4.5. **Functions of the class C^1.**

Definition. Let $G \subset \mathbf{R}^n$ be a nonempty open set. Let a function $f\colon G \to \mathbf{R}$ have at each point of the set G all partial derivatives continuous (i.e., function $\boldsymbol{x} \mapsto \frac{\partial f}{\partial x_j}(\boldsymbol{x})$ are continuous on G for each $j \in \{1, \dots, n\}$). Then we say that f is of the *class* $\mathcal{C}^1$ *on* G. The set of all these functions is denoted by $\mathcal{C}^1(G)$.

Remark. If $G \subset \mathbf{R}^n$ is a nonempty open set and and $f, g \in \mathcal{C}^1(G)$, then $f + g \in \mathcal{C}^1(G)$, $f - g \in \mathcal{C}^1(G)$, and $fg \in \mathcal{C}^1(G)$. If moreover for each $\boldsymbol{x} \in G$ we have : $g(\boldsymbol{x}) \neq 0$, then $f/g \in \mathcal{C}^1(G)$.

Proposition 4.13 (weak Lagrange theorem)**.** *Let $n \in \mathbf{N}$, $I_1, \dots, I_n \subset \mathbf{R}$ be open intervals, $I = I_1 \times I_2 \times \cdots \times I_n$, $f \in \mathcal{C}^1(I)$, $\boldsymbol{a}, \boldsymbol{b} \in I$. Then there exist points $\boldsymbol{\xi}^1, \dots, \boldsymbol{\xi}^n \in I$ with $\xi^i_j \in \langle a_j, b_j \rangle$ for each $i, j \in \{1, \dots, n\}$, such that*

$$f(\boldsymbol{b}) - f(\boldsymbol{a}) = \sum_{i=1}^{n} \frac{\partial f}{\partial x_i}(\boldsymbol{\xi}^i)(b_i - a_i).$$

Definition. Let $G \subset \mathbf{R}^n$ be an open set, $\boldsymbol{a} \in G$, and $f \in \mathcal{C}^1(G)$. Then the graph of the function

$$T\colon \boldsymbol{x} \mapsto f(\boldsymbol{a}) + \frac{\partial f}{\partial x_1}(\boldsymbol{a})(x_1 - a_1) + \frac{\partial f}{\partial x_2}(\boldsymbol{a})(x_2 - a_2) + \cdots + \frac{\partial f}{\partial x_n}(\boldsymbol{a})(x_n - a_n), \quad \boldsymbol{x} \in \mathbf{R}^n,$$

is called *tangent hyperplane* to the graph of the function f at the point $[\boldsymbol{a}, f(\boldsymbol{a})]$.

Theorem 4.14. *Let $G \subset \mathbf{R}^n$ be an open set, $\boldsymbol{a} \in G$, $f \in \mathcal{C}^1(G)$, and T be a function, such that its graph is the tangent hyperplane of the function f at the point $[\boldsymbol{a}, f(\boldsymbol{a})]$. Then*

$$\lim_{\boldsymbol{x} \to \boldsymbol{a}} \frac{f(\boldsymbol{x}) - T(\boldsymbol{x})}{\rho(\boldsymbol{x}, \boldsymbol{a})} = 0.$$

Theorem 4.15. *Let $G \subset \mathbf{R}^n$ be an open nonempty set and $f \in \mathcal{C}^1(G)$. Then f is continuous on G.*

Theorem 4.16. *Let $r, s \in \mathbf{N}$, $G \subset \mathbf{R}^s$, $H \subset \mathbf{R}^r$ be open sets. Let $\varphi_1, \dots, \varphi_r \in \mathcal{C}^1(G)$, $f \in \mathcal{C}^1(H)$ and $[\varphi_1(\boldsymbol{x}), \dots, \varphi_r(\boldsymbol{x})] \in H$ for each $\boldsymbol{x} \in G$. Then the composed function $F\colon G \to \mathbf{R}$ defined by*

$$F(\boldsymbol{x}) = f\big(\varphi_1(\boldsymbol{x}), \varphi_2(\boldsymbol{x}), \dots, \varphi_r(\boldsymbol{x})\big), \quad \boldsymbol{x} \in G,$$

is of the class $\mathcal{C}^1$ on G. Let $\boldsymbol{a} \in G$ and $\boldsymbol{b} = [\varphi_1(\boldsymbol{a}), \dots, \varphi_r(\boldsymbol{a})]$. Then for each $j \in \{1, \dots, s\}$ we have

$$\frac{\partial F}{\partial x_j}(\boldsymbol{a}) = \sum_{i=1}^{r} \frac{\partial f}{\partial y_i}(\boldsymbol{b}) \frac{\partial \varphi_i}{\partial x_j}(\boldsymbol{a}).$$

Definition. Let $G \subset \mathbf{R}^n$ be an open set, $\boldsymbol{a} \in G$, and $f \in \mathcal{C}^1(G)$. *Gradient of f at the point $\boldsymbol{a}$* is defined as the vector

$$\nabla f(\boldsymbol{a}) = \left[\frac{\partial f}{\partial x_1}(\boldsymbol{a}), \frac{\partial f}{\partial x_2}(\boldsymbol{a}), \dots, \frac{\partial f}{\partial x_n}(\boldsymbol{a}) \right].$$

Definition. Let $G \subset \mathbf{R}^n$ be an open set, $\boldsymbol{a} \in G$, $f \in \mathcal{C}^1(G)$, and $\nabla f(\boldsymbol{a}) = \boldsymbol{o}$. Then the point $\boldsymbol{a}$ is called *stationary* (or also *critical*) *point* of the function f.

Definition. Let $G \subset \mathbf{R}^n$ be an open set, $f \colon G \to \mathbf{R}$, $i, j \in \{1, \dots, n\}$, and $\frac{\partial f}{\partial x_i}(\boldsymbol{x})$ exists for each $\boldsymbol{x} \in G$. Then *partial derivative of the second order of the function f according to i-th and j-th variable at the point $\boldsymbol{a} \in G$* is defined by

$$\frac{\partial^2 f}{\partial x_i \partial x_j}(\boldsymbol{a}) = \frac{\partial}{\partial x_j}\left(\frac{\partial f}{\partial x_i}\right)(\boldsymbol{a}).$$

If $i = j$ then we use the notation

$$\frac{\partial^2 f}{\partial x_i^2}(\boldsymbol{a}).$$

Similarly we define higher order partial derivatives.

Theorem 4.17. *Let $i, j \in \{1, \dots, n\}$ and let both partial derivatives $\frac{\partial^2 f}{\partial x_i \partial x_j}$ and $\frac{\partial^2 f}{\partial x_j \partial x_i}$ be continuous at a point $\boldsymbol{a} \in \mathbf{R}^n$. Then we have*

$$\frac{\partial^2 f}{\partial x_i \partial x_j}(\boldsymbol{a}) = \frac{\partial^2 f}{\partial x_j \partial x_i}(\boldsymbol{a}).$$

Definition. Let $G \subset \mathbf{R}^n$ be an open set and $k \in \mathbf{N}$. We say that a function f is of the *class $\mathcal{C}^k$ on G*, if all partial derivatives of f till k-th order are continuous on G. The set of all these functions is denoted by $\mathcal{C}^k(G)$. We say that a function f is of the *class $\mathcal{C}^\infty$ on G*, if all partial derivatives of all orders of f are continuous on G. The set of all functions of the class $\mathcal{C}^\infty$ on G is denoted by $\mathcal{C}^\infty(G)$.