To verify that (T, P_T, Σ'_T) is a finite element, it suffices to prove the unisolvence since other properties are obvious. Thus, consider any $\alpha_1, \dots, \alpha_4 \in \mathbb{R}$. We want to show that there is a unique function $p \in P_T$ such that $\phi_i(p) = \alpha_i$, $i = 1, \dots, 4$.
Since $p = \sum_{j=1}^{3} \beta_j \lambda_j + \beta_4 \lambda_1 \lambda_2 \lambda_3$ for some $\beta_1, \dots, \beta_4 \in \mathbb{R}$, one gets, for $i = 1, 2, 3$, $\alpha_i = \phi_i(p) = p(a_i) = \beta_i$. The coefficient β_4 is uniquely determined by the condition $\alpha_4 = \phi_4(p) = \sum_{j=1}^{3} \alpha_j \phi_4(\lambda_j) + \beta_4 \phi_4(\lambda_1 \lambda_2 \lambda_3)$.

The space X_h is defined by

$$X_h = \{ v \in L^2(\Omega);\ v|_T \in P_T\ \forall T \in \mathcal{T}_h,\ v|_T(z_i) = v|_{T'}(z_i)\ \forall T \in \mathcal{T}_{h,i},\ i = 1, \dots, N \},$$

where $z_1, \dots, z_N$ are the vertices of $\mathcal{T}_h$ and $\mathcal{T}_{h,i} = \{T \in \mathcal{T}_h;\ T \ni z_i\}$, $i = 1, \dots, N$.
The degrees of freedom of X_h are

$$\Sigma'_h = \{ \text{values at the vertices } z_1, \dots, z_N, \text{ mean values on elements of } \mathcal{T}_h \}.$$

Thus, $\dim X_h = N + \operatorname{card} \mathcal{T}_h$. The basis functions of X_h are functions $\{p_i\}_{i=1}^{N}$ assigned to vertices and functions $\{p_T\}_{T \in \mathcal{T}_h}$ assigned to elements of $\mathcal{T}_h$. One has $\operatorname{supp} p_i = \bigcup_{T \ni z_i} T$, $\operatorname{supp} p_T = T$, $p_i(z_j) = \delta_{ij}$ $(i,j = 1, \dots, N)$, $\int_T p_i \, dx = 0$ $\forall T \in \mathcal{T}_h$, $p_T = \lambda_1 \lambda_2 \lambda_3$ in T. To show that $X_h \subset C(\overline{\Omega})$, consider any $v_h \in X_h$ and any interior edge of $\mathcal{T}_h$. Let $T, \tilde{T} \in \mathcal{T}_h$ be the two triangles adjacent to E and denote $q = (v_h|_T)|_E - (v_h|_{\tilde{T}})|_E$. Then $q \in P_1(E)$ and $q = 0$ at the endpoints of E. Thus, $q = 0$ on E and hence v_h is continuous across E. Since E was arbitrary, $v_h \in C(\overline{\Omega})$.

Denote the degrees of freedom in Σ'_h in the following way:

$$\phi_i(v) = v(z_i), \quad i = 1, \dots, N, \qquad \phi_T(v) = \frac{1}{|T|} \int_T v \, dx, \quad T \in \mathcal{T}_h.$$

The interpolation operator Π_h is defined by the following conditions:

$$\forall v \in C(\overline{\Omega}): \quad \Pi_h v \in X_h, \quad \phi_i(\Pi_h v) = \phi_i(v), \ i = 1, \dots, N, \quad \phi_T(\Pi_h v) = \phi_T(v) \ \forall T \in \mathcal{T}_h.$$

One has $(\Pi_h v)|_T = \Pi_T(v|_T)$, where Π_T is the P_T-interpolation operator defined by the conditions

$$\forall v \in C(T): \ \Pi_T v \in P_T, \quad \phi_i(\Pi_T v) = \phi_i(v), \quad i = 1, \dots, 4,$$

where $\phi_i(v) = v(a_i)$, $i = 1, 2, 3$, $\phi_4(v) = \frac{1}{|T|} \int_T v \, dx$. Let $(\hat{T}, \hat{P}, \hat{\Sigma})$ be the reference element defined analogously as (T, P_T, Σ'_T). Then

$$\widehat{\Pi_T v} = \hat{\Pi} \hat{v} \quad \forall v \in C(T), \ T \in \mathcal{T}_h,$$

where $\hat{\Pi}$ is the $\hat{P}$-interpolation operator.

Denoting by $\hat{p}_1, \dots, \hat{p}_4$ the basis functions of the finite element $(\hat{T}, \hat{P}, \hat{\Sigma}_1)$, one has $\hat{\Pi}\hat{v} = \sum_{i=1}^{4} \hat{\Phi}_i(\hat{v})\,\hat{p}_i$. Consider $\hat{v} \in H^2(\hat{T})$. Since $H^2(\hat{T}) \hookrightarrow C(\hat{T})$, one gets $|\hat{\Phi}_i(\hat{v})| = |\hat{v}(\hat{a}_i)| \le \|\hat{v}\|_{0,\infty,\hat{T}} \le C\|\hat{v}\|_{2,\hat{T}}$, $i = 1,2,3$. Furthermore, $|\hat{\Phi}_4(\hat{v})| \le \frac{1}{\sqrt{|\hat{T}|}}\|\hat{v}\|_{0,\hat{T}} \le \frac{1}{\sqrt{|\hat{T}|}}\|\hat{v}\|_{2,\hat{T}}$. Thus, for any norm $\|\cdot\|$ on $\hat{P}$, one gets
$\|\hat{\Pi}\hat{v}\| \le \sum_{i=1}^{4} |\hat{\Phi}_i(\hat{v})| \cdot \|\hat{p}_i\| \le C\|\hat{v}\|_{2,\hat{T}} \quad \forall\, \hat{v} \in H^2(\hat{T})$.

We will also use the fact that $\hat{\Pi}\hat{v} = \hat{v} \quad \forall\, \hat{v} \in \hat{P}$. For any $T \in \mathcal{T}_h$ and $m \in \{0,1\}$, one has

$$|v - \Pi_T v|_{m,T} \le C h_T^{1-m} |\widehat{v - \Pi_T v}|_{m,\hat{T}} = C h_T^{1-m} |\hat{v} - \hat{\Pi}\hat{v}|_{m,\hat{T}} =$$
$$= C h_T^{1-m} \inf_{\hat{q} \in P_1(\hat{T})} |\hat{v} + \hat{q} - \hat{\Pi}(\hat{v} + \hat{q})|_{m,\hat{T}} \le \tilde{C} h_T^{1-m} \inf_{\hat{q} \in P_1(\hat{T})} \|\hat{v} + \hat{q}\|_{2,\hat{T}} \le \bar{C} h_T^{1-m} |\hat{v}|_{2,\hat{T}} \le$$
$$\le \hat{C} h_T^{2-m} |v|_{2,T} \qquad \forall\, v \in H^2(T).$$

Thus, for $v \in H^2(\Omega)$, one obtains

$$\|v - \Pi_h v\|_{0,\Omega} \le C h^2 |v|_{2,\Omega}, \qquad |v - \Pi_h v|_{1,\Omega} \le C h |v|_{2,\Omega}.$$

The weak formulation of (1) reads: find $u \in H_0^1(\Omega)$ such that

$$a(u,v) = (f,v) \quad \forall\, v \in H_0^1(\Omega), \qquad \text{where } a(u,v) = \int_\Omega \nabla u \cdot \nabla v \, dx.$$

The discrete problem reads: find $u_h \in V_h := \{v_h \in X_h;\ v_h = 0 \text{ on } \partial\Omega\}$ such that

$$a(u_h, v_h) = (f, v_h) \quad \forall\, v_h \in V_h.$$

Since $V_h \subset H_0^1(\Omega)$, one has $a(u - u_h, v_h) = 0 \quad \forall\, v_h \in V_h$. Consequently

$$|u - u_h|_{1,\Omega}^2 = a(u - u_h, u - u_h) = a(u - u_h, u - \underbrace{\Pi_h u}_{\in V_h}) \le |u - u_h|_{1,\Omega} |u - \Pi_h u|_{1,\Omega}$$

$$\Rightarrow \|u - u_h\|_{1,\Omega} \le C|u - u_h|_{1,\Omega} \le C|u - \Pi_h u|_{1,\Omega} \le C h |u|_{2,\Omega}.$$

We have used that problem (1) is regular in the sense that

$$\forall\, f \in L^2(\Omega): \quad u \in H^2(\Omega), \qquad \|u\|_{2,\Omega} \le C\|f\|_{0,\Omega}.$$

To prove an estimate in the L^2 norm, we denote by $\varphi \in H_0^1(\Omega)$ the solution of

$$a(v, \varphi) = (u - u_h, v) \quad \forall\, v \in H_0^1(\Omega).$$

The regularity implies that $\varphi \in H^2(\Omega)$ and $\|\varphi\|_{2,\Omega} \le C\|u - u_h\|_{0,\Omega}$.
Since (see above) $a(u - u_h, \Pi_h \varphi) = 0$, one has

$$\|\underbrace{u - u_h}_{\in H_0^1(\Omega)}\|_{0,\Omega}^2 = a(u - u_h, \varphi) = a(u - u_h, \varphi - \Pi_h \varphi) \le |u - u_h|_{1,\Omega} |\varphi - \Pi_h \varphi|_{1,\Omega} \le$$
$$\le C h |u - u_h|_{1,\Omega} |\varphi|_{2,\Omega} \le \tilde{C} h |u - u_h|_{1,\Omega} \|u - u_h\|_{0,\Omega}$$

$$\Rightarrow \quad \|u - u_h\|_{0,\Omega} \le \tilde{C} h |u - u_h|_{1,\Omega} \le C h^2 |u|_{2,\Omega}.$$