VII. More on locally convex topologies

Reminder:

- A **locally convex space** is a vector space X over $\mathbb{F}$ equipped with a topology $\mathcal{T}$ with the properties:
  - The mapping $(x,y) \mapsto x+y$ is a continuous mapping $X \times X \to X$.
  - The mapping $(t,x) \mapsto t \cdot x$ is a continuous mapping $\mathbb{F} \times X \to X$.
  - There exists a base of neighborhoods of zero formed by convex sets.
- Let X be a vector space over $\mathbb{F}$ and let $\mathcal{U}$ be a nonempty system of its subsets with the properties:
  (a) Elements of $\mathcal{U}$ are absolutely convex and absorbing.
  (b) For any $U \in \mathcal{U}$ there exists $V \in \mathcal{U}$ satisfying $2V \subset U$.
  (c) For any two elements $U, V \in \mathcal{U}$ there exists $W \in \mathcal{U}$ satisfying $W \subset U \cap V$.

  Then there exists a unique locally convex topology on X such that $\mathcal{U}$ is a base of its neighborhoods of zero. This topology is Hausdorff if and only if $\bigcap \mathcal{U} = \{\boldsymbol{o}\}$.

  Conversely, any locally convex space has a base of neighborhoods of zero $\mathcal{U}$ with the properties (a)-(c). Moreover, $\mathcal{U}$ can consist of open sets.
- Let X be a vector space over $\mathbb{F}$ and let $\mathcal{P}$ be a nonempty family of seminorms on X. Then the family

$$\mathcal{U} = \{\{x \in X; p_1(x) < c_1, \dots, p_n(x) < c_n\}; p_1, \dots, p_n \in \mathcal{P}, c_1, \dots, c_n \in (0,\infty)\}$$

  is a base of neighborhoods of zero of some (uniquely determined) locally convex topology on X.

  Conversely, any locally convex topology on X is defined in this way by a family of seminorms, for example by the family of all the continuous seminorms.

  Moreover, if the topology $\mathcal{T}$ is generated by a family of seminorms $\mathcal{P}$, then a seminorm p is $\mathcal{T}$-continuous if and only if there exist $p_1, \dots, p_n \in \mathcal{P}$ and $c > 0$ such that $p \le c \cdot \max\{p_1, \dots, p_n\}$.

VII.1 Lattice of locally convex topologies and topologies agreeing with duality

Notation: Let X be a vector space. Denote by the symbol $\mathscr{LC}(X)$ the family of all locally convex topologies on X.

Proposition 1. *Let X be a vector space. Then $\mathscr{LC}(X)$ is a complete lattice. I.e., whenever $\mathscr{F} \subset \mathscr{LC}(X)$ is a nonempty subfamily, there exist the weakest locally convex topology finer than all the elements of $\mathscr{F}$ (we denote it $\sup \mathscr{F}$) and the finest locally convex topology weaker than all the elements of $\mathscr{F}$ (we denote it $\inf \mathscr{F}$). They can be described as follows:*

- *$\sup \mathscr{F}$ is generated by the family of all the seminorms which are continuous in some topology from $\mathscr{F}$.*
- *$\inf \mathscr{F}$ is generated by the family of all the seminorms which are continuous in all topologies from $\mathscr{F}$.*

Remarks:

(1) If at least one element of $\mathscr{F}$ is a Hausdorff topology, then $\sup \mathscr{F}$ is a Hausdorff topology as well.
(2) $\sup \mathscr{LC}(X)$ is the strongest locally convex topology. A base of neighborhoods of zero is formed by all the absorbing absolutely convex sets. All the seminorms are continuous in it, so it is generated by the family of all the seminorms on X. All linear functionals are continuous in it, hence $(X, \sup \mathscr{LC}(X))^* = X^\#$ (the algebraic dual of X).
(3) $\inf \mathscr{LC}(X)$ is the indiscrete topology, the unique neighborhood of zero is the whole space X, the unique continuous seminorm is the zero one and the unique continuous linear functional is the zero one.
(4) If $\dim X < \infty$, then X admits a unique Hausdorff locally convex topology.
(5) Let $\dim X = \infty$. Then $\inf \mathscr{F}$ need not be a Hausdorff topology, even if all the elements of $\mathscr{F}$ are Hausdorff. In fact, the infimum of the family of all Hausdorff locally convex topologies is the indiscrete topology.

Lemma 2. *Let X be vector space, $f : X \to \mathbb{F}$ linear functional and $p_1, \dots, p_n$ seminorms on X. If $|f| \le \max\{p_1, \dots, p_n\}$, then there exist linear functionals $f_1, \dots, f_n$ and numbers $t_1, \dots, t_n \in [0,1]$ satisfying*

(i) $|f_j| \le p_j$ *for* $j = 1, \dots, n$;
(ii) $f = t_1 f_1 + t_2 f_2 + \dots + t_n f_n$;
(iii) $t_1 + t_2 + \dots + t_n = 1$.

Proposition 3. *Let X be a vector space and let $\mathcal{F} \subset \mathscr{LC}(X)$ be any nonempty subfamily. Then*

$$(X, \sup \mathcal{F})^* = \operatorname{span}\left(\bigcup_{\mathcal{T}\in\mathcal{F}} (X,\mathcal{T})^*\right), \quad (X, \inf \mathcal{F})^* = \bigcap_{\mathcal{T}\in\mathcal{F}} (X,\mathcal{T})^*.$$

Definition. Let X be a vector space and $M \subset\subset X^\#$.

- Denote
$$\mathscr{LC}(X,M) = \{\mathcal{T} \in \mathscr{LC}(X); (X,\mathcal{T})^* = M\}.$$
If X is a locally convex space and $M = X^*$, then the topologies from the family $\mathscr{LC}(X,X^*)$ are called **admissible topologies** or **topologies agreeing with the duality**.
- By Proposition 3 the family $\mathscr{LC}(X,M)$ has the smallest and the largest element, i.e.,
$$\inf \mathscr{LC}(X,M) \in \mathscr{LC}(X,M) \qquad \text{and} \qquad \sup \mathscr{LC}(X,M).$$
The smallest element is called the **weak topology generated by** M and is denoted by $\sigma(X,M)$ (it coincides with the weak topology from Section II.1). The largest element is called the **Mackey topology generated by** M, we will denote it by $\mu(X,M)$. (The symbol $\tau(X,M)$ is often used as well.)

Lemma 4. *Let $(X,\mathcal{T})$ be a LCS. Consider X^* as a subspace of $X^\#$ and the topologies $\sigma(X^*,X)$ on X^* and $\sigma(X^\#,X)$ on $X^\#$. Then:*

(a) *The topology $\sigma(X^\#,X)$ is Hausdorff. The topology $\sigma(X^*,X)$ coincides with the subspace topology generated by $\sigma(X^\#,X)$.*
(b) *If $\mathcal{T}$ is Hausdorff, then X^* is a $\sigma(X^\#,X)$-dense subspace of $X^\#$.*
(c) *Let $A \subset X^*$. Then A is relatively compact in $(X^*,\sigma(X^*,X))$ (i.e., its closure is compact) if and only if the following two conditions hold:*
  - *A is $\sigma(X^*,X)$-bounded.*
  - *$\overline{A}^{\sigma(X^\#,X)} \subset X^*$.*

Definition. Let X be a vector space.

- Let $A \subset X^\#$ be a $\sigma(X^\#,X)$-bounded set. By the symbol q_A we will denote the seminorm on X defined by
$$q_A(x) = \sup\{|f(x)|; f \in A\}, \quad x \in X.$$
- Let $\mathcal{A}$ be a nonempty family of $\sigma(X^\#,X)$-bounded subsets of $X^\#$. By the **topology of uniform convergence on elements of** $\mathcal{A}$ we mean the locally convex topology on X generated by the family of seminorms $\{q_A; A \in \mathcal{A}\}$.

Lemma 5. *Let X be a vector space, $A \subset X^\#$ a $\sigma(X^\#,X)$-bounded set and $f \in X^\#$. Then*

$$|f| \le q_A \Leftrightarrow f \in \overline{\operatorname{aco} A}^{\sigma(X^\#,X)}.$$

Theorem 6 (Mackey-Arens). *Let X be a vector space and $M \subset\subset X^\#$. Then the topology $\mu(X,M)$ coincides with the topology of uniform convergence on absolutely convex $\sigma(M,X)$-compact subsets of M.*

Proposition 7. *Let $(X,\mathcal{T})$ be a metrizable LCS. Then:*

(a) *$(X^*,\sigma(X^*,X))$ is σ-compact.*
(b) *$\mu(X,X^*) = \mathcal{T}$.*

Corollary 8. *Let X be a normed linear space. Then the topology $\mu(X,X^*)$ is the norm topology on X.*

Example 9. *Let X be a Banach space.*

(1) *The topology $\mu(X^*,X)$ coincides with the topology of uniform convergence on absolutely convex weakly compact subsets of X. Moreover, the topology $\mu(X^*,X)$ coincides with the norm topology on X if and only if X is reflexive.*
(2) *Consider on X the topology of uniform convergence on absolutely convex weakly compact subsets of X^*, denote it by ρ. Then ρ is an admissible topology on X, i.e., $(X,\rho)^* = X^*$.*