Proof of Theorem XIII.8

E ... abstract spectral measure in H

$\mathcal{A}$ .. the domain σ-algebra

$f: \mathbb{C} \to \mathbb{C}$ a bdd $\mathcal{A}$-measurable function

① $B(x,y) := \int f \, dE_{x,y}$, $x,y \in H$

- $B(x,y)$ well-defined (f measurable, bdd, $E_{x,y}$ finite complex measure)
- B is sesquilinear ($x \mapsto B(x,y)$ linear, $y \mapsto B(x,y)$ conj. linear) by L 6 (a,b)
- $|B(x,y)| \le \int |f| \, d|E_{x,y}| \le \|f\|_\infty \cdot \|E_{x,y}\| \le \|f\|_\infty \|x\| \|y\|$

So, by Lax-Milgram lemma there is (a unique) $T \in \mathcal{L}(H)$ s.t. $\langle Tx, y \rangle = \int f \, dE_{x,y}$, $x,y \in H$

Moreover, $\|T\| \le \|f\|_\infty$

Denote $\Phi_0(f) := T$

② If f, g are bdd $\mathcal{A}$-measurable, $f = g$ E-a.e.
(i.e. $\{\lambda \in \mathbb{C}; f(\lambda) \neq g(\lambda)\} \in \mathcal{N} = \{A \in \mathcal{A}, E(A) = 0\}$
$\Rightarrow \Phi_0(f) = \Phi_0(g)$

[$f = g$ E-a.e. $\Rightarrow$ $\forall x,y \in H$ $f = g$ $|E_{x,y}|$-a.e.,

so $\int f \, dE_{x,y} = \int g \, dE_{x,y}$.

By ① we see $\Phi_0(f) = \Phi_0(g)$

Therefore, Φ_0 is a well-defined mapping $L^\infty(E) \to \mathcal{L}(H)$
$\|\Phi_0(f)\| \le \|f\|$, $f \in L^\infty(E)$]

(3) Clearly, Φ_0 is linear

(4) $\Phi_0(f)^* = \Phi_0(\bar{f})$

$$\Big[\langle \Phi_0(f)^* x, x\rangle = \langle x, \Phi_0(f)x\rangle = \overline{\langle \Phi_0(f)x, x\rangle} =$$

$$= \overline{\int f \, dE_{x,x}} \underset{\substack{\uparrow \\ E_{x,x}\geq 0}}{=} \int \bar{f} \, dE_{x,x} = \langle \Phi_0(\bar{f})x, x\rangle \text{ for } x \in H \Big]$$

(5) $A \in \mathcal{A} \Rightarrow \Phi_0(\chi_A) = E(A)$

$$\Big[\langle \Phi_0(\chi_A)x, y\rangle = \int \chi_A \, dE_{x,y} = E_{x,y}(A) = \langle E(A)x, y\rangle \Big]$$

(6) $\Phi_0(f \cdot g) = \Phi_0(f)\Phi_0(g)$

[a] $f = \chi_A$, $g = \chi_B \Rightarrow \Phi_0(fg) \overset{(5)}{=} \Phi_0(\chi_{A\cap B}) = E(A\cap B) =$

$$\underset{(V)}{=} E(A)E(B) \underset{(5)}{=} \Phi_0(\chi_A)\Phi_0(\chi_B) = \Phi_0(f)\Phi_0(g)$$

[b] f, g simple $\mathcal{A}$-measurable

f given $\Rightarrow \{ g \, ; \, \Phi_0(fg) = \Phi_0(f)\Phi_0(g) \}$ is a linear space
$\{ g \, ; \, \Phi_0(gf) = \Phi_0(g)\Phi_0(f) \}$ —″—

so, by [a] we deduce the validity for simple functions

[c] g simple, f general
Fix $x, y \in H$. Find (f_n) simple Borel measurable
s.t. $\|f_n\|_\infty \leq \|f\|_\infty$ and
$f_n \to f$ $|E_{x,y}| + |E_{\Phi_0(g)x,y}|$ - a.e.

Lebesgue dom. conv.

Then $\langle \Phi_0(f)\Phi_0(g)x, y\rangle = \int f \, dE_{\Phi_0(g)x,y} =$

$= \lim_{n\to\infty} \int f_n \, dE_{\Phi_0(g)x,y} = \lim_{n\to\infty} \langle \Phi_0(f_n)\Phi_0(g)x, y\rangle =$

Lebesgue dom. conv.

$\overset{\boxed{b}}{=} \lim_{n\to\infty} \langle \Phi_0(f_n g)x, y\rangle = \lim_{n\to\infty} \int f_n g \, dE_{x,y} = \int fg \, dE_{x,y}$

$= \langle \Phi_0(fg)x, y\rangle$

$\boxed{d}$ f, g general. Fix $x, y \in H$

Find (g_n) simple Borel measurable, $\|g_n\|_\infty \leq \|g\|_\infty$

$g_n \to g$ $\quad |E_{x,y}| + |E_{x,\Phi_0(f)^* y}|$ - a.e.

Then $\langle \Phi_0(f)\Phi_0(g)x, y\rangle = \langle \Phi_0(g)x, \Phi_0(f)^* y\rangle =$

$= \int g \, dE_{x,\Phi_0(f)^* y} = \lim_{n\to\infty} \int g_n \, dE_{x,\Phi_0(f)^* y} =$

$= \lim_{n\to\infty} \langle \Phi_0(g_n)x, \Phi_0(f)^* y\rangle = \lim_{n\to\infty} \langle \Phi_0(f)\Phi_0(g_n)x, y\rangle$

$\overset{\boxed{c}}{=} \lim_{n\to\infty} \langle \Phi_0(fg_n)x, y\rangle = \lim_{n\to\infty} \int fg_n \, dE_{x,y} = \int fg \, dE_{x,y}$

$= \langle \Phi_0(fg)x, y\rangle$

(7) $\|\Phi_0(f)x\|^2 = \langle \Phi_0(f)x, \Phi_0(f)x\rangle = \langle \Phi_0(f)^* \Phi_0(f)x, x\rangle$

$\overset{(4),(6)}{=} \langle \Phi_0(\bar{f} \cdot f)x, x\rangle = \int |f|^2 \, dE_{x,x}$

(This proves (d))

(8) Φ_0 is one-to-one : $\Phi(f) = 0 \overset{(7)}{\Rightarrow} \forall x \in H \ \int |f|^2 \, dE_{x,x} = 0$

$\Rightarrow \forall x \in H : f = 0 \ E_{x,x}$ - a.e. $\Rightarrow f = 0$ E - a.e.

$\Rightarrow f = 0$ in $L^\infty(E)$

(9) proof of (a): By (3), (4), (6) Φ_0 is a *-homomorphism. By (7) it is even a *-isomorphism. So, it is an isometry.

(10) $\sigma(\Phi_0(f)) = \text{ess-rng}(f)$

[• It is easy to see that $\sigma(f) = \text{ess-rng}(f)$ in $L^\infty(E)$

• Set $\mathcal{B} := \Phi_0(L^\infty(E))$. Then $\mathcal{B}$ is a C^*-subalgebra of $L(H)$ containing $I = \Phi_0(1)$. So, for each $f \in L^\infty(E)$ we have

$$\sigma_{L(H)}(\Phi_0(f)) = \sigma_{\mathcal{B}}(\Phi_0(f)) = \sigma(f) = \text{ess-rng}(f)$$]

(11) $\Phi_0(f)$ is always normal, as $\mathcal{B}$ is commutative

$\Phi_0(f)$ self-adjoint $\Leftrightarrow$ f self-adjoint, i.e. real-valued.

$$\Phi_0(f) \geq 0 \overset{\text{XII.5 (c)}}{\Longleftrightarrow} \Phi_0(f) \text{ self-adjoint and } \sigma(\Phi_0(f)) \subset [0,\infty)$$

$\Updownarrow$

$f \geq 0$ E-a.e. by properties of $L^\infty(E)$.

(12) $f \in L^\infty(E)$, $g \in \mathcal{C}(\sigma(\Phi_0(f))) \Rightarrow$

$\Rightarrow \Phi_0(g \circ f) = \tilde{g}(\Phi_0(f))$

$\ulcorner$ $\sigma(\Phi_0(f)) = $ ess-rg(f) is a compact set in $\mathbb{C}$

Let $\Upsilon = \{ g \in \mathcal{C}(\sigma(\Phi_0(f))) \,;\, \Phi_0(g \circ f) = \tilde{g}(\Phi_0(f)) \}$

Then • $g \mapsto \tilde{g}(\Phi_0(f))$ is a $*$-isomorphism
$g \mapsto \Phi_0(g \circ f)$ is a $*$-homomorphism

So, Υ is a closed $*$-subalgebra of $\mathcal{C}(\sigma(\Phi_0(f)))$

Moreover $1 \in \Upsilon$, as

$\tilde{1}(\Phi_0(f)) = I$, $\Phi_0(1 \circ f) = \Phi_0(1) = I$,

hence Υ contains constants

Finally, $id \in \Upsilon$, as

$\widetilde{id}(\Phi_0(f)) = \Phi_0(f) = \Phi_0(id \circ f)$

So, Υ separates points of $\sigma(\Phi_0(f))$

Stone-Weierstrass theorem shows that $\Upsilon = \mathcal{C}(\sigma(\Phi_0(f)))$.