Basic notation

- $\mathbb{R}$ ... the set of real numbers
- $\mathbb{C}$ ... the set of complex numbers
- $\overline{\mathbb{C}}$ ... the extended complex plane, i.e. $\mathbb{C} \cup \{\infty\}$
- $H(G)$ ... the algebra of functions holomorphic (=analytic) on G, where $G \subset \overline{\mathbb{C}}$ is a nonempty open subset.
- $U(a,r)$ ($a \in \mathbb{C}$, $r > 0$) ... the open disc with center a and radius r
- $P(a,r)$ ($a \in \mathbb{C}$, $r > 0$) ... the reduced neighborhood $U(a,r) \setminus \{a\}$
- $P(a,r,R)$ ($a \in \mathbb{C}$, $0 \le r < R \le +\infty$)
  ... the annulus $\{z \in \mathbb{C} : r < |z-a| < R\}$
- $\operatorname{ind}_\gamma a$ ... the index of the point a with respect to the closed path γ (= the winding number of γ around a)
- $\operatorname{res}_a f$ ... the residue of the function f at the point a

I.1 Harmonic functions on $\mathbb{R}^2$ and their connections to holomorphic ones

Definition. Let $G \subset \mathbb{R}^2$ be an open set. A function $f : G \to \mathbb{R}$ is said to be **harmonic**, if it is continuous on G and satisfies on G the equality

$$\Delta f = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} = 0.$$

Remark. Complex-valued harmonic functions are defined similarly. Then obviously, a complex function f is harmonic if and only if both $\operatorname{Re} f$ and $\operatorname{Im} f$ are harmonic.

Proposition 1. *Let $G \subset \mathbb{C}$ be an open set.*

(i) *If $f \in H(G)$, then functions f_1, f_2 defined by the formulas*

$$f_1(x,y) = \operatorname{Re} f(x+iy), \qquad f_2(x,y) = \operatorname{Im} f(x+iy)$$

are harmonic on G (if we identify $\mathbb{C}$ and $\mathbb{R}^2$).

(ii) *Let $f : G \to \mathbb{R}$ be a harmonic function (if we identify $\mathbb{C}$ and $\mathbb{R}^2$). If moreover $f \in C^2(G)$, then the following assertions hold:*

- *The function*

$$g(x+iy) = \frac{\partial f}{\partial x}(x,y) - i\frac{\partial f}{\partial y}(x,y)$$

is holomorphic on G.

- *If G is simply connected, then there is $\tilde{f} \in H(G)$ such that $\operatorname{Re} \tilde{f}(x+iy) = f(x,y)$ on G.*

Corollary. *Let $G \subset \mathbb{C}$ be an open set and f be a holomorphic function on G, which does not attain zero on G. Then the function $g(x,y) = \ln|f(x+iy)|$ is harmonic on G (if we identify $\mathbb{C}$ and $\mathbb{R}^2$).*

Remark. It follows from Theorem 6 below that harmonic functions are automatically C^∞.

Definition. By the **Poisson kernel** we understand the function defined by the formula

$$P_r(t) = \sum_{n=-\infty}^{\infty} r^{|n|} e^{int}, \qquad t \in \mathbb{R}, r \in [0,1).$$

Proposition 2 (properties of the Poisson kernel).

(i) $P_r(\theta - t) = \operatorname{Re} \frac{e^{it}+re^{i\theta}}{e^{it}-re^{i\theta}} = \frac{1-r^2}{1-2r\cos(\theta-t)+r^2}$ *for* $r \in [0,1)$, $t, \theta \in \mathbb{R}$.

(ii) $\frac{1}{2\pi}\int_{-\pi}^{\pi} P_r(t)\,\mathrm{d}t = 1$ *for* $r \in [0,1)$.

(iii) P_r *is a strictly positive even* 2π*-periodic function for each* $r \in [0,1)$. *For* $r > 0$ *the function* P_r *is strictly decreasing on* $[0,\pi]$.

(iv) $\lim_{r\to 1-} P_r(t) = 0$ *unless* t *is a multiple of* 2π.

Remark. By $\mathbb{T}$ we denote the unit circle, i.e., $\{e^{it}, t \in \mathbb{R}\}$. Functions on $\mathbb{T}$ are canonically identified with 2π-peridodic functions on $\mathbb{R}$, measures on $\mathbb{T}$ are identified with measures on $[-\pi,\pi)$ (sometimes on $[\alpha, \alpha+2\pi)$ for some $\alpha \in \mathbb{R}$). On $\mathbb{T}$ we consider the normalized Lebesgue measure. The spaces $L^p(\mathbb{T})$ are considered with respect to this measure.

Definition.

- Let $f \in L^1(\mathbb{T})$. By the **Poisson integral** of the function f we mean the function $P[f]$ defined on $U(0,1)$ by the formula

$$P[f](re^{i\theta}) = \frac{1}{2\pi}\int_{-\pi}^{\pi} P_r(\theta - t) f(t)\,\mathrm{d}t, \qquad r \in [0,1), \theta \in \mathbb{R}.$$

- Let μ be a (signed or complex-valued) Borel measure on $\mathbb{T}$. By the **Poisson integral** of the measure μ we mean the function $P[\,\mathrm{d}\mu]$ defined on $U(0,1)$ by the formula

$$P[\,\mathrm{d}\mu](re^{i\theta}) = \int_{[-\pi,\pi)} P_r(\theta - t)\,\mathrm{d}\mu(t), \qquad r \in [0,1), \theta \in \mathbb{R}.$$

Proposition 3. *$P[\mathrm{d}\mu]$ is a harmonic function on $U(0,1)$ for any complex Borel measure μ on $\mathbb{T}$. In particular, $P[f]$ is a harmonic function on $U(0,1)$ for any $f \in L^1(\mathbb{T})$.*

Further, if μ is a real-valued measure, the function $P[\mathrm{d}\mu]$ is real-valued as well. If μ is non-negative, the function $P[\mathrm{d}\mu]$ is non-negative as well. Similarly for f and $P[f]$.

Proposition 4 (a version of the residue theorem). *Let $a \in \mathbb{C}$, $R > 0$ and $M \subset U(a,R)$ be a finite set. Let f be a complex function continuous on $\overline{U(a,R)} \setminus M$ and holomorphic on $U(a,R) \setminus M$. If φ is the positively oriented circle with center a and radius R, then*

$$\int_\varphi f = 2\pi i \sum_{a \in M} \operatorname{res}_a f.$$

Corollary (Poisson integral of a holomorphic function). *Let $a \in \mathbb{C}$, $R > 0$ and f be a complex function continuous on $\overline{U(a,R)}$ and holomorphic on $U(a,R)$. Then for each $r \in [0,R)$ and $\theta \in \mathbb{R}$ the following formulas hold:*

- $\frac{1}{2\pi}\int_{-\pi}^{\pi} f(a+Re^{it}) \cdot \frac{Re^{it}+re^{i\theta}}{Re^{it}-re^{i\theta}}\,\mathrm{d}t = 2f(a+re^{i\theta}) - f(a);$
- $\frac{1}{2\pi}\int_{-\pi}^{\pi} f(a+Re^{it}) \cdot \frac{Re^{-it}+re^{-i\theta}}{Re^{-it}-re^{-i\theta}}\,\mathrm{d}t = f(a);$
- $f(a+re^{i\theta}) = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(a+Re^{it}) \cdot \operatorname{Re}\frac{Re^{it}+re^{i\theta}}{Re^{it}-re^{i\theta}}\,\mathrm{d}t.$

Theorem 5 (solution of the Dirichlet problem on the disc). *Let f be a function continuous on $\mathbb{T}$. Let us define a function Hf by the formula*

$$Hf(re^{i\theta}) = \begin{cases} f(e^{i\theta}), & r = 1, \theta \in \mathbb{R}, \\ P[f](re^{i\theta}), & r \in [0,1), \theta \in \mathbb{R}. \end{cases}$$

Then the function Hf is continuous on $\overline{U(0,1)}$ (and also harmonic on $U(0,1)$ and equal to f on $\mathbb{T}$).

Theorem 6 (expressing a harmonic function by the Poisson integral). *Let f be a complex function continuous on $\overline{U(0,1)}$ and harmonic on $U(0,1)$. Then $f = P[f|_{\mathbb{T}}]$ on $U(0,1)$.*

Corollary.

- *If f is a complex function continuous on $\overline{U(a,R)}$ and harmonic on $U(a,R)$, then for $r \in [0,R)$ and $\theta \in \mathbb{R}$ the following formula holds:*

$$\begin{aligned} f(a+re^{i\theta}) &= \frac{1}{2\pi}\int_{-\pi}^{\pi} \frac{R^2-r^2}{R^2-2Rr\cos(\theta-t)+r^2} f(a+Re^{it})\,\mathrm{d}t \\ &= \frac{1}{2\pi}\int_{-\pi}^{\pi} f(a+Re^{it}) \cdot \operatorname{Re}\frac{Re^{it}+re^{i\theta}}{Re^{it}-re^{i\theta}}\,\mathrm{d}t. \end{aligned}$$

- *A real-valued harmonic function on $U(a,R)$ is the real part of a holomorphic function on $U(a,R)$.*
- *Harmonic functions are C^∞.*
- *Let f be a function continuous on $\overline{U(a,R)}$ and harmonic on $U(a,R)$. Then $f(a) = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(a+Re^{it})\,\mathrm{d}t$.*

Theorem 7 (Harnack). *Let $G \subset \mathbb{R}^2$ be a domain and let (f_n) be a sequence of harmonic functions on G.*

(i) *It the sequence (f_n) is locally uniformly convergent on G, the limit function is harmonic on G.*
(ii) *Suppose that the functions f_n are real-valued and the sequence $(f_n(z))$ is non-decreasing for each $z \in G$. Then either the sequence (f_n) is locally uniformly convergent on G or $f_n(z) \to +\infty$ for each $z \in G$.*

Definition. Let $G \subset \mathbb{R}^2$ be an open set and let f be a continuous function on G. We say that f enjoys the **mean value property**, if for any $a \in G$ there is a sequence $r_n \searrow 0$ such that for any $n \in \mathbb{N}$ the following formula holds:

$$f(a) = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(a+r_n e^{it})\,\mathrm{d}t$$

Věta 8. *Let $G \subset \mathbb{R}^2$ be an open set and let f be a continuous function on G. If f enjoys the mean value property, then f is harmonic on G.*

Theorem 9 (Schwarz reflection principle). *Let $\Omega \subset \mathbb{C}$ be a domain, which is symmetric with respect to reflection through the real axis. Denote by Ω^+ the intersection of Ω with the half-plane $\{z : \operatorname{Im} z > 0\}$ and Ω^- the intersection with the half-plane $\{z : \operatorname{Im} z < 0\}$. Let f be a holomorphic function on Ω^+ such that for each $x \in \Omega \cap \mathbb{R}$ we have*

$$\lim_{z\to x, z\in\Omega^+} \operatorname{Im} f(z) = 0.$$

Then there is $F \in H(\Omega)$ such that $F = f$ na Ω^+. Moreover, this F satisfies $F(\overline{z}) = \overline{F(z)}$ for $z \in \Omega$.