Exercises for week 13

Problem 1. Prove that the algebras $(\mathbb{R}_+, \cdot)$ and $(\mathbb{R}, +)$ are isomorphic. Here $\mathbb{R}_+$ are all positive real numbers.

Problem 2 (from David Stanovský)**.** Prove that the algebras $(\mathbb{R}^2, \cdot)$ and $(\mathbb{R}^3, \cdot)$ are not isomorphic.

Problem 3 (from Libor Barto)**.** Take p prime, $n \in \mathbb{N}$ and consider the algebra $\mathbf{A}$ with the base set $\{0, 1, \dots, p-1\}$ and one ternary operation $m(x, y, z) = x - y + z \pmod{p}$. Prove that $R \subset A^n$ a subuniverse of $\mathbf{A}^n$ if and only if R is an affine subspace of $\mathbb{Z}_p^n$ (ie. R is closed under affine combinations: $\mathbf{x}, \mathbf{y} \in R, a \in \mathbb{Z}_p \Rightarrow a\mathbf{x} + (1-a)\mathbf{y} \in R$).

Problem 4. Take an algebra $\mathbf{A}$. Prove that an equivalence relation $\alpha \subseteq A^2$ is a congruence if and only if for every n-ary basic operation t of $\mathbf{A}$ and every choice of $a, b, c_1, \dots, c_n \in A$ such that $(a, b) \in \alpha$ we have

$$
\begin{aligned}
(t(a, c_2, \dots, c_{n-1}, c_n), t(b, c_2, \dots, c_{n-1}, c_n)) &\in \alpha \\
(t(c_1, a, \dots, c_{n-1}, c_n), t(c_1, b, \dots, c_{n-1}, c_n)) &\in \alpha \\
&\vdots \\
(t(c_1, c_2, \dots, c_{n-1}, a), t(b, c_2, \dots, c_{n-1}, b)) &\in \alpha.
\end{aligned}
$$

Problem 5. Let α, β be congruences of an algebra $\mathbf{A}$. How does the smallest congruence that contains both α and β (also denoted by $\alpha \vee \beta$) look like?

Problem 6 (Even more general Chinese Remainder Theorem)**.** Let α_1, α_2 be congruences of an algebra $\mathbf{A}$ such that $\alpha_1 \circ \alpha_2 = \mathbf{1}$ (here $\mathbf{1}$ is the trivial congruence that glues together everything; see pg. 40 of Burris, Sankappanavar for what $\circ$ means). Then

$$A/\alpha_1 \times A/\alpha_2 \simeq A/(\alpha_1 \cap \alpha_2)$$

Why is the CRT for two ideals a special case of this theorem?