Operators of multiplication – spectrum, spectral measures etc.

① Let (Ω, Σ, μ) be a complete semifinite measure space (positive, σ-additive):
semifinite means: $\forall A \in \Sigma$, $\mu(A) > 0$ $\exists B \in \Sigma$, $B \subset A$, $0 < \mu(B) < \infty$

[This includes • μ finite
• μ σ-finite
• Σ = the power set of Ω, μ = the counting measure]

Let $\varphi : \Omega \to \mathbb{C}$ be a Σ-measurable function (bdd or not)
Define

$$D(M_\varphi) = \{ f \in L^2(\mu) ; \varphi \cdot f \in L^2(\mu) \}$$
$$M_\varphi(f) = \varphi \cdot f, \quad f \in D(M_\varphi)$$

Then clearly $D(M_\varphi) \subset\subset L^2(\mu)$ and M_φ is an operator on $L^2(\mu)$ with domain $D(M_\varphi)$

② φ essentially bounded $\Rightarrow$ $D(M_\varphi) = L^2(\mu)$, $M_\varphi \in L(L^2(\mu))$, $\|M_\varphi\| = \|\varphi\|_\infty$

• φ ess. bdd, i.e. $\varphi \in L^\infty(\mu)$ Then:

$$f \in L^2(\mu) \Rightarrow \int |\varphi \cdot f|^2 d\mu \underset{\uparrow\, |\varphi| \le \|\varphi\|_\infty \ \mu\text{-a.e.}}{\le} \int \|\varphi\|_\infty^2 \cdot |f|^2 d\mu = \|\varphi\|_\infty^2 \cdot \|f\|_2$$

It follows that $D(M_\varphi) = L^2(\mu)$ & $\|M_\varphi\| \le \|\varphi\|_\infty$

• $\|M_\varphi\| \ge \|\varphi\|_\infty$:

Take any $c \in \mathbb{R}$ s.t. $0 < c < \|\varphi\|_\infty$ (If $\|\varphi\|_\infty = 0$, the inequality is obvious)
It means that the set $A := \{\omega \in \Omega ; |\varphi(\omega)| > c\}$ belongs to Σ and $\mu(A) > 0$
μ semifinite $\Rightarrow$ $\exists B \in \Sigma$, $B \subset A$, $0 < \mu(B) < \infty$
Define
$$f(\omega) = \begin{cases} \frac{1}{\sqrt{\mu(B)}} \cdot \frac{\varphi(\omega)}{|\varphi(\omega)|} & \omega \in B \\ 0 & \omega \in \Omega \setminus B \end{cases}$$

Then f is Σ-measurable, $|f(\omega)| = \begin{cases} \frac{1}{\sqrt{\mu(B)}}, & \omega \in B \\ 0 & \omega \in \Omega \setminus B \end{cases}$

$$\Rightarrow \int |f|^2 d\mu = \int_B \frac{1}{\mu(B)} d\mu = 1, \text{ so } \|f\|_2 = 1$$

$$\|M_\varphi(f)\|_2^2 = \|\varphi\cdot f\|_2^2 = \int_B \frac{|\varphi(\omega)|^2}{\mu(B)}\,d\mu \geq \int_B \frac{c^2}{\mu(B)}\,d\mu = c^2$$

So $\|M_\varphi\| \geq c$.

③ $D(M_\varphi) = L^2(\mu) \Rightarrow \varphi$ is essentially bdd

$\lceil D(M_\varphi) = L^2(\mu)$ means: $\forall f \in L^2(\mu)$: $\varphi\cdot f \in L^2(\mu)$

For $n \in \mathbb{N}$ set

$A_n := \{\omega \in \Omega;\ |\varphi(\omega)| \leq n\}$. Then $A_n \in \Sigma$, $A_n \nearrow \Omega$

So, for each $f \in L^2(\mu)$ we have

$$f\cdot \chi_{A_n} \to f \text{ in } L^2(\mu)$$

$$\lceil \|f\cdot\chi_{A_n} - f\|_2^2 = \int \underbrace{|f|^2 \chi_{\Omega\setminus A_n}}_{\to 0 \text{ pointwise}}\, d\mu \longrightarrow 0$$

Lebesgue dom. conv. thm as

and $|f|^2$ is integrable majorant

In particular:

$$\forall f \in L^2(\mu): \varphi\cdot f = \lim_{n\to\infty} \varphi\cdot f\cdot \chi_{A_n} = \lim_{n\to\infty} M_{\varphi\cdot\chi_{A_n}}(f)$$

(limits in $L^2(\mu)$)

Hence, by the uniform bddness principle $\exists C > 0\ \forall n\ \|M_{\varphi\cdot\chi_{A_n}}\| \leq C$

Hence, by ② we deduce $\forall n$: $\|\varphi\cdot\chi_{A_n}\|_\infty \leq C$,

i.e. $|\varphi\cdot\chi_{A_n}| \leq C$ μ-a.e. (for each $n \in \mathbb{N}$)

$\Rightarrow |\varphi| \leq C$ μ-a.e.

So, φ is essentially bdd.

④ M_φ is densely defined and closed:

- φ ess. bdd $\overset{②}{\Rightarrow} D(M_\varphi) = L^2(\mu)$, so it is densely defined
- φ not ess. bdd $\Rightarrow$ Let A_n be as in ③

Then $\forall f \in L^2(\mu)$: $f\cdot\chi_{A_n} \to f$ in $L^2(\mu)$

and $f\cdot\chi_{A_n} \in D(M_\varphi)$ (see ③)

So, M_φ is densely defined

- $(f_n) \subset D(M_\varphi)$: $f_n \longrightarrow f$ in $L^2(\mu)$
$M_\varphi(f_n) \longrightarrow g$ in $L^2(\mu)$

Then $\exists\, (f_{n_k})$ subsequence s.t. $f_{n_k} \longrightarrow f$ μ-a.e.

Then $M_\varphi(f_{n_k}) = \varphi \cdot f_{n_k} \longrightarrow \varphi \cdot f$ μ-a.e.

$\searrow g$ in $L^2(\mu)$

It follows that $\varphi \cdot f = g$ μ-a.e.

Thus $f \in D(M_\varphi)$ & $g = M_\varphi(f)$.

(5) $M_{\varphi+\psi} \supset M_\varphi + M_\psi$. If ψ is ess. bdd, then $M_{\varphi+\psi} = M_\varphi + M_\psi$

Γ. $f \in D(M_\varphi + M_\psi) = D(M_\varphi) \cap D(M_\psi) \Rightarrow \varphi f, \psi f \in L^2(\mu)$

$\Rightarrow (\varphi+\psi) f = \varphi f + \psi f \in L^2(\mu)$, so $f \in D(M_{\varphi+\psi})$

and $M_{\varphi+\psi}(f) = M_\varphi(f) + M_\psi(f)$

- Assume ψ ess. bdd. Then $D(M_\psi) = L^2(\mu)$ and hence

$f \in D(M_\varphi + M_\psi) \Leftrightarrow \varphi f, \psi f \in L^2(\mu) \Leftrightarrow \varphi f \in L^2(\mu) \Leftrightarrow (\varphi+\psi) f \in L^2(\mu)$

$\psi f \in L^2(\mu)$ automatically $\qquad \Updownarrow$

$f \in D(M_{\varphi+\psi})$ $\lrcorner$

(6) $M_{\varphi \cdot \psi} \supset M_\varphi \cdot M_\psi$. If ψ is ess bdd, then $M_{\varphi\cdot\psi} = M_\varphi \cdot M_\psi$

Γ. $f \in D(M_\varphi M_\psi) \Leftrightarrow f \in D(M_\psi)$ & $M_\psi(f) \in D(M_\varphi) \Leftrightarrow \psi \cdot f \in L^2(\mu)$ & $\varphi \cdot \psi f \in L^2(\mu)$

$\Rightarrow \varphi \psi f \in L^2(\mu) \Leftrightarrow f \in D(M_{\varphi\psi})$. Then clearly $M_{\varphi\psi}(f) = \varphi\psi f = M_\varphi M_\psi f$

- If ψ is ess bdd, then $D(M_\psi) = L^2(\mu)$, so here $\Leftarrow$ holds $\lrcorner$

(7) $(M_\varphi)^* = M_{\bar\varphi}$, $D((M_\varphi)^*) = D(M_\varphi)$

- clearly $D(M_{\bar\varphi}) = D(M_\varphi)$
- $f, g \in D(M_\varphi) \Rightarrow \langle M_\varphi f, g\rangle = \langle \varphi f, g\rangle = \int \varphi f \bar g \, d\mu$

$$= \int f \cdot \overline{\bar\varphi g}\, d\mu = \langle f, M_{\bar\varphi} g\rangle$$

So, $M_{\bar\varphi} \subset (M_\varphi)^*$

- Conversely, assume $g \in D((M_\varphi)^*)$. Then $\exists\, h \in L^2(\mu)$: $\langle M_\varphi f, g\rangle = \langle f, h\rangle$ for $f \in D(M_\varphi)$

So $\forall f \in D(M_\varphi)$: $0 = \langle M_\varphi f, g\rangle - \langle f, h\rangle = \int \varphi f \bar g \, d\mu - \int f \bar h \, d\mu$

$$= \int f (\varphi \bar g - \bar h)\, d\mu$$

Then $\varphi \bar g - \bar h = 0$ μ-a.e.

[Assume not: $B = \{\omega;\ \varphi(\omega)\overline{g(\omega)} - \bar h(\omega) \neq 0\}$, assume $\mu(B) > 0$

Define A_n as in (3). then $A_n \cap B \nearrow B$, so there is some $n \in \mathbb{N}$ s.t. $\mu(B \cap A_n) > 0$.

μ semi-finite $\Rightarrow \exists\ C \in \Sigma$, $C \subset B \cap A_n$, $0 < \mu(C) < \infty$

Define

$$f(\omega) = \begin{cases} \dfrac{\overline{\varphi(\omega)}\, g(\omega) - h(\omega)}{|\varphi(\omega)\overline{g(\omega)} - \overline{h(\omega)}|}, & \omega \in C \\ 0, & \omega \in \Omega \setminus C \end{cases}$$

Then $f \in D(M_\varphi)$ [$|f| = 1$ on C, $|f| = 0$ on $\Omega\setminus C$, $\mu(C) < \infty$ $\Rightarrow f \in L^2(\mu)$

$|\varphi| \le n$ on $C \Rightarrow \varphi \cdot f \in L^2(\mu)$

and $\int f(\varphi \bar g - \bar h)\, d\mu = \int_C \underbrace{|\varphi \bar g - \bar h|}_{>0 \text{ on } C}\, d\mu > 0$ as $\mu(C) > 0$ and]

So, $h = \bar\varphi g$ μ-a.e., hence $g \in D(M_{\bar\varphi})$ and $(M_\varphi)^* g = h$

(8) $\sigma(M_\varphi) = \text{ess-rng}(\varphi)$, $\lambda \in \sigma_p(M_\varphi) \Leftrightarrow \mu(\varphi^{-1}(\{\lambda\})) > 0$

- Assume $\mu(\varphi^{-1}(\{\lambda\})) > 0$. Let $B \in \Sigma$, $B \subset \varphi^{-1}(\lambda)$ be such that $0 < \mu(B) < \infty$.
  Then $\chi_B \in L^2(\mu) \setminus \{0\}$, $M_\varphi(\chi_B) = \lambda \chi_B$, so λ is an eigenvalue and χ_B an eigenvector.

- Assume $\mu(\varphi^{-1}(\{\lambda\})) = 0$ and $f \in L^2(\mu)$, $M_\varphi f = \lambda f$.
  Thus $\varphi f = \lambda f$ μ-a.e.
  $$\underbrace{(\varphi(\omega) - \lambda)}_{\neq 0\ \mu\text{-a.e.}} f(\omega) = 0 \quad \mu\text{-a.e.}$$
  $\Rightarrow f(\omega) = 0$ μ-a.e., so $f = 0$. Hence $\lambda \notin \sigma_p(M_\varphi)$

- Assume $\lambda \notin \text{ess-rng}\ \varphi$. Then $\exists\, r > 0$: $\mu(\varphi^{-1}(U(\lambda, r))) = 0$
  So $|\varphi - \lambda| \geq r$ μ-a.e.
  $\Rightarrow \frac{1}{|\varphi - \lambda|} \leq \frac{1}{r}$ μ-a.e. $\Rightarrow \frac{1}{\lambda - \varphi}$ is essentially bdd
  $$\Rightarrow M_{\frac{1}{\lambda-\varphi}} \in L(L^2(\mu)). \qquad M_{\frac{1}{\lambda-\varphi}}(\lambda I - M_\varphi) = M_{\frac{1}{\lambda-\varphi}} M_{\lambda-\varphi} \subset M_1 = I$$
  $$(\lambda I - M_\varphi) M_{\frac{1}{\lambda-\varphi}} = M_{\lambda-\varphi} M_{\frac{1}{\lambda-\varphi}} \overset{(6)}{=} M_1 = I$$
  So, $\lambda I - M_\varphi$ is one-to-one and onto, so $\lambda \in \rho(M_\varphi)$

- Assume $\lambda \in \rho(M_\varphi)$
  - if $\mu(\varphi^{-1}(\lambda)) > 0$, then $\varphi \in \sigma_p(M_\varphi)$, a contradiction
  - Hence $\mu(\varphi^{-1}(\lambda)) = 0$, i.e. $\varphi \neq \lambda$ μ-a.e.
    Then $\lambda I - M_\varphi$ is one-to-one and $M_{\frac{1}{\lambda-\varphi}}$ is defined.
    Moreover, by (6) we know $M_{\frac{1}{\lambda-\varphi}}(\lambda I - M_\varphi) = M_{\frac{1}{\lambda-\varphi}} M_{\lambda-\varphi} \subset M_1 = I$

    Therefore $(\lambda I - M_\varphi)^{-1}$ must be an extension of $M_{\frac{1}{\lambda-\varphi}}$

But this means that $M_{\frac{1}{\lambda-\varphi}}$ must be cts, so $\frac{1}{\lambda-\varphi}$ is ess. bdd, i.e. $\exists\, c>0$ $\left|\frac{1}{\lambda-\varphi}\right| \le c$ μ-a.e.

$$\Rightarrow |\lambda-\varphi| \ge \frac{1}{c} \ \mu\text{-a.e.} \Rightarrow \mu\left(\varphi^{-1}\left(U(\lambda,\tfrac{1}{c})\right)\right)=0$$
$$\Rightarrow \lambda \notin \text{ess rng } \varphi$$

(9) φ ess. bdd, $f\in C(\sigma(M_\varphi)) \Rightarrow \tilde{f}(M_\varphi) = M_{f\circ\varphi}$

- $f \mapsto \tilde{f}(M_\varphi)$ is a $*$-isomorphism [by properties of functional calculus]
- $\varphi \mapsto M_\varphi$ is a $*$-isomorphism $L^\infty(\mu) \longrightarrow L(L^2(\mu))$ [by (2), (5), (6), (7)]
- $f \mapsto f\circ\varphi$ is a $*$-homomorphism $C(\sigma(M_\varphi)) \longrightarrow L^\infty(\mu)$
  [easy: it is linear, $(f\circ\varphi)(g\circ\varphi) = (fg)\circ\varphi$, $\overline{f\circ\varphi} = \overline{f}\circ\varphi$]
- So, $f \mapsto M_{f\circ\varphi}$ is a $*$-homomorphism $C(\sigma(M_\varphi)) \longrightarrow L(L^2(\mu))$
  [composition of two $*$-homomorphisms]

So, we have two $*$-homomorphisms $C(\sigma(M_\varphi)) \longrightarrow L(L^2(\mu))$

We claim they are equal:

$\mathcal{A} = \{f\in C(\sigma(M_\varphi))\ ;\ \tilde{f}(M_\varphi) = M_{f\circ\varphi}\}$ is a C^*-subalgebra

$1\in\mathcal{A}$: $\tilde{1}(M_\varphi) = I$, $M_{1\circ\varphi} = M_1 = I$

$id\in\mathcal{A}$: $\widetilde{id}(M_\varphi) = M_\varphi$, $M_{id\circ\varphi} = M_\varphi$

So, $\mathcal{A}$ contains constants and separates points $\overset{S-W}{\Rightarrow}$ $\mathcal{A} = C(\sigma(M_\varphi))$.

(10) Spectral measures $E_{g,h}$:

Fix $g,h\in L^2(\mu)$. By definition we have (for $f\in C(\sigma(M_\varphi))$):

$$\int f\, dE_{g,h} = \langle \tilde{f}(M_\varphi)g, h\rangle = \langle M_{f\circ\varphi}\, g, h\rangle = \int (f\circ\varphi) g\bar{h}\, d\mu$$

$$= \int f\circ\varphi\, d\mu_{g,h} = \int f\, d\varphi(\mu_{g,h})$$

$d\mu_{g,h} = g\bar{h}\, d\mu$,
measure with density $g\bar{h}$ w.r.t. μ

image of $\mu_{g,h}$ under φ

The equality holds for any f cts. It follows that $E_{g,h}$ is the completion of $\varphi(\mu_{g,h})$ restricted to the Borel σ-algebra.

$\ulcorner$ Recall $\varphi(\mu_{g,h})(B) = \mu_{g,h}(\varphi^{-1}(B))$, $B \in \mathcal{A}_0 = \{A;\ \varphi^{-1}(A) \in \Sigma\}$

$\mathcal{A}_0$ contains Borel sets, as φ is measurable.

So, $E_{g,h}$ is obtained as follows:

- Take $\varphi(\mu_{g,h})$ on $\mathcal{A}_0$ as above
- Restrict $\varphi(\mu_{g,h})$ to the Borel σ-algebra
- take the completion $\lrcorner$

(11) In some cases $E_{g,h} = \varphi(\mu_{g,h})$, the additional procedure is not necessary.

This applies, for example, in the following cases:

- $\Omega \subset \mathbb{R}^d$ open, Σ = Lebesgue-measurable sets, μ = the d-dim. Lebesgue measure

  $\ulcorner$ One may use the method of the proof of Lemma XIII.16, part (1) $\lrcorner$
- μ = the counting measure on Ω.

  $\ulcorner$ Then measures $\mu_{g,h}$ are countably supported (carried by a ctble set) $\lrcorner$

(12) The spectral measure in the special cases mentioned in (11) is $E(A) = M_{\chi_{\varphi^{-1}(A)}}$, $A \in \mathcal{A} = \{A \subset \sigma(M_\varphi);\ \varphi^{-1}(A) \in \Sigma\}$

$$\ulcorner \langle E(A)g, h\rangle = E_{g,h}(A) = \varphi(\mu_{g,h})(A) = \mu_{g,h}(\varphi^{-1}(A)) =$$

$$= \int_{\varphi^{-1}(A)} g\bar{h}\, d\mu = \langle M_{\chi_{\varphi^{-1}(A)}} g, h\rangle \lrcorner$$

(13) In the cases mentioned in (11) we have (E is from (12), f is $\mathcal{A}$-measurable)

$$\int f\, dE = M_{f\circ\varphi}$$

- $\int |f|^2 dE_{g,g} = \int |f|^2 d\varphi(\mu_{g,g}) = \int |f\circ\varphi|^2 d\mu_{g,g} = \int |f\circ\varphi|^2 |g|^2$

So, $g \in D(\int f dE) \Leftrightarrow \int |f|^2 dE_{gg} < \infty \Leftrightarrow (f\circ\varphi)g \in L^2(\mu) \Leftrightarrow g \in D(M_{f\circ\varphi})$

- $g, h \in D(\int f dE) \Rightarrow \langle (\int f dE)g, h\rangle = \int f dE_{g,h} = \int f d\varphi(\mu_{g,h}) =$

$= \int f\circ\varphi \, d\mu_{g,h} = \int (f\circ\varphi) g \bar{h} \, d\mu = \langle M_{f\circ\varphi} g, h\rangle$

(14) Let (Ω, Σ, μ) be one of the special cases from (11)
Let $\varphi: \Omega \to \mathbb{C}$ be Σ-measurable.
Then the spectral measure of M_φ is defined by

$E(A) = M_{1_{\varphi^{-1}(A)}}, \quad A \in \mathcal{B} = \{A \subset \mathbb{C}, \ \varphi^{-1}(A) \in \Sigma\}$

Case 1: φ ess. bdd ... use (12)

Case 2: φ general: $\psi(\omega) := \frac{\varphi(\omega)}{1+|\varphi(\omega)|}$, $f(\lambda) := \frac{\lambda}{1-|\lambda|}$

Then
- ψ is bdd, Σ-measurable $\Rightarrow$ the formula for E_{M_ψ} works
- $|\psi| < 1 \Rightarrow \sigma(M_\psi) \subset \overline{U(0,1)}$
- $E_{M_\psi}(\mathbb{T}) = \mu(\psi^{-1}(\mathbb{T})) = \mu(\emptyset) = 0$

$\uparrow |\psi| < 1$ everywhere

- f cts on $U(0,1) \Rightarrow f$ Borel on $\sigma(M_\psi)$

(defined E_{M_ψ}-a.e.)

$\overset{(13)}{\Longrightarrow} \int f \, dE_{M_\psi} = M_{f\circ\psi} = M_\varphi$

Define $F(A) = E_{M_\psi}(f^{-1}(A))$; $A \in \tilde{\mathcal{B}} = \{A \subset \mathbb{C}, \ f^{-1}(A) \in \mathcal{B}_{M_\psi}\}$

Then $M_\varphi = \int \mathrm{id} \, dF$ by Lemma XIII.16

Moreover, $F(A) = E_{M_\psi}(f^{-1}(A)) = \mu(\psi^{-1}(f^{-1}(A))) = \mu(\varphi^{-1}(A))$