Proof of Theorem XIII.12

(a) $\Phi(f)+\Phi(g)\subset\Phi(f+g)$

⌈ $x\in D(\Phi(f)+\Phi(g)) = D(\Phi(f))\cap D(\Phi(g))$

$\Rightarrow f,g\in\mathcal{L}^2(E_{x,x}) \Rightarrow f+g\in\mathcal{L}^2(E_{x,x}) \Rightarrow x\in D(\Phi(f+g))$

Moreover, for $x,y\in D(\Phi(f)+\Phi(g))$ we have

$$\langle\Phi(f)x+\Phi(g)x, y\rangle = \langle\Phi(f)x,y\rangle+\langle\Phi(g)x,y\rangle =$$

$$=\int f\,dE_{x,y}+\int g\,dE_{x,y} = \int(f+g)\,dE_{x,y} = \langle\Phi(f+g)x,y\rangle$$

Since $D(\Phi(f))\cap D(\Phi(g)) = D(\Phi(|f|+|g|))$ is dense,

we deduce $\Phi(f)x+\Phi(g)x=\Phi(f+g)x$ ⌋.

(b) $\Phi(f)\Phi(g)\subset\Phi(fg)$ & $D(\Phi(f)\Phi(g)) = D(\Phi(g))\cap D(\Phi(fg))$

⌈ $x\in D(\Phi(f)\Phi(g)) \Leftrightarrow x\in D(\Phi(g))$ & $\Phi(g)x\in D(\Phi(f))$

$\Leftrightarrow g\in\mathcal{L}^2(E_{x,x})$ & $f\in\mathcal{L}^2(E_{\Phi(g)x,\Phi(g)x}) \Leftrightarrow (*)$

Compute $E_{\Phi(g)x,\Phi(g)x}(A) = \langle E(A)\Phi(g)x,\Phi(g)x\rangle =$

$$=\|E(A)\Phi(g)x\|^2 = \lim_{n\to\infty}\|E(A)\Phi_0(g_n)x\|^2 =$$

E(A) is an OS projection

g_n as in Proof of Thm 11 ⑤

$$=\lim_{n\to\infty}\|\Phi_0(\chi_A g_n)\|^2 = \lim_{n\to\infty}\int_A|g_n|^2\,dE_{x,x} = \int_A|g|^2\,dE_{x,x}$$

Thm 8(a) Thm 8(d)

So, $E_{\Phi(g)x, \Phi(g)x}$ is absolutely continuous with respect to $E_{x,x}$, with density $|g|^2$

So, $\int |f|^2 \, dE_{\Phi(g)x, \Phi(g)x} = \int |fg|^2 \, dE_{x,x}$

Hence, continue:

$(*) \Leftrightarrow g \in L^2(E_{x,x})$ & $fg \in L^2(E_{x,x}) \Leftrightarrow x \in D(\Phi(g)) \cap D(\Phi(fg))$

The equality $\Phi(fg)x = \Phi(f)\Phi(g)x$:

- f, g bdd ... by Thm 8 (a)

- f bdd, g not necessarily ... take g_n as in proof of Thm 28 5

Then $\Phi(g)x = \lim_{n\to\infty} \Phi_0(g_n)x$, $x \in D(\Phi(g))$

f bdd $\Rightarrow \Phi(f) \in L(H)$, so

$\Rightarrow \Phi(f)\Phi(g)x = \lim_{n\to\infty} \Phi(f)\Phi_0(g_n)x =$

$= \lim_{n\to\infty} \Phi_0(fg_n)x \underset{(*)}{=} \Phi(fg)x$

$\left(fg_n \to fg \text{ in } L^1(E_{x,x}) \right)$

- f, g general ... take f_n as in proof of Thm 11 5

$\Rightarrow \Phi(f)\Phi(g)x = \lim_{n\to\infty} \Phi_0(f_n)\Phi(g)x = \lim_{n\to\infty} \Phi_0(f_n g)x \overset{(*)}{=}$

$= \Phi(fg)x$

In (*) we use Lebesgue dominated convergence theorem

$$\langle \Phi_0(f g_n)x, y\rangle = \int f g_n \, dE_{x,y} \longrightarrow \int fg \, dE_{x,y}$$

$$\int fg\, dE_{x,y} = \langle \Phi(fg)x, y\rangle$$

$|fg|$ is an integrable majorant.

(c) $\Phi(f)^* = \Phi(\bar f)$, $\Phi(f)\Phi(f)^* = \Phi(f)^*\Phi(f) = \Phi(|f|^2)$

- $D(\Phi(\bar f)) = D(\Phi(f))$ .... clear, as $|\bar f|^2 = |f|^2$

- $x, y \in D(\Phi(f)) \Rightarrow$

$$\langle \Phi(f)x, y\rangle = \lim_n \langle \Phi_0(f_n)x, y\rangle = \lim_n \langle x, \Phi_0(f_n)^* y\rangle$$

$$= \lim_n \langle x, \Phi_0(\bar f_n) y\rangle = \langle x, \Phi(\bar f) y\rangle$$

So, $\Phi(\bar f) \subset \Phi(f)^*$

- Conversely. Let $A_n := \{\lambda ;\ |f(\lambda)| \leq n\}$. Then

$$E(A_n)\Phi(f)^* \subset (\Phi(f) E(A_n))^* = (\Phi(f)\Phi_0(\chi_{A_n}))^*$$

$$\overset{(5)}{=} (\Phi(f\chi_{A_n}))^* = \Phi(\bar f \chi_{A_n})$$

$x \in D(\Phi(f)^*) \Rightarrow E(A_n)\Phi(f)^* x \longrightarrow \Phi(f)^* x$

$$\Rightarrow \Phi(f)^* x = \lim_{n\to\infty} \Phi(\bar f \chi_{A_n}) x$$

$\Rightarrow \infty > \langle \Phi(f)^* x, x \rangle = \lim_{n\to\infty} \langle \Phi(\bar{f}\chi_{A_n}) x, x \rangle = \lim_{n\to\infty} \int |\bar{f}\chi_{A_n}|^2 \, dE_{x,x} =$

$= \int |f|^2 \, dE_{x,x}$, so $x \in D(\Phi(f))$.

Hence $D(\Phi(f)^*) \subset D(\Phi(f))$. It follows that $\Phi(f)^* = \Phi(\bar{f})$

- $\Phi(f)^* \Phi(f) = \Phi(f)\Phi(f)^* \subset \Phi(|f|^2)$ by (b)

$\ulcorner$ $D(\Phi(f)^*\Phi(f)) = D(\Phi(f)) \cap D(\Phi(|f|^2))$

$D(\Phi(f)\Phi(f)^*) = D(\Phi(\bar{f})) \cap D(\Phi(|f|^2))$ $\lrcorner$

- $D(\Phi(|f|^2)) \subset D(\Phi(f))$

$\ulcorner$ $x \in D(\Phi(|f|^2)) \Rightarrow |f|^2 \in L^2(E_{x,x}) \Rightarrow |f|^2 \in L^1(E_{x,x})$

$\Rightarrow f \in L^2(E_{x,x})$, so $x \in D(\Phi(f))$ $\lrcorner$

(d) $\Phi(f)$ is a closed operator

$\ulcorner$ $\Phi(f) = \Phi(\bar{f})^*$ $\lrcorner$

(e) $\Phi(f)$ is cts $\Leftrightarrow$ f is essentially bdd

$\ulcorner$ $\Leftarrow$: by Thm 8

$\Rightarrow$: $\Phi(f)$ bdd $\Rightarrow D(\Phi(f)) = H$, f_n as above

(i.e. $f_n = f \cdot \chi_{A_n}$, $A_n = \{\lambda : |f(\lambda)| \le n\}$)

$\Rightarrow \Phi_0(f_n)x \longrightarrow \Phi(f)x$, $x \in H$

uniform bddness principle $\Rightarrow$ $(\Phi_0(f_n))$ is uniformly bdd

but $\|\Phi_0(f_n)\| = \|f_n\|_\infty \Rightarrow f$ ess. bdd $\lrcorner$