

Theorem XIII.11 Let E be an abstract spectral measure in H , defined on \mathcal{A} , $f: \mathbb{C} \rightarrow \mathbb{C}$ \mathcal{A} -measurable function

set $D(\underline{\Phi}(f)) := \{x \in H; \int |f|^2 dE_{x,x} < \infty\}$

Then $D(\underline{\Phi}(f))$ is a dense linear subspace of H .

Further, $\exists! \underline{\Phi}(f)$, an operator on H with domain $D(\underline{\Phi}(f))$

s.t. $\langle \underline{\Phi}(f)x, y \rangle = \int f dE_{x,y}$, $x, y \in D(\underline{\Phi}(f))$.

Moreover, $\| \underline{\Phi}(f)x \|^2 = \left(\int |f|^2 dE_{x,x} \right)$, $x \in D(\underline{\Phi}(f))$.

Proof (1) $D(\underline{\Phi}(f))$ is a linear subspace of H :

- clearly $0 \in D(\underline{\Phi}(f))$, as $E_{0,0} = 0$

- $x \in D(\underline{\Phi}(f))$, $d \in \mathbb{C} \Rightarrow dx \in D(\underline{\Phi}(f))$

as $E_{dx,dx} = |d|^2 E_{x,x}$ ($\int |d|^2 dE_{x,x} < \infty$)

- $x, y \in D(\underline{\Phi}(f)) \Rightarrow x+y \in D(\underline{\Phi}(f))$

as $E_{x+y,x+y} \leq 2(E_{x,x} + E_{y,y})$ ($\int |f|^2 dE_{x+y,x+y} < \infty$)

(2) For $n \in \mathbb{N}$ set $A_n := \{ \lambda \in \mathbb{C}, |f(\lambda)| \leq n \}$

Then $A_n \in \mathcal{A}$. Moreover, $R(E(A_n)) \subset D(\underline{\Phi}(f))$

$\{ x \in R(E(A_n)) \}$, i.e. $\underline{\Phi}(f) \underbrace{x}_{\in D(\underline{\Phi}(f))} = x$

Then $E_{x,x}(A_n) = \langle E(A_n)x, x \rangle = \langle x, x \rangle = \langle Ix, x \rangle$
 $= \langle E(\mathbb{C})x, x \rangle = E_{x,x}(\mathbb{C})$

So, $E_{x,x}(\mathbb{C} \setminus A_n) = 0$, in other words $|f| \leq n$ $E_{x,x}$ -a.e.

So, $\int |f|^2 dE_{x,x} \leq \int n^2 dE_{x,x} < \infty$

Hence, $x \in D(\underline{\Phi}(f))$ \downarrow

③ $\forall x \in H: E(A_n)x \rightarrow x$, hence $D(\Phi(t))$ is dense in H

$$\Gamma \quad \|x - E(A_n)x\|^2 \stackrel{(iii)}{=} \|E(\Omega)x - E(A_n)x\|^2 \stackrel{(vi)}{=}$$

$$= \|E(\Omega \setminus A_n)x\|^2 = \langle E(\Omega \setminus A_n)x, E(\Omega \setminus A_n)x \rangle =$$

$$= \langle E(\Omega \setminus A_n)x, x \rangle = E_{x,x}(\Omega \setminus A_n) \xrightarrow{n \rightarrow \infty} 0$$

\nearrow
 $E(\Omega \setminus A_n)$ is an OG projection

\nearrow
 $\Omega \setminus A_n \downarrow, \bigcap_n (\Omega \setminus A_n) = \emptyset$
 $E_{x,x}$ is a finite measure \searrow

④ $x, y \in D(\Phi(t)) \Rightarrow \int f dE_{x,y}$ is well defined

Γ By Lemma 6 (f): $|E_{x,y}(A)| \leq \frac{1}{2} (E_{x,x}(A) + E_{y,y}(A)), A \in \mathcal{B}$,

so by the definition of absolute variation: $|E_{x,y}| \leq \frac{1}{2} (E_{x,x} + E_{y,y})$

Since $x, y \in D(\Phi(t)) \Rightarrow \left. \begin{array}{l} f \in L^2(E_{x,x}) \subset L^1(E_{x,x}) \\ f \in L^2(E_{y,y}) \subset L^1(E_{y,y}) \end{array} \right\} \Rightarrow f \in L^1(|E_{x,y}|)$

\uparrow
 $E_{x,x}, E_{y,y}$ finite measures \perp

⑤ Set $f_n := f \cdot \chi_{A_n}$ (A_n -defined in ②) above

The f_n is odd \mathcal{A} -measurable \Rightarrow we have $\Phi_0(f_n) \in L(H)$ by Thm 8.

For $x \in D(\Phi(t))$ set $\Phi(t)x := \lim_{n \rightarrow \infty} \Phi_0(f_n)x$.

The limit exists, as the sequence $(\Phi_0(f_n)x)_{n=1}^{\infty}$ is Cauchy.

linearity of Φ_0

$$m < n \Rightarrow \|\Phi_0(f_n)_x - \Phi_0(f_m)_x\|^2 = \|\Phi_0(f_n - f_m)_x\|^2 =$$

$$\stackrel{V.P.(a)}{=} \int |f_n - f_m|^2 dE_{x,x} = \int_{A_n \setminus A_m} |f|^2 dE_{x,x} \leq \int_{A \setminus A_m} |f|^2 dE_{x,x} \xrightarrow{m \rightarrow \infty} 0,$$

as $f \in L^2(\mathbb{R}_{x,x}) \in \mathcal{C} \setminus A_m \downarrow \emptyset$.

$$\begin{aligned} \textcircled{6} \text{ Then } \|\Phi(f)_x\|^2 &= \lim_{n \rightarrow \infty} \|\Phi_0(f_n)_x\|^2 \stackrel{V.P.(a)}{=} \lim_{n \rightarrow \infty} \int |f_n|^2 dE_{x,x} = \\ &= \lim_{n \rightarrow \infty} \int_{A_n} |f|^2 dE_{x,x} = \int |f|^2 dE_{x,x} \end{aligned}$$

$$\textcircled{7} \langle \Phi(f)_x, y \rangle = \lim_{n \rightarrow \infty} \langle \Phi_0(f_n)_x, y \rangle = \lim_{n \rightarrow \infty} \int f_n dE_{x,y} \quad (t_{ij} \in \mathcal{D}(\Phi(f)))$$

$$= \lim_{n \rightarrow \infty} \int_{A_n} f dE_{x,y} = \int f dE_{x,y} \quad (\text{as } f \in \mathcal{C}^1(\mathbb{R}_{x,y}) \text{ by } \textcircled{4} \text{ and } t_n \uparrow \mathcal{C})$$

$\textcircled{8} \Phi(f)$ unique ;

Fix $x \in \mathcal{D}(\Phi(f))$. If $z \in H$ is such that

$$\langle \Phi(f)_x, y \rangle = \langle z, y \rangle, \quad y \in \mathcal{D}(\Phi(f)), \text{ then necessarily}$$

$$z = \Phi(f)_x \text{ as } \mathcal{D}(\Phi(f)) \text{ is dense in } H.$$