Lemma XIII.6 E ... abstract spectral measure on a Hilbert space H

$\mathcal{A}$ ... domain σ-algebra of E

For $x, y \in H$ ... $E_{x,y}(A) = \langle E(A)x, y\rangle$, $A \in \mathcal{A}$

(a) $x \mapsto E_{x,y}$ is linear
(b) $y \mapsto E_{x,y}$ is conjugate linear

} clear, using properties of the inner product

(c) $E_{x,y} = \overline{E_{y,x}}$

$E(A)$ is an OG projection, hence $E(A)^* = E(A)$

$A \in \mathcal{A} \Rightarrow$ $E_{x,y}(A) = \langle E(A)x, y\rangle = \langle x, E(A)y\rangle = \overline{\langle E(A)y, x\rangle} = \overline{E_{y,x}(A)}$

(d) $E_{x,x} \geq 0$ [$E_{x,x}(A) = \langle E(A)x, x\rangle = \|E(A)x\|^2 \geq 0$ by properties of OG projections]

(e) $E_{x,y} = \frac{1}{4}\left(E_{x+y,x+y} - E_{x-y,x-y} + i E_{x+iy,x+iy} - i E_{x-iy,x-iy}\right)$

[polarization identity – direct computation using (a), (b)]

(f) $\forall x, y \in H\ \forall A \in \mathcal{A}: |E_{x,y}(A)| \leq \sqrt{E_{x,x}(A)\, E_{y,y}(A)} \leq \frac{1}{2}(E_{x,x}(A) + E_{y,y}(A))$

[• $B(x,y) := E_{x,y}(A)$ By (a,b,c,d) B is a „semi-inner product", i.e. positive semi-definite hermitian sesquilinear form. Such forms satisfy Cauchy-Schwarz inequality (with the same proof as for an inner product), i.e.

$$|B(x,y)| \leq \sqrt{B(x,x) \cdot B(y,y)}$$, which is the first inequality

The second one is just the inequality between the geometric and arithmetic means]

Remark to (f): It follows that $|E_{x,y}| \leq \frac{1}{2}(E_{x,x} + E_{y,y})$

$$|E_{x,y}|(A) = \sup\left\{ \sum_1^n |E_{x,y}(B_j)| : B_j \text{ disjoint}, B_j \in \mathcal{A}, \bigcup_{j=1}^n B_j = A \right\}$$

$$\leq \sup\left\{ \sum_1^n \frac{1}{2}\big(E_{x,x}(B_j) + E_{y,y}(B_j)\big) : \dots \right\} = \frac{1}{2}\big(E_{x,x}(A) + E_{y,y}(A)\big)$$

(g) $E_{x+y,x+y} \leq 2(E_{x,x} + E_{y,y})$

$$\Gamma\ E_{x+y,x+y}(A) \overset{(a),(b)}{=} E_{x,x}(A) + E_{x,y}(A) + E_{y,x}(A) + E_{y,y}(A)$$

$$\overset{(c)}{=} E_{x,x}(A) + 2\,\mathrm{Re}\, E_{x,y}(A) + E_{y,y}(A) \overset{(f)}{\leq} 2\big(E_{x,x}(A) + E_{y,y}(A)\big) \rfloor$$

(h) $\|E_{x,y}\| \leq \|x\| \cdot \|y\|$

Recall $\|E_{x,y}\| = |E_{x,y}|(\mathbb{C}) = \sup\left\{ \sum_1^n |E_{x,y}(A_j)| \,;\, A_j \in \mathcal{A} \text{ disjoint} \right\}$

So, fix $A_1, \dots, A_n \in \mathcal{A}$ disjoint and compute:

$$\sum_1^n |E_{x,y}(A_j)| = \sum_1^n |\langle E(A_j)x, y\rangle| = \sum_1^n d_j \cdot \langle E(A_j)x, y\rangle =$$

$d_1, \dots, d_n$ suitable complex units

$$\overset{(a)}{=} \left\langle \sum_1^n d_j \cdot E(A_j)x, y \right\rangle \overset{\text{Cauchy-Schwarz}}{\leq} \left\| \sum_{j=1}^n d_j E(A_j)x \right\| \cdot \|y\| =$$

$A_1 \dots A_n$ disjoint $\Rightarrow E(A_j)x$, $j = 1 \dots n$ mutually orthogonal by (v) in the definition of abstract sp. meas.

$$= \left(\sum_{j=1}^n \|d_j \cdot E(A_j)x\|^2 \right)^{1/2} \cdot \|y\| =$$

$$\underset{|d_j|=1}{=} \left(\sum_{j=1}^n \|E(A_j)x\|^2 \right)^{1/2} \|y\| = \left\| \sum_{j=1}^n E(A_j)x \right\| \|y\| =$$

$$\underset{(vi)}{=} \left\| E\Big(\bigcup_{j=1}^n A_j\Big)x \right\| \|y\| \leq \|x\| \cdot \|y\|.$$