Exercises for week 12

Problem 1. Let us order monomials in x, y, z lexicographicaly by powers: We say that $x^i y^j z^k > x^{i'} y^{j'} z^{k'}$ if one of the following happens: $i > i'$, or $i = i'$ and $j > j'$ or $i = i', j = j'$ and $k > k'$. Prove that there is no infinite descending sequence of monomials $m_1 > m_2 > m_3 > \dots$

(This is the first step in showing that rewriting according to a Gröbner basis always finishes in a finite number of steps.)

Problem 2. Say we have the polynomial

$$f = x^3y^2 + x$$

and the Gröbner basis G consisting of polynomials $y^3 - 2y^2$, $xy - 2x$, $2x^2 - y^2$. We will be using the lexicographic ordering from Problem 1. We keep rewriting f using G. What polynomial will we end up with? Don't use a computer here.

Problem 3. Prove that $H = \{xy + 1, x + y^2\}$ is not a Gröbner basis, ie. find a polynomial that H rewrites to two different terminal forms. Use the lexicographical ordering from Problem 1.

Problem 4. Find a basis (it need not be Gröbner) of the ideal of $\mathbb{C}[x, y]$ of all polynomials p such that $p(0, 0) = p(1, 1) = 0$.

Problem 5. Find a basis (it need not be Gröbner) of the ideal of $\mathbb{R}[x, y]$ of all polynomials p that are zero everywhere on the curve $\{(x, y)\colon x = \pm\sqrt{y + 1}, y \geq 1\}$.