \sum_{j=1}^{n} v_{i,j+1/2}^2\,, \qquad i = 1, \dots, n\,.$$

Thus, we have

$$\sum_{i,j=1}^{n} s_{ij} \ge \frac{2}{n^2} \sum_{i,j=1}^{n} \{u_{i+1/2,j}^2 + v_{i,j+1/2}^2\}\,. \tag{29}$$

Using the fact that for any $i, j \in \{1, \dots, n\}$

$$f_{i,j+1/2}^2 + v_{i,j+1/2}^2 \ge \frac{1}{2}\, u_{i,j+1/2}^2\,, \qquad f_{i+1/2,j}^2 + u_{i+1/2,j}^2 \ge \frac{1}{2}\, v_{i+1/2,j}^2\,,$$

and combining (28) and (29), we obtain the lemma. □

As corollaries of the above lemma we can now state the following two Korn inequalities representing the main result of this section.

Theorem 1. *We have*

$$\|\boldsymbol{v}_h\|_{0,\Omega}^2 \leq \bar{C} \sum_{K\in\mathcal{T}_h} \|\nabla \boldsymbol{v}_h + (\nabla \boldsymbol{v}_h)^\mathsf{T}\|_{0,K}^2 \qquad \forall\ \boldsymbol{v}_h \in V_h\,, \tag{30}$$

where $\bar{C} = 18\,\|\widehat{q}_1\|_{0,\widehat{K}}^2 / \min\{1, 3\,\varepsilon\}$.

Proof. For any $\boldsymbol{v}_h \in V_h$ and any $K \in \mathcal{T}_h$, one easily derives that

$$\|\boldsymbol{v}_h\|_{0,K}^2 \leq h^2\,\|\widehat{q}_1\|_{0,\widehat{K}}^2 \sum_{E\subset\partial K} |J_E(\boldsymbol{v}_h)|^2\,.$$

Summing up these inequalities over all elements $K \in \mathcal{T}_h$ and applying (21) and (23), we obtain (30). □

Theorem 2. *There exists a constant* C_0 *independent of* h *such that*

$$\sum_{K\in\mathcal{T}_h} |\boldsymbol{v}_h|_{1,K}^2 \leq C_0\,h^{-2} \sum_{K\in\mathcal{T}_h} \|\nabla \boldsymbol{v}_h + (\nabla \boldsymbol{v}_h)^\mathsf{T}\|_{0,K}^2 \qquad \forall\ \boldsymbol{v}_h \in V_h\,. \tag{31}$$

Proof. It follows from the equivalence of norms on finite–dimensional spaces that there exists a constant $\widehat{C}$ such that, for any $K \in \mathcal{T}_h$ and any $\boldsymbol{v}_h \in V_h$,

$$|\boldsymbol{v}_h|_{1,K} = |\boldsymbol{v}_h \circ F_K|_{1,\widehat{K}} \leq \widehat{C}\,\|\boldsymbol{v}_h \circ F_K\|_{0,\widehat{K}} = 2\,\widehat{C}\,h^{-1}\,\|\boldsymbol{v}_h\|_{0,K}\,.$$

Thus, we have

$$\sum_{K\in\mathcal{T}_h} |\boldsymbol{v}_h|_{1,K}^2 \leq 4\,\widehat{C}^2\,h^{-2}\,\|\boldsymbol{v}_h\|_{0,\Omega}^2$$

and the theorem follows from (30). □

The results of Section 3 show that the estimate (31) is sharp.

5. Convergence analysis for a nonconforming finite element discretization of the Stokes equations

In this section we analyze the convergence of the discrete solutions of the Stokes equations (8)–(12) discretized using a suitable subspace of the space V_h introduced in Section 2. We shall only consider the mean–value–oriented degrees of freedom, i.e.,

$$J_E(v) = \frac{1}{|E|} \int_E v\,\mathrm{d}\gamma\,.$$

The subspace of V_h will be used for approximating the velocity $\boldsymbol{u}$ whereas the pressure p will be approximated by piecewise constant functions from the space

$$Q_h = \{q_h \in L_0^2(\Omega);\ q_h|_K \in P_0(K)\ \ \forall\, K \in \mathcal{T}_h\}\,.$$

We shall assume that the data of the problem (8)–(12) satisfy $\boldsymbol{f} \in L^2(\Omega)^2$ and

$$\boldsymbol{u}_b \cdot \boldsymbol{n} = 0 \quad \text{on } \Gamma^N\,, \qquad\qquad \int_{\Gamma^D} \boldsymbol{u}_b \cdot \boldsymbol{n}\,\mathrm{d}\gamma = 0\,.$$

Moreover, from now on, $\boldsymbol{u}_b$ will denote a lifting of the boundary condition satisfying $\boldsymbol{u}_b \in H^1(\Omega)^2$.

To introduce a weak formulation of (8)–(12), we first define a test function space

$$W = \{\boldsymbol{v} \in V;\ \boldsymbol{v}\cdot\boldsymbol{n} = 0 \ \text{ on } \Gamma^N\}\,.$$

Then the standard weak formulation of (8)–(12) reads: Find $\boldsymbol{u} \in H^1(\Omega)^2$ and $p \in L^2_0(\Omega)$ such that $\boldsymbol{u} - \boldsymbol{u}_b \in W$ and

$$a(\boldsymbol{u},\boldsymbol{v}) + b(\boldsymbol{v},p) - b(\boldsymbol{u},q) = (\boldsymbol{f},\boldsymbol{v}) \qquad \forall\, \boldsymbol{v} \in W,\, q \in L^2_0(\Omega)\,.$$

Here, $(\cdot,\cdot)$ denotes the usual inner product in $L^2(\Omega)^2$ and

$$a(\boldsymbol{u},\boldsymbol{v}) = 2\,(D(\boldsymbol{u}),D(\boldsymbol{v}))\,, \qquad b(\boldsymbol{v},p) = -(p,\operatorname{div}\boldsymbol{v})\,.$$

Since the Korn first inequality (1) implies that the bilinear form a is W–elliptic and since the spaces W and $L^2_0(\Omega)$ satisfy an inf–sup condition (see [12]), it can be proved that there always exists a unique weak solution of (8)–(12) (cf. e.g. [4] or [12]). In what follows we shall still assume that this solution possesses at least the regularity $\boldsymbol{u} \in H^2(\Omega)^2$, $p \in H^1(\Omega)$, which implies that the functions $\boldsymbol{u}$, p satisfy the equations (8)–(12) almost everywhere.

For piecewise H^1 vector–valued functions $\boldsymbol{v}$, we define the 'elementwise' differential operators D_h and div_h by

$$D_h(\boldsymbol{v})|_K = \frac{1}{2}\Big(\nabla(\boldsymbol{v}|_K) + (\nabla\boldsymbol{v}|_K)^{\mathsf{T}}\Big)\,, \qquad (\operatorname{div}_h\boldsymbol{v})|_K = \operatorname{div}(\boldsymbol{v}|_K) \qquad \forall\, K \in \mathcal{T}_h$$

and we set

$$a_h(\boldsymbol{u},\boldsymbol{v}) = 2\,(D_h(\boldsymbol{u}),D_h(\boldsymbol{v}))\,, \qquad b_h(\boldsymbol{v},p) = -(p,\operatorname{div}_h\boldsymbol{v})\,.$$

Further, we define a discrete analogue of $|\cdot|_{1,\Omega}$ by

$$|\boldsymbol{v}|_{1,h} = \left(\sum_{K\in\mathcal{T}_h} |\boldsymbol{v}|^2_{1,K}\right)^{1/2}\,.$$

Obviously, it is a norm on V_h and the discrete Korn inequality (31) assures that, for each h, the bilinear form a_h is V_h–elliptic with respect to this norm.

We introduce the spaces

$$\begin{aligned} W_h &= \{\boldsymbol{v}_h \in V_h;\ J_E(\boldsymbol{v}_h)\cdot\boldsymbol{n}|_E = 0 \ \ \forall\, E \in \mathcal{E}^N_h\}\,,\\ \widetilde{W}_h &= \{\boldsymbol{v}_h \in L^2(\Omega)^2;\ \boldsymbol{v}_h|_K \in Q(K)^2 \ \ \forall\, K \in \mathcal{T}_h,\ \ J_E([|\boldsymbol{v}_h|]_E) = \mathbf{0} \ \ \forall\, E \in \mathcal{E}^{in}_h\} \end{aligned}$$

and an interpolation operator $i_h : H^1(\Omega)^2 \to \widetilde{W}_h$ such that

$$J_E(i_h\,\boldsymbol{v}) = J_E(\boldsymbol{v}) \qquad \forall\, E \in \mathcal{E}_h,\, \boldsymbol{v} \in H^1(\Omega)^2\,.$$

Further we define an operator $j_h : L^2_0(\Omega) \to Q_h$ by

$$(j_h\,q)|_K = \frac{1}{|K|}\int_K q\,\mathrm{d}x\,\mathrm{d}y \qquad \forall\, K \in \mathcal{T}_h,\, q \in L^2_0(\Omega)\,.$$

Then (cf. e.g. [3] or [6])

$$\|\boldsymbol{v} - i_h\,\boldsymbol{v}\|_{0,\Omega} + h\,|\boldsymbol{v} - i_h\,\boldsymbol{v}|_{1,h} \ \le\ C\,h^2\,|\boldsymbol{v}|_{2,\Omega} \qquad \forall\, \boldsymbol{v} \in H^2(\Omega)^2\,, \tag{32}$$

$$\|q - j_h\,q\|_{0,\Omega} \ \le\ C\,h\,|q|_{1,\Omega} \qquad \forall\, q \in L^2_0(\Omega)\cap H^1(\Omega)\,. \tag{33}$$

Note also that $i_h\,\boldsymbol{v} \in W_h$ for any $\boldsymbol{v} \in W$ and that $i_h\,\boldsymbol{v}$ is discretely divergence–free if $\operatorname{div}\boldsymbol{v} = 0$, i.e.,

$$b_h(i_h\,\boldsymbol{v},q_h) = 0 \qquad \forall\, q_h \in Q_h,\, \boldsymbol{v} \in H^1(\Omega)^2,\, \operatorname{div}\boldsymbol{v} = 0\,.$$

The spaces W_h and Q_h satisfy the inf–sup condition

$$\sup_{\boldsymbol{v}_h\in W_h\setminus\{\mathbf{0}\}} \frac{b_h(\boldsymbol{v}_h,q_h)}{|\boldsymbol{v}_h|_{1,h}} \ge \beta\,\|q_h\|_{0,\Omega} \qquad \forall\, q_h \in Q_h \tag{34}$$

with a positive constant β independent of h (cf. [17], [7]).

It is natural to define the discrete solution of (8)–(12) as functions $\boldsymbol{u}_h \in \widetilde{W}_h$ and $p_h \in Q_h$ such that

$$\boldsymbol{u}_h - i_h\,\boldsymbol{u}_b \in W_h$$

and

$$a_h(\boldsymbol{u}_h, \boldsymbol{v}_h) + b_h(\boldsymbol{v}_h, p_h) - b_h(\boldsymbol{u}_h, q_h) = (\boldsymbol{f}, \boldsymbol{v}_h) \qquad \forall\, \boldsymbol{v}_h \in W_h,\ q_h \in Q_h\,. \tag{35}$$

The properties of the bilinear forms a_h and b_h immediately imply that the discrete problem always has a unique solution.

Since the space W_h is nonconforming, the exact solution $\boldsymbol{u}$, p does not solve the discrete problem. Indeed, multiplying (8) by a function $\boldsymbol{v}_h \in W_h$, integrating by parts over each $K \in \mathcal{T}_h$ and summing up, we obtain

$$a_h(\boldsymbol{u}, \boldsymbol{v}_h) + b_h(\boldsymbol{v}_h, p) = (\boldsymbol{f}, \boldsymbol{v}_h) + e_h(\boldsymbol{u}, p; \boldsymbol{v}_h) \quad \forall\, \boldsymbol{v}_h \in W_h\,, \tag{36}$$

where the consistency error e_h is defined by

$$e_h(\boldsymbol{u}, p; \boldsymbol{v}_h) = \sum_{K\in\mathcal{T}_h} \int_{\partial K} \boldsymbol{v}_h|_K \cdot \sigma(\boldsymbol{u}, p)\, \boldsymbol{n}_{\partial K}\, \mathrm{d}\gamma$$

with $\boldsymbol{n}_{\partial K}$ being the outer unit normal vector to the boundary of K. For estimating the consistency error we shall use the following lemma.

Lemma 3. *Consider any $K \in \mathcal{T}_h$ and denote for any $z, v \in H^1(K)$*

$$e_K(z, v) = \sum_{E\subset\partial K} \int_E z\,(v - J_E(v))\, \boldsymbol{n}_{\partial K}\, \mathrm{d}\gamma\,.$$

Then $e_K(z, v) = \mathbf{0}$ for any $v \in Q(K)$ and any function $z \in H^1(K)$ having the form $z(x, y) = z_1(x) + z_2(y)$.

Proof. Let us denote $\widehat{z}_i = z_i \circ F_K$, $i = 1, 2$, and set $\widehat{z}(\widehat{x}, \widehat{y}) = \widehat{z}_1(\widehat{x}) + \widehat{z}_2(\widehat{y})$. Then $e_K(z, v) = \frac{1}{2}\, h\, e_{\widehat{K}}(\widehat{z}, v \circ F_K)$. Denoting

$$\vartheta(\xi) = -\frac{1}{4} + \frac{1}{\kappa}\,(\theta(1) - \theta(\xi))\,,$$

we have

$$\begin{aligned} e_{\widehat{K}}(\widehat{z}, \widehat{q}_1) \;&=\; (1,0) \int_{-1}^{1} \vartheta(\widehat{y})\,(\widehat{z}(1, \widehat{y}) - \widehat{z}(-1, \widehat{y}))\, \mathrm{d}\widehat{y} \\ &+\; (0,1) \int_{-1}^{1} (\frac{\widehat{x}}{2} - \vartheta(\widehat{x}))\,(\widehat{z}(\widehat{x}, 1) - \widehat{z}(\widehat{x}, -1))\, \mathrm{d}\widehat{x}\,. \end{aligned}$$

Since $\widehat{z}(1, \widehat{y}) - \widehat{z}(-1, \widehat{y}) = \widehat{z}_1(1) - \widehat{z}_1(-1)$, $\widehat{z}(\widehat{x}, 1) - \widehat{z}(\widehat{x}, -1) = \widehat{z}_2(1) - \widehat{z}_2(-1)$ and

$$\int_{-1}^{1} \vartheta(\xi)\, \mathrm{d}\xi = \int_{\widehat{E}_3} \widehat{q}_1\, \mathrm{d}\gamma = 0\,,$$

we obtain $e_{\widehat{K}}(\widehat{z}, \widehat{q}_1) = \mathbf{0}$. It is easy to see that the basis functions $\widehat{q}_2$, $\widehat{q}_3$ and $\widehat{q}_4$ can be obtained from $\widehat{q}_1$ by rotating the coordinate system through angles of $\pi/2$, π and $3\,\pi/2$. Since these rotations preserve the additive form of $\widehat{z}$, we deduce that $e_{\widehat{K}}(\widehat{z}, \widehat{q}_i) = \mathbf{0}$ for $i = 1, \ldots, 4$. $\square$

Now we can prove an error estimate for the above–defined discrete solution $\boldsymbol{u}_h$, p_h.

Theorem 3. *Let the weak solution of (8)–(12) satisfy $\boldsymbol{u} \in H^{2+\alpha}(\Omega)^2$ and $p \in H^{1+\alpha}(\Omega)$ for some $\alpha \in [0,1]$. Then*

$$\|\boldsymbol{u}-\boldsymbol{u}_h\|_{0,\Omega} + \|D_h(\boldsymbol{u}-\boldsymbol{u}_h)\|_{0,\Omega} + \|p-p_h\|_{0,\Omega} \le C\,h^\alpha\,(|\boldsymbol{u}|_{2+\alpha,\Omega} + |p|_{1+\alpha,\Omega})\,,$$

where C is a constant independent of h, $\boldsymbol{u}$ and p.

Proof. According to (35) and (36), we have

$$a_h(\boldsymbol{u}-\boldsymbol{u}_h,\boldsymbol{v}_h) + b_h(\boldsymbol{v}_h,p-p_h) = e_h(\boldsymbol{u},p;\boldsymbol{v}_h) \qquad \forall\ \boldsymbol{v}_h \in W_h\,. \tag{37}$$

Since $\boldsymbol{w}_h \equiv \boldsymbol{u}_h - i_h\,\boldsymbol{u}$ belongs to W_h and is discretely divergence–free, we derive from (37) with $\boldsymbol{v}_h = \boldsymbol{w}_h$ that

$$2\,\|D_h(\boldsymbol{w}_h)\|_{0,\Omega}^2 = a_h(\boldsymbol{u}-i_h\,\boldsymbol{u},\boldsymbol{w}_h) + b_h(\boldsymbol{w}_h,p-j_h\,p) - e_h(\boldsymbol{u},p;\boldsymbol{w}_h)\,. \tag{38}$$

Using (32) and (33), we get

$$a_h(\boldsymbol{u}-i_h\,\boldsymbol{u},\boldsymbol{w}_h) \;\le\; 2\,|\boldsymbol{u}-i_h\,\boldsymbol{u}|_{1,h}\,\|D_h(\boldsymbol{w}_h)\|_{0,\Omega} \le C\,h\,|\boldsymbol{u}|_{2,\Omega}\,\|D_h(\boldsymbol{w}_h)\|_{0,\Omega}\,, \tag{39}$$

$$b_h(\boldsymbol{w}_h,p-j_h\,p) \;\le\; \|p-j_h\,p\|_{0,\Omega}\,\|\mathrm{div}_h\,\boldsymbol{w}_h\|_{0,\Omega} \le C\,h\,|p|_{1,\Omega}\,\|D_h(\boldsymbol{w}_h)\|_{0,\Omega}\,. \tag{40}$$

Further, we have for any $\boldsymbol{v}_h \in W_h$

$$e_h(\boldsymbol{u},p;\boldsymbol{v}_h) = \sum_{K\in\mathcal{T}_h}\sum_{E\subset\partial K}\int_E (\boldsymbol{v}_h|_K - J_E(\boldsymbol{v}_h))\cdot\sigma(\boldsymbol{u},p)\,\boldsymbol{n}_{\partial K}\,\mathrm{d}\gamma\,.$$

Let us denote by $P_1(K)^{2\times2}$ the space of 2×2 matrices whose entries are linear functions on K. Applying Lemma 3 componentwise, we get

$$e_h(\boldsymbol{u},p;\boldsymbol{v}_h) = \sum_{K\in\mathcal{T}_h}\inf_{z\in P_1(K)^{2\times2}}\sum_{E\subset\partial K}\int_E (\boldsymbol{v}_h|_K - J_E(\boldsymbol{v}_h))\cdot(\sigma(\boldsymbol{u},p)-z)\,\boldsymbol{n}_{\partial K}\,\mathrm{d}\gamma\,.$$

In view of Lemma 3 from [7], we derive

$$\int_E (\boldsymbol{v}_h|_K - J_E(\boldsymbol{v}_h))\cdot(\sigma(\boldsymbol{u},p)-z)\,\boldsymbol{n}_{\partial K}\,\mathrm{d}\gamma \le C\,h\,|\sigma(\boldsymbol{u},p)-z|_{1,K}\,|\boldsymbol{v}_h|_{1,K}\,.$$

Since each component of $\sigma(\boldsymbol{u},p)$ belongs to $H^{1+\alpha}(\Omega)$, we get by virtue of [11] and [6]

$$\inf_{z\in P_1(K)^{2\times2}} |\sigma(\boldsymbol{u},p)-z|_{1,K} \le C\,h^\alpha\,|\sigma(\boldsymbol{u},p)|_{1+\alpha,K}\,.$$

Thus, using the Cauchy–Schwarz inequality, we deduce that

$$e_h(\boldsymbol{u},p;\boldsymbol{v}_h) \le C\,h^{1+\alpha}\,|\sigma(\boldsymbol{u},p)|_{1+\alpha,\Omega}\,|\boldsymbol{v}_h|_{1,h} \le C\,h^{1+\alpha}\,(|\boldsymbol{u}|_{2+\alpha,\Omega} + |p|_{1+\alpha,\Omega})\,|\boldsymbol{v}_h|_{1,h}$$

and, applying the discrete Korn inequality (31), we obtain

$$e_h(\boldsymbol{u},p;\boldsymbol{v}_h) \le C\,h^\alpha\,(|\boldsymbol{u}|_{2+\alpha,\Omega} + |p|_{1+\alpha,\Omega})\,\|D_h(\boldsymbol{v}_h)\|_{0,\Omega}\,. \tag{41}$$

Combining (38)–(41), we get

$$\|D_h(\boldsymbol{w}_h)\|_{0,\Omega} \le C\,h^\alpha\,(|\boldsymbol{u}|_{2+\alpha,\Omega} + |p|_{1+\alpha,\Omega})\,, \tag{42}$$

which, together with (32), immediately implies the estimate for $\|D_h(\boldsymbol{u}-\boldsymbol{u}_h)\|_{0,\Omega}$. Applying the discrete Korn inequality (30) to (42) and using (32) once again, we derive the estimate for $\|\boldsymbol{u}-\boldsymbol{u}_h\|_{0,\Omega}$. Finally, in view of (37), we have

$$b_h(\boldsymbol{v}_h,p_h-j_h\,p) = a_h(\boldsymbol{u}-\boldsymbol{u}_h,\boldsymbol{v}_h) + b_h(\boldsymbol{v}_h,p-j_h\,p) - e_h(\boldsymbol{u},p;\boldsymbol{v}_h) \qquad \forall\ \boldsymbol{v}_h \in W_h$$

and the estimate for $\|p-p_h\|_{0,\Omega}$ follows from (34), (33) and (41) using the estimate for $\|D_h(\boldsymbol{u}-\boldsymbol{u}_h)\|_{0,\Omega}$. $\square$

Remark 2. It is easy to see that Lemma 3 also holds for any parallelogram K if we consider $z \in P_1(K)$. Consequently, Theorem 3 remains valid for triangulations consisting of shape–regular parallelograms provided that Theorems 1 and 2 still hold.

Remark 3. We were not able to prove the convergence of $\boldsymbol{u}_h$ with respect to the seminorm $|\cdot|_{1,h}$. For this, it would be sufficient to prove the convergence of $\|\mathrm{rot}(\boldsymbol{u}-\boldsymbol{u}_h)\|_{0,\Omega}$, which, however, seems to be a difficult task. At least, using (42), (31) and (32), we can prove that $|\boldsymbol{u}_h|_{1,h}$ is bounded.

Remark 4. For the midpoint–oriented degrees of freedom $J_E(v) = v(C_E)$ Lemma 3 does not hold for $z \in P_1(K)$, which prevents us from proving convergence results using the techniques applied above. Therefore, we consider only the meanvalue–oriented degrees of freedom in this section.

If $\boldsymbol{u} \in H^3(\Omega)^2$ and $p \in H^2(\Omega)$, then it follows from Theorem 3 that

$$\|D_h(\boldsymbol{u}-\boldsymbol{u}_h)\|_{0,\Omega} + \|p - p_h\|_{0,\Omega} \le C\, h\, (\|\boldsymbol{u}\|_{3,\Omega} + \|p\|_{2,\Omega})\,.$$

This estimate is optimal with respect to the convergence order but the required regularity of the weak solution is higher than usually. In fact, one would expect that there exists a constant C such that

$$\|D_h(\boldsymbol{u}-\boldsymbol{u}_h)\|_{0,\Omega} + \|p - p_h\|_{0,\Omega} \le C\, h\, (\|\boldsymbol{u}\|_{2,\Omega} + \|p\|_{1,\Omega})$$

as soon as the weak solution satisfies $\boldsymbol{u} \in H^2(\Omega)^2$, $p \in H^1(\Omega)$. However, according to (37), this would imply that

$$e_h(\boldsymbol{u}, p; \boldsymbol{v}_h) \le C\, h\, (\|\boldsymbol{u}\|_{2,\Omega} + \|p\|_{1,\Omega})\, \|D_h(\boldsymbol{v}_h)\|_{0,\Omega} \qquad \forall\ \boldsymbol{v}_h \in W_h$$

for any $\boldsymbol{u} \in W \cap H^2(\Omega)^2$ and $p \in L^2_0(\Omega) \cap H^1(\Omega)$. Unfortunately, such an estimate of the consistency error does not hold as it follows from the following theorem.

Theorem 4. *There exists a constant C independent of h such that*

$$\forall\ h = \frac{1}{n},\ n \ge 2, \quad \exists\ p \in L^2_0(\Omega) \cap H^1(\Omega),\ \boldsymbol{v}_h \in W_h :$$
$$e_h(\boldsymbol{0}, p; \boldsymbol{v}_h) > C\, \|p\|_{1,\Omega}\, \|D_h(\boldsymbol{v}_h)\|_{0,\Omega}\,.$$

Proof. Consider any $h = 1/n$, $n \ge 2$. For any $E \in \mathcal{E}_h$, let $\boldsymbol{t}_E = \widetilde{\boldsymbol{v}}_h(C_E)$ be a tangent vector to E, see Fig. 1. For any element $K \in \mathcal{T}_h$ we introduce a triangulation depicted in Fig. 4, where the distances between the vertices of K and the nearest points in the interiors of the edges are $h/4$. We define a function $p \in L^2_0(\Omega) \cap H^1(\Omega)$

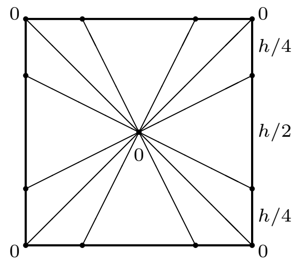

FIGURE 4. Triangulation of an element K.

which is piecewise linear with respect to the triangulation of any element K. We set $p = 0$ at the vertices and centres of the elements K and $p = 1$ or $p = -1$ at the points in the interiors of edges. These values are chosen in such a way that, for any $E \in \mathcal{E}_h$,

$$p|_E \quad \text{is odd} \qquad \text{and} \qquad \frac{\partial p}{\partial \boldsymbol{t}_E}(C_E) > 0\,.$$

Then $\|p\|_{0,\Omega} = 1/\sqrt{6}$ and $|p|_{1,\Omega} = 2\sqrt{6}/h$ so that $\|p\|_{1,\Omega} < 5/h$.

Let $\overline{\boldsymbol{v}}_h \in W_h$ be the function constructed in Section 3. Let E be any inner edge of $\mathcal{T}_h$. Then (possibly after a rotation of the coordinate system), the directions of the degrees of freedom of $\overline{\boldsymbol{v}}_h$ associated with the edges of the two elements adjacent

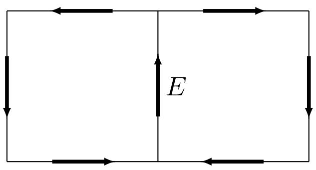

FIGURE 5. Directions of the degrees of freedom of $\overline{\boldsymbol{v}}_h$.

to E are like in Fig. 5. Let us denote by $E_1, \dots, E_4$ the four edges sharing a vertex with E which are perpendicular to E. Let ζ_E vanish at the midpoint of E and satisfy

$$\frac{\partial \zeta_E}{\partial \boldsymbol{t}_E} = \frac{1}{h}\,.$$

Then

$$[|\overline{\boldsymbol{v}}_h|]_E = -\left(\sum_{k=1}^{4} |J_{E_k}(\overline{\boldsymbol{v}}_h)|\right) \zeta_E\, \boldsymbol{n}_E + \boldsymbol{\xi}_E(\overline{\boldsymbol{v}}_h)\,,$$

where the function $\boldsymbol{\xi}_E(\overline{\boldsymbol{v}}_h)$ is even along E. Since p is odd along E, we see that

$$-\int_E p\,[|\overline{\boldsymbol{v}}_h|]_E \cdot \boldsymbol{n}_E \,\mathrm{d}\gamma = \left(\sum_{k=1}^{4} |J_{E_k}(\overline{\boldsymbol{v}}_h)|\right) \int_E p\,\zeta_E \,\mathrm{d}\gamma\,.$$

Consequently,

$$-\int_E p\,[|\overline{\boldsymbol{v}}_h|]_E \cdot \boldsymbol{n}_E \,\mathrm{d}\gamma = \frac{h}{8} \sum_{k=1}^{4} |J_{E_k}(\overline{\boldsymbol{v}}_h)|$$

and hence

$$-\int_E p\,[|\overline{\boldsymbol{v}}_h|]_E \cdot \boldsymbol{n}_E \,\mathrm{d}\gamma \ge \frac{h}{4}\, i \qquad \forall\ E \in \mathcal{E}_h^i,\ i = 1, \dots, n-1\,.$$

If $E \subset \partial\Omega$, then we have

$$-\int_E p\,[|\overline{\boldsymbol{v}}_h|]_E \cdot \boldsymbol{n}_E \,\mathrm{d}\gamma \ge \frac{h}{8}\,.$$

Thus, we derive using (18)

$$e_h(\mathbf{0}, p; \overline{\boldsymbol{v}}_h) \ge \frac{h}{2}\, n + h \sum_{i=1}^{n-1} (n-i)\, i = \frac{1}{2} + \frac{n^2-1}{6} \ge \frac{n^2}{6} > \frac{1}{60\,M}\, \|p\|_{1,\Omega}\, \|D_h(\overline{\boldsymbol{v}}_h)\|_{0,\Omega}\,.$$

$\square$

The above theorem shows that for $p \in L^2_0(\Omega) \cap H^1(\Omega)$ the estimate

$$e_h(\mathbf{0}, p; \boldsymbol{v}_h) \leq C\, \|p\|_{1,\Omega}\, \|D_h(\boldsymbol{v}_h)\|_{0,\Omega} \qquad \forall\, \boldsymbol{v}_h \in W_h$$

cannot be improved. This is also the reason why it is impossible to improve the results of Theorem 3 using the Aubin–Nitsche duality technique.

Remark 5. A triangular counterpart of the quadrilateral finite elements considered in this paper is the piecewise linear Crouzeix–Raviart element, see [7]. Thus, let us now assume that $\mathcal{T}_h$ consists of triangles and V_h is defined as in Section 2 with $Q(K)$ replaced by $P_1(K)$. It is known that, for certain types of triangulations, the finite element spaces V_h contain functions $\boldsymbol{v}_h$ for which $D_h(\boldsymbol{v}_h) = 0$ so that the inequality (2) cannot hold for any constant C_h (cf. [10], [1]). However, if we construct $\mathcal{T}_h$ from the mesh in Fig. 3 by dividing each square by its diagonal from the lower left vertex to the upper right vertex, then $D_h(\boldsymbol{v}_h) \neq 0$ for any $\boldsymbol{v}_h \in V_h \setminus \{\mathbf{0}\}$ and hence the natural question is whether one can prove similar results as above. This question was investigated in [14] where it was proved using similar techniques as above that Theorems 1 and 2 still hold and that the dependence on h in Theorem 2 is optimal. However, for the consistency error, one can only prove that

$$|e_h(\boldsymbol{u}, p; \boldsymbol{v}_h)| \leq C\, \|\boldsymbol{v}_h\|_{0,\Omega}\,, \qquad |e_h(\boldsymbol{u}, p; \boldsymbol{v}_h)| \leq C\, \|D_h(\boldsymbol{v}_h)\|_{0,\Omega} \qquad \forall\, \boldsymbol{v}_h \in W_h$$

with C independent of h. It was shown in [14] that these estimates cannot be improved (also not for infinitely smooth functions). That leads to error estimates of the type $\|\boldsymbol{u} - \boldsymbol{u}_h\|_{0,\Omega} = \mathcal{O}(1)$, $\|D_h(\boldsymbol{u} - \boldsymbol{u}_h)\|_{0,\Omega} = \mathcal{O}(1)$, $|\boldsymbol{u} - \boldsymbol{u}_h|_{1,h} = \mathcal{O}(h^{-1})$ and $\|p - p_h\|_{0,\Omega} = \mathcal{O}(1)$ which were also confirmed by our unpublished numerical experiments. Moreover, in the numerical tests, the convergence of the used solver was very bad compared to the case when the bilinear form $\sum_{K \in \mathcal{T}_h} \int_K \nabla\, \boldsymbol{u} \cdot \nabla\, \boldsymbol{v}\, \mathrm{d}x\, \mathrm{d}y$ is used instead of a_h.

References

[1] D. N. Arnold, *On nonconforming linear-constant elements for some variants of the Stokes equations*, Istit. Lombardo Accad. Sci. Lett. Rend. A., 127 (1993), pp. 83–93.

[2] S. C. Brenner, *Korn's inequalities for piecewise H^1 vector fields*, Math. Comput., 73 (2004), pp. 1067–1087.

[3] S. C. Brenner and L. R. Scott, *The Mathematical Theory of Finite Element Methods*, Springer–Verlag, New York, 1994.

[4] F. Brezzi and M. Fortin, *Mixed and Hybrid Finite Element Methods*, Springer–Verlag, New York, 1991.

[5] Z. Cai, J. Douglas Jr., J. E. Santos, D. Sheen, and X. Ye, *Nonconforming quadrilateral finite elements: a correction*, Calcolo, 37 (2000), pp. 253–254.

[6] P. G. Ciarlet, *Basic Error Estimates for Elliptic Problems*, in Handbook of Numerical Analysis, v. 2 – Finite Element Methods (pt. 1), P. G. Ciarlet and J. L. Lions, eds., North–Holland, Amsterdam, 1991, pp. 17–351.

[7] M. Crouzeix and P.–A. Raviart, *Conforming and nonconforming finite element methods for solving the stationary Stokes equations I*, RAIRO, 7 (1973), R-3, pp. 33–76.

[8] J. Douglas Jr., J. E. Santos, D. Sheen, and X. Ye, *Nonconforming Galerkin methods based on quadrilateral elements for second order elliptic problems*, M^{2}AN, 33 (1999), pp. 747–770.

[9] R. S. Falk, *Nonconforming finite element methods for the equations of linear elasticity*, Math. Comput., 57 (1991), pp. 529–550.

[10] R. S. Falk and M. E. Morley, *Equivalence of finite element methods for problems in elasticity*, SIAM J. Numer. Anal., 27 (1990), pp. 1486–1505.

[11] M. Feistauer, *On the finite element approximation of functions with noninteger derivatives*, Numer. Funct. Anal. Optimization, 10 (1989), pp. 91–110.

[12] V. Girault and P.–A. Raviart, *Finite Element Methods for Navier–Stokes Equations*, Springer–Verlag, Berlin, 1986.

[13] P. Knobloch, *On Korn's inequality for nonconforming finite elements*, Technische Mechanik, 20 (2000), pp. 205–214 and 375 (errata).

[14] P. Knobloch, *Influence of mesh–dependent Korn's inequality on the convergence of nonconforming finite element schemes*, in Proceedings of Czech–Japanese Seminar in Applied Mathematics 2004, M. Beneš, J. Mikyška, and T. Oberhuber, eds., Faculty of Nuclear Sciences and Physical Engineering, Czech Technical University in Prague, pp. 85–95, 2005.

[15] Q. Lin, L. Tobiska, and A. Zhou, *Superconvergence and extrapolation of nonconforming low order finite elements applied to the Poisson equation*, IMA J. Numer. Anal., 25 (2005), pp. 160–181.

[16] J. Nečas and I. Hlaváček, *Mathematical Theory of Elastic and Elastico–Plastic Bodies: An Introduction*, Elsevier, Amsterdam, 1981.

[17] R. Rannacher and S. Turek, *Simple nonconforming quadrilateral Stokes element*, Numer. Methods Partial Different. Equations, 8 (1992), pp. 97–111.

[18] V. Ruas, *Circumventing discrete Korn's inequalities in convergence analyses of nonconforming finite element approximations of vector fields*, Z. angew. Math. Mech., 76 (1996), pp. 483–484.

[19] S. Turek, A. Ouazzi and R. Schmachtel, *Multigrid methods for stabilized nonconforming finite elements for incompressible flow involving the deformation tensor formulation*, J. Numer. Math., 10 (2002), pp. 235–248.

Charles University, Faculty of Mathematics and Physics, Department of Numerical Mathematics, Sokolovská 83, 186 75 Praha 8, Czech Republic
E-mail: knobloch@karlin.mff.cuni.cz

Otto–von–Guericke–Universität Magdeburg, Institut für Analysis und Numerik, Postfach 4120, 39016 Magdeburg, Germany
E-mail: tobiska@mathematik.uni-magdeburg.de
URL: http://www-ian.math.uni-magdeburg.de/home/tobiska/