Exercises for week 7

Let R be a commutative ring. We define the multiplicative group $R^\star$ as $(S, \cdot)$ where $S = \{s \in R \colon \exists r \in R,\, rs = 1\}$ are the invertible elements of R.

Problem 1. Verify that $R^\star$ is an Abelian group whenever R is a commutative ring (with a unit).

Problem 2. Find all generators of $\mathbb{Z}_7^\star$.

Problem 3. Find all generators of $\big(\mathbb{Z}_3[x]/(x^2+1)\big)^\star$ (yes, the domain in parentheses is a field).

Problem 4. Let $\mathbb{F}$ be a field of characteristic p. Prove that the mapping $x \mapsto x^p$ is an automorphism of $\mathbb{F}$.

Problem 5. Prove or disprove the following statement: Let $n \in \mathbb{N}$. The group $(\mathbb{Z}_n)^\star$ is cyclic if and only if n is prime.