Proposition XIII. 13

$T = \int f \, dE \Rightarrow \sigma(T) = \text{ess-rng}(f)$;

$\lambda \in \mathbb{C} \Rightarrow \ker(\lambda I - T) = R(E(f^{-1}(\{\lambda\})))$

so, $\lambda \in \mathbb{C}$ is an eigenvalue of $T \Leftrightarrow E(f^{-1}(\lambda)) \neq 0$

Proof: (1) $\lambda I - \Phi(f) = \Phi(\lambda - f)$, so look just at $\Phi(f)$

(2) $A := f^{-1}(\{0\}) \Rightarrow R(E(A)) = \ker \Phi(f)$

$\ulcorner \subset$: $\Phi(f) \upharpoonright R(E(A)) = 0$, as

$f = 0$ on A

$$\Phi(f) E(A) = \Phi(f) \Phi_0(\chi_A) \underset{\text{Thm 12 (6)}}{=} \Phi(f \chi_A) \overset{\downarrow}{=} \Phi(0) = 0$$

$D(\Phi(f)\Phi_0(\chi_A)) = D(\Phi_0(\chi_A)) \cap$

$\cap D(\Phi(f\chi_A)) = D(\Phi(f\chi_A))$

$\supset$: Let $g(t) = \begin{cases} 0 & t \in A \\ \frac{1}{f(t)} & t \notin A \end{cases}$

$\Rightarrow g$ is $\mathcal{A}$-measurable, $g \cdot f = \chi_{\mathbb{C} \setminus A}$

by Thm 12 (5): $\Phi(g)\Phi(f) \subset \Phi(\chi_{\mathbb{C}\setminus A}) = E(\mathbb{C} \setminus A)$

So, if $x \in \ker \Phi(f) \Rightarrow \Phi(f)x = 0 \Rightarrow \Phi(g)\Phi(f)x = 0$

$\Rightarrow E(\mathbb{C} \setminus A)x = 0 \Rightarrow x = E(A)x \Rightarrow x \in R(E(A)) \lrcorner$

(3) By (2) we see: 0 is an eigenvalue $\Leftrightarrow E(A) \neq 0$

(4) Suppose $E(A) = 0$. Then $\frac{1}{f}$ is $\mathcal{A}$-measurable

$\Phi(\frac{1}{f})\Phi(f) \subset \Phi(1) = I$ (Thm 12)

$D(\Phi(\frac{1}{f})\Phi(f)) = D(\Phi(f))$ (Thm 12(5)) $\Rightarrow \Phi(\frac{1}{f}) = \Phi(f)^{-1}$

$0 \notin \sigma(\Phi(f)) \Leftrightarrow \Phi(\frac{1}{f}) \in \mathcal{L}(H) \Leftrightarrow \frac{1}{f}$ ess. bdd (Thm 12 (e))

$\Leftrightarrow 0 \notin \text{ess-rng}(f)$