Exercises for week 4

Problém 1. Let R be a domain that is not a UFD. Show that then $R[x]$ is not a UFD.

Problém 2. Is the domain $\mathbb{Q}[e]$ a field? How about $\mathbb{Q}[\sqrt{2}]$?

Note: You can use without proof the (nontrivial) fact that the number e is not a root of any polynomial with rational coefficients.

Problém 3. Are the following polynomials irreducible?

1. $2x - 6$ in $\mathbb{Z}[x]$
2. $x^2 - 2$ in $\mathbb{Q}[x, y]$
3. $xy + x + y + 1$ in $\mathbb{Q}[x, y]$

Problém 4. What is the gcd of $x^n - 1$ and $x^m - 1$ in $\mathbb{Z}[x]$ (depending on the parameters m, n)?

Problém 5. Let R be a commutative ring and $r \in R$ be fixed element. Show that the map $f\colon R[x] \to R[x]$ that sends $p(x) = p_0 + p_1 x + p_2 x^2 + \cdots + p_n(x)^n$ to the polynomial $p(x + r) = p_0 + p_1(x + r) + p_2(x + r)^2 + \cdots + p_n(x + r)^n$ is an automorphism of the ring $R[x]$.

Problém 6. Let R be a domain, $r \in R$ be any element and $p(x) \in R[x]$ be an irreducible polynomial. Prove that then $q(x) = p(x + r)$ is also an irreducible polynomial in $R[x]$.

Problém 7. Show that for each prime p the polynomial that is the result of evaluating $(x^p - 1)/(x - 1)$ is irreducible in $\mathbb{Z}[x]$. Hint: Eisenstein's criterion helps here.