VI.2 Integral with respect to a spectral measure

Definition. An **abstract spectral measure** in a Hilbert space H is a mapping E with the following properties:

(i) The domain of E is a σ-algebra $\mathcal{A}$ of subsets of $\mathbb{C}$ containing all Borel sets.
(ii) $E(A)$ is an orthogonal projection on H for each $A \in \mathcal{A}$.
(iii) $E(\emptyset) = 0$, $E(\mathbb{C}) = I$.
(iv) If $A \in \mathcal{A}$ satisfies $E(A) = 0$, then $B \in \mathcal{A}$ (and $E(B) = 0$) for each $B \subset A$.
(v) $E(A \cap B) = E(A)E(B)$ for $A, B \in \mathcal{A}$.
(vi) $E(A \cup B) = E(A) + E(B)$ whenever $A, B \in \mathcal{A}$, $A \cap B = \emptyset$.
(vii) For each pair $x, y \in H$ the mapping $E_{x,y} : A \mapsto \langle E(A)x, y\rangle$ is a complex Borel measure on $\mathbb{C}$.

The spectral measure E is called **compactly supported** if there is a compact set $K \subset \mathbb{C}$ such that $E(\mathbb{C} \setminus K) = 0$.

Recall that μ is a **Borel measure** if it is a σ-additive measure defined on a σ-algebra $\mathcal{A}_\mu$ containing all Borel sets such that for any $A \in \mathcal{A}_\mu$ there are Borel sets B, C such that $B \subset A \subset C$ and $|\mu|(B \setminus C) = 0$.

Lemma 5. *If $T \in L(H)$ is a normal operator, then E_T is a compactly supported abstract spectral measure.*

Lemma 6 (properties of a spectral measure). *Let E be an abstract spectral measure in a Hilbert space H defined on a σ-algebra $\mathcal{A}$. Then the following holds:*

(a) *The mapping $x \mapsto E_{x,y}$ is linear for each $y \in H$.*
(b) *The mapping $y \mapsto E_{x,y}$ is conjugate linear for each $x \in H$.*
(c) $E_{y,x} = \overline{E_{x,y}}$ *for $x, y \in H$.*
(d) $E_{x,x}$ *is a nonnegative measure for each $x \in H$.*
(e) $E_{x,y} = \frac{1}{4}(E_{x+y,x+y} - E_{x-y,x-y} + iE_{x+iy,x+iy} - iE_{x-iy,x-iy})$ *for $x, y \in H$.*
(f) $|E_{x,y}(A)| \le \sqrt{E_{x,x}(A) \cdot E_{y,y}(A)} \le \frac{1}{2}(E_{x,x}(A) + E_{y,y}(A))$ *for $x, y \in H$ and $A \in \mathcal{A}$.*
(g) $E_{x+y,x+y} \le 2(E_{x,x} + E_{y,y})$ *for $x, y \in H$.*
(h) $\|E_{x,y}\| \le \|x\| \cdot \|y\|$ *for $x, y \in H$.*

Remark. In the definition of an abstract spectral measure, in (vii) it is enough to assume that $E_{x,x}$ is a Borel measure on $\mathbb{C}$ for any $x \in H$.

Proposition 7. *Let E be an abstract spectral measure in a separable Hilbert space H. Then for any $A \in \mathcal{A}$ there are Borel sets B and C such that $B \subset A \subset C$ and $E(C \setminus B) = 0$.*

Remark. Spectral measure is sometimes defined only for separable Hilbert spaces H. Then it is defined only on the σ-algebra of Borel sets and the condition (iv) is omitted. For nonseparable H the above approach is necessary.

Definition. Let E be an abstract spectral measure in a Hilbert space H defined on a σ-algebra $\mathcal{A}$.

- Set $\mathcal{N} = \{A \in \mathcal{A}; E(A) = 0\}$.
- We denote by $L^\infty(E)$ the space of all bounded $\mathcal{A}$-measurable functions on $\mathcal{C}$, where we identify functions, which are equal except on a set from $\mathcal{N}$ (i.e., E-**almost everywhere**). Equip $L^\infty(E)$ the the norm

$$\|f\| = \operatorname*{ess\,sup}_{\lambda \in \mathbb{C}} |f(\lambda)| = \inf\{c > 0; \{\lambda \in \mathbb{C}; f(\lambda) > c\} \in \mathcal{N}\}.$$

Then $L^\infty(E)$ is a commutative C^*-algebra (with pointwise multiplication and involution defined as complex conjugation).

Theorem 8 (integral of a bounded function with respect to a spectral measure). *If E is an abstract spectral measure in H defined on a σ-albegra $\mathcal{A}$ and $f : \mathbb{C} \to \mathbb{C}$ is a bounded $\mathcal{A}$-measurable function, then there is a unique operator $\Phi_0(f) \in L(H)$ such that*

$$\langle \Phi_0(f)x, y \rangle = \int f \, \mathrm{d}E_{x,y} \qquad x, y \in H.$$

Moreover:

(a) *Φ_0 is an isometric $*$-isomorphism of the C^*-algebra $L^\infty(E)$ into $L(H)$.*
(b) *$\sigma(\Phi_0(f)) = \operatorname{ess\,rng}(f)$ for each $f \in L^\infty(E)$.*
(c) *For any $f \in L^\infty(E)$ the operator $\Phi_0(f)$ is normal. Moreover $\Phi_0(f)$ is self-adjoint if and only if f is real-valued (E-almost everywhere) and $\Phi_0(f)$ is positive if and only if $f \geq 0$ E-almost everywhere.*
(d) *$\|\Phi_0(f)x\| = \sqrt{\int |f|^2 \, \mathrm{d}E_x}$ for $x \in H$.*
(e) *If $f \in L^\infty(E)$ and $g \in \mathcal{C}(\sigma(\Phi_0(f)))$, then $\Phi_0(g \circ f) = \tilde{g}(\Phi_0(f))$.*

Notation: The operator $\Phi_0(f)$ from the previous theorem is denoted by $\int f \, \mathrm{d}E$ and is called the **integral of the function f with respect to the spectral measure E.**

Lemma 9. *Let E be an abstract spectral measure, $f \in L^\infty(E)$ and $T = \int f \, \mathrm{d}E$. Then the spectral measure E_T of T is given by $E_T(A) = E(f^{-1}(A))$.*

Corollary 10 (spectral decomposition of a bounded normal operator). *Let H be a Hilbert space and $T \in L(H)$ a normal operator. Then there is a unique abstract spectral measure such that $T = \int \mathrm{id} \, \mathrm{d}E$. Moreover, this is the measure E_T.*

Theorem 11 (integral of a (not necessarily bounded) function with respect to a spectral measure). *Let E be an abstract spectral measure in H defined on a σ-albegra $\mathcal{A}$, let $f : \mathbb{C} \to \mathbb{C}$ be an $\mathcal{A}$-measurable function. Set*

$$D(\Phi(f)) = \{x \in H : \int |f|^2 \, \mathrm{d}E_{x,x} < \infty\}.$$

Then $D(\Phi(f))$ is a dense linear subspace of H. Further, there exists a unique operator $\Phi(f)$ on H with domain $D(\Phi(f))$ satisfying.

$$\langle \Phi(f)x, y \rangle = \int f \, \mathrm{d}E_{x,y}, \qquad x, y \in D(\Phi(f)).$$

Moreover,

$$\|\Phi(f)x\| = \sqrt{\int |f|^2 \, dE_{x,x}}, \qquad x \in D(\Phi(f)).$$

Remark: If f is bounded, then $D(\Phi(f)) = H$ and $\Phi(f) = \Phi_0(f)$.

Notation: The operator $\Phi(f)$ from the previous theorem is denoted by $\int f \, \mathrm{d}E$ and is called the **integral of the function f with respect to the spectral measure E.**

Theorem 12 (properties of $\int f \, \mathrm{d}E$). *If E is an abstract spectral measure in H and f, g are $\mathcal{A}$-measurable functions, then:*

(a) *$\Phi(f) + \Phi(g) \subset \Phi(f + g)$;*
(b) *$\Phi(f)\Phi(g) \subset \Phi(fg)$ and $D(\Phi(f)\Phi(g)) = D(\Phi(g)) \cap D(\Phi(fg))$.*
(c) *$\Phi(f)^* = \Phi(\overline{f})$ and $\Phi(f)\Phi(f)^* = \Phi(|f|^2) = \Phi(f)^*\Phi(f)$, in particular $\Phi(f)$ is* **normal.**
(d) *$\Phi(f)$ is a closed operator.*
(e) *$\Phi(f)$ is continuous if and only if f is essentially bounded, i.e., there exists $A \in \mathcal{A}$, such that $E(\mathbb{C} \setminus A) = 0$ and f is bounded on A.*

Proposition 13 (spectrum of $\int f \, \mathrm{d}E$). *If E is an abstract spectral measure, f is an $\mathcal{A}$-measurable function and $T = \inf f \, \mathrm{d}E$, then*

$$\sigma(T) = \operatorname{ess\,rng}(f) := \mathbb{C} \setminus \bigcup \{G \subset \mathbb{C} : G \text{ open}, \, E(f^{-1}(G)) = 0\}.$$

Moreover, for any $\lambda \in \mathbb{C}$ we have $\ker(\lambda I - T) = R(E(f^{-1}(\{\lambda\})))$. In particular, λ is an eigenvalue of T if and only if $E(f^{-1}(\{\lambda\})) \neq 0$.