

SELF-ADJOINT LAPLACE OPERATORS

Let $\Omega \subset \mathbb{R}^d$ be an open set.

① Denote $Z := W^{1,2}(\Omega) = \{f \in L^2(\Omega), \forall j=1 \dots d: \partial_j f \in L^2(\Omega)\}$
↑
in $\mathcal{D}'(\Omega)$

Then Z is a Hilbert space, the inner product

is defined by $\langle f, g \rangle_Z = \int_{\Omega} f \bar{g} + \sum_{j=1}^d \int_{\Omega} \partial_j f \partial_j \bar{g}$

Sketch: $\nabla f = (\partial_1 f, \dots, \partial_d f)$... a vector function $\Omega \rightarrow \mathbb{C}$

Then $\langle f, g \rangle_Z = \int_{\Omega} f \bar{g} + \int_{\Omega} \langle P f(x), P g(x) \rangle_{\mathbb{C}^d} dx$

Let $Z_0 := \overline{\mathcal{D}(\Omega)}^Z$. Fix a closed subspace Y s.t.
 $Z_0 \subset Y \subset Z$

② Let $J: Y \rightarrow L^2(\Omega)$ be the "identity" mapping, i.e.
 $Jf = f, f \in Y$

Then J is a bounded linear operator, $\|J\| \leq 1$

$$\sqrt{\|Jf\|_{L^2(\Omega)}^2} = \langle Jf, Jf \rangle_{L^2(\Omega)} = \int_{\Omega} |f|^2 \leq$$

$$\leq \int_{\Omega} |f|^2 + \int_{\Omega} \langle P f(x), P f(x) \rangle dx = \|f\|_Y^2$$

$\bullet R(J)$ is dense in $L^2(\Omega)$, as it contains $\mathcal{D}(\Omega)$

③ Let $J^*: L^2(\Omega) \rightarrow Y$ be the adjoint operator, i.e.

$$\langle Jf, g \rangle_{L^2(\Omega)} = \langle f, J^*g \rangle_Y \quad \text{for } f \in Y, g \in L^2(\Omega)$$

Then $\|J^*\| = \|J\| \leq 1$, and J^* is one-to-one (as $R(J)$ is dense)

(4) The operator $JJ^* : L^2(\Omega) \rightarrow L^2(\Omega)$ is a self-adjoint ($\|JJ^*\| \leq 1$) self-adjoint ($(JJ^*)^* = (J^*)^*J^* = JJ^*$) operator. Moreover, JJ^* is one-to-one:

$$\begin{aligned} \bullet \quad JJ^*f = 0 &\Rightarrow 0 = \langle JJ^*f, f \rangle = \langle J^*f, J^*f \rangle = \|J^*f\|^2 \\ &\Rightarrow J^*f = 0 \Rightarrow f = 0 \text{ (as } J^* \text{ is one-to-one)}. \end{aligned}$$

Hence, $R(JJ^*)$ is dense and $T := (JJ^*)^{-1}$ is self-adjoint (Prop. XI.23 (e))

(5) $DCT) = R(JJ^*) = J(RJ^*)$. Since J is "identity", let us describe $R(J^*)$

Let $f \in Y$. Then $f \in R(J^*) \Leftrightarrow \exists g \in L^2(\Omega) : J^*g = f$

$$\text{Further, } J^*g = f \Leftrightarrow \forall h \in Y : \langle J^*g, h \rangle_Y = \langle f, h \rangle_Y$$

$$\text{We have } \langle f, h \rangle_Y = \int_{\Omega} f \bar{h} + \int_{\Omega} \langle Pf, Ph \rangle$$

$$\langle J^*g, h \rangle_Y = \langle g, Jh \rangle_Y = \int_{\Omega} g \bar{h}$$

$$\text{So, } \langle f, h \rangle_Y = \langle J^*g, h \rangle_Y \Leftrightarrow \int_{\Omega} \langle Pf, Ph \rangle = \int_{\Omega} (g-f) \bar{h}$$

Conclusion:

$$f \in R(J^*) \Leftrightarrow \exists g \in L^2(\Omega) \forall h \in Y : \int_{\Omega} \langle Pf, Ph \rangle = \int_{\Omega} (g-f) \bar{h}$$

(6) Necessary condition: $f \in \mathcal{D}'(\mathbb{R}^d) \Rightarrow \Delta f$ (in $\mathcal{D}'(\mathbb{R}^d)$) belongs to $L^2(\mathbb{R}^d)$ and $Tf = f - \Delta f$

Apply the characterization from (5) to $h \in \mathcal{D}(\mathbb{R}^d)$.

Note, that, $\mathcal{D}(\mathbb{R}^d) \subset \mathcal{C}_c^\infty$

$$\text{So, } f \in \mathcal{R}(\mathcal{D}') \Rightarrow \exists g \in L^2(\mathbb{R}^d) \quad \forall h \in \mathcal{D}(\mathbb{R}^d) \quad \int_{\mathbb{R}^d} \langle \mathcal{D}f, \mathcal{D}h \rangle = \int_{\mathbb{R}^d} (g-f)\bar{h}$$

So, for $h \in \mathcal{D}(\mathbb{R}^d)$ we have

$$\int_{\mathbb{R}^d} (g-f)\bar{h} = \int_{\mathbb{R}^d} \langle \mathcal{D}f, \mathcal{D}h \rangle = \sum_{j=1}^d \int_{\mathbb{R}^d} \partial_j f \partial_j \bar{h} =$$

$$= \sum_{j=1}^d \langle \partial_j f, \partial_j \bar{h} \rangle = - \sum_{j=1}^d \langle \partial_{jj} f, \bar{h} \rangle =$$

$$= \langle -\Delta f, \bar{h} \rangle \quad (\text{where } \partial_{jj} \text{ are applications of a distribution to a test function}).$$

It follows $g-f = -\Delta f$ in $\mathcal{D}'(\mathbb{R}^d)$. Thus $\Delta f \in L^2(\mathbb{R}^d)$

$$\text{and } g = f - \Delta f$$

Since $f - \Delta f = g$ and $f = \mathcal{D}^* g$, we have

$$\mathcal{D}f = \mathcal{D}^* g \Rightarrow g = T(\mathcal{D}f)$$

So, $T(\mathcal{D}f) = f - \Delta f$. This is the sought formula. \square

(7) Sufficient condition: $\mathcal{D}(\Delta) \subset \mathcal{D}(T)$

$$\Gamma f \in \mathcal{D}(\Delta) \Rightarrow g := f - \Delta f \in \mathcal{D}(\Delta) \subset \mathcal{C}^2(\Omega)$$

and for any $h \in Y$ we have

$$\int_{\Omega} (g - f) \bar{h} = \int_{\Omega} -\Delta f \bar{h} = - \sum_{j=1}^d \int_{\Omega} \bar{h} \cdot \partial_{j_j}^2 f =$$

$$= - \sum_{j=1}^d \langle \bar{h}, \partial_{j_j}^2 f \rangle = \sum_{j=1}^d \langle \partial_{j_j}^2 \bar{h}, \partial_{j_j}^2 f \rangle =$$

application of a distribution to a test function

$$= \sum_{j=1}^d \int_{\Omega} \partial_{j_j}^2 \bar{h} \partial_{j_j}^2 f = \int_{\Omega} \langle \mathcal{P}f, \mathcal{P}h \rangle$$

$$h \in Y \subset \mathcal{C} \Rightarrow \partial_{j_j} h \in \mathcal{C}^2(\Omega)$$

So, $f \in \mathcal{D}(T)$ by (5). \square

(8) Characterization of $\mathcal{D}(T)$:

$$f \in \mathcal{D}(T) \Leftrightarrow f \in Y \text{ (necessarily, by (1)) ; } \Delta f \in \mathcal{C}^2(\Omega)$$

$$\& \forall h \in Y: \int_{\Omega} (\langle \mathcal{P}f, \mathcal{P}h \rangle + \Delta f \bar{h}) = 0$$

$$\Rightarrow: f \in \mathcal{D}(T) \Rightarrow f \in \mathcal{D}(Y) \text{ (by construction), } \Delta f \in \mathcal{C}^2(\Omega) \text{ (by (6))}$$

$$\text{and } Tf = f - \Delta f \text{ (by (6)). By (5) we have}$$

$$\forall h \in Y: \int_{\Omega} \langle \mathcal{P}f, \mathcal{P}h \rangle = \int_{\Omega} -\Delta f \bar{h}$$

$$\Leftarrow: \text{Let } g = f - \Delta f \in \mathcal{C}^2(\Omega). \text{ Then by (5) we deduce } f \in \mathcal{D}(T)$$

(9) Conclusion:

$$\text{We have } \mathcal{D}(T) = \{ f \in \mathcal{C}^2(\Omega) ; f \in Y, \Delta f \in \mathcal{C}^2(\Omega), \forall h \in Y: \int_{\Omega} (\langle \mathcal{P}f, \mathcal{P}h \rangle + \Delta f \bar{h}) = 0 \}$$

$$Tf = f - \Delta f, f \in \mathcal{D}(T) \text{ is self-adjoint.}$$

$$\text{Hence } f \mapsto \Delta f, f \in \mathcal{D}(T) \text{ is also self-adjoint, as it is } I - T.$$