V. Bounded and unbounded operators on a Hilbert space

Convention. In this chapter we consider the Banach spaces over the complex field (except in Section V.2 or unless the converse is explicitly stated). In particular, the Hilbert spaces we deal with are the complex ones.

V.1 Various types of bounded operators on Hilbert spaces and their properties

Reminder: Let H and K be Hilbert spaces.

(1) By $L(H, K)$ we denote the Banach space of all the bounded linear operators $T : H \to K$ equipped with the operator norm. $L(H)$ is a shortcut for $L(H, H)$.

(2) For any $T \in L(H, K)$ there is a unique operator $T^* \in L(K, H)$, called **the adjoint of** T satisfying
$$\langle Tx, y\rangle_K = \langle x, T^*y\rangle_H \quad \text{for } x \in H \text{ and } y \in K.$$

(3) The mapping $T \mapsto T^*$ is an involution on $L(H)$ it turns $L(H)$ to be a C^*-algebra. Thus the notions and the results from Chapter IV could be applied to $L(H)$. This applies, in particular, to the notions of spectrum, spectral radius, resolvent set, resolvent function, holomorphic functional calculus, self-adjoint, normal and unitary elements and continuous functional calculus for normal elements.

(4) For $x, y \in H$ the following **polarization identity** holds:
$$\langle x, y\rangle = \tfrac{1}{4}\left(\|x+y\|^2 - \|x-y\|^2 + i\,\|x+iy\|^2 - i\,\|x-iy\|^2\right).$$

Definition. Let H and K be Hilbert spaces. An operator $T \in L(H, K)$ is called **unitary** if $T^* = T^{-1}$, i.e., if $T^*T = I_H$ and $TT^* = I_K$.

Proposition 1 (a characterization of unitary operators). *Let H and K be Hilbert spaces and $T \in L(H, K)$. Consider the following assertions:*

(i) *T is unitary.*
(ii) *T is an isometry of H onto K.*
(iii) *T is an isometry of H into K.*
(iv) *$\langle Tx, Ty\rangle_K = \langle x, y\rangle_H$ for $x, y \in H$.*

Then (i) $\Leftrightarrow$ (ii) $\Rightarrow$ (iii) $\Leftrightarrow$ (iv). If T is assumed to be onto, all the assertions are equivalent.

Definition. Let X be a Banach space, $T \in L(X)$ and $\lambda \in \sigma(T)$.

- We say that λ is an **eigenvalue** of T if $\lambda I - T$ is not one-to-one, i.e., whenever there is $x \in X \setminus \{\boldsymbol{o}\}$ such that $Tx = \lambda x$ (then x is an **eigenvector** associated to λ). The set of all the eigenvalues is called the **point spectrum** of T and is denoted by $\sigma_p(T)$.
- We say that λ is an **approximate eigenvalue** of T if there is a sequence of vectors (x_n) of norm one such that $(\lambda I - T)x_n \to \boldsymbol{o}$. The set of all the approximate eigenvalues is called the **approximate point spectrum** of T and is denoted by $\sigma_{ap}(T)$.
- We say that λ belongs to the **continuous spectrum** $\sigma_c(T)$ if $\lambda I - T$ is one-to-one, has dense range but is not onto.
- We say that λ belongs to the **residual spectrum** $\sigma_r(T)$ (also called **compression spectrum**) if $\lambda I - T$ is one to one and its range is not dense.

Proposition 2 (on subsets of the spectrum). *Let X be a Banach space and $T \in L(X)$. Then the following assertions hold:*

(a) *$\sigma_p(T) \subset \sigma_{ap}(T)$.*
(b) *$\lambda \in \mathbb{C} \setminus \sigma_{ap}(T)$ if and only if $\lambda I - T$ is an isomorphism of X into X.*
(c) *$\sigma(T) = \sigma_{ap}(T) \cup \sigma_r(T)$.*
(d) *$\sigma_c(T) = \sigma_{ap}(T) \setminus (\sigma_p(T) \cup \sigma_r(T))) = \sigma(T) \setminus (\sigma_p(T) \cup \sigma_r(T))$.*
(e) *$\lambda \in \sigma_r(T) \setminus \sigma_{ap}(T)$ if and only if $\lambda I - T$ is an isomorphism of X onto a proper closed subspace of X.*

Definition. Let H be a Hilbert space and $T \in L(H)$.

- The **numerical range** of T is the set $W(T) = \{\langle Tx, x\rangle\,; x \in H, \|x\| = 1\}$.
- The **numerical radius** of T is defined by $w(T) = \sup\{|\lambda|\,; \lambda \in W(T)\} = \sup\{|\langle Tx, x\rangle|\,; x \in H, \|x\| = 1\}$.

Lemma 3 (polarization formula for an operator). *Let H be a Hilbert space and $T \in L(H)$. For each $x, y \in H$ the following formula holds:*
$$\langle Tx, y\rangle = \frac{1}{4}\left(\langle T(x+y), x+y\rangle - \langle T(x-y), x-y\rangle + i\,\langle T(x+iy), x+iy\rangle - i\,\langle T(x-iy), x-iy\rangle\right)$$

Proposition 4 (properties of the numerical radius). *Let H be a Hilbert space.*

(a) *The numerical radius w is an equivalent norm on $L(H)$ satisfying $\frac{1}{2}\|T\| \le w(T) \le \|T\|$ for $T \in L(H)$.*
(b) *If $T \in L(H)$ satisfies $\langle Tx, x\rangle = 0$ for all $x \in H$, then $T = 0$.*
(c) *If $S, T \in L(H)$ satisfy $\langle Tx, x\rangle = \langle Sx, x\rangle$ for all $x \in H$, then $S = T$.*
(d) *$W(T)$ is a connected subset of $\mathbb{C}$ for $T \in L(H)$.*
(e) *$\sigma_p(T) \subset W(T)$ and $\sigma(T) \subset \overline{W(T)}$ for $T \in L(H)$.*
(f) *$w(T) \ge r(T)$ for $T \in L(H)$.*

Proposition 5 (structure of normal operators). *Let H be a Hilbert space and $T \in L(H)$. The operator T is normal if and only if $\|Tx\| = \|T^*x\|$ for each $x \in H$. If T is normal, then the following assertions hold.*

(a) $\ker T = \ker T^*$ *and* $\ker T = (R(T))^\perp$.
(b) *$R(T)$ is dense if and only if T is one-to-one. Hence, $\sigma_r(T) = \emptyset$ and $\sigma(T) = \sigma_{ap}(T)$.*
(c) *If $\lambda \in \mathbb{C}$ and $x \in H$ then $Tx = \lambda x$ if and only if $T^*x = \overline{\lambda}x$. In particular, $\sigma_p(T^*) = \{\overline{\lambda}; \lambda \in \sigma_p(T)\}$.*
(d) *If $\lambda_1, \lambda_2 \in \sigma_p(T)$ are distinct, then $\ker(\lambda_1 I - T) \perp \ker(\lambda_2 I - T)$.*

Proposition 6 (characterization of orthogonal projections). *Let H be a Hilbert space and let $P \in L(H)$ be a projection (i.e., $P^2 = P$). The following assertions are equivalent:*

(i) *P is an* **orthogonal projection**, *i.e.,* $\ker P \perp R(P)$.
(ii) *P is self-adjoint.*
(iii) *P is normal.*
(iv) $\langle Px, x\rangle = \|Px\|^2$ *for* $x \in H$.
(v) $\langle Px, x\rangle \geq 0$ *for* $x \in H$.
(vi) $\|P\| \leq 1$.

Moreover, if $P, Q \in L(H)$ are two orthogonal projections, then $R(P) \perp R(Q)$ if and only if $PQ = 0$. In this case P and Q are called **mutually orthogonal**.

Proposition 7 (spectrum of a self-adjoint operator). *Let H be a Hilbert space and $T \in L(H)$.*

(a) *T is self-adjoint if and only if $W(T) \subset \mathbb{R}$.*
(b) *Suppose that T is self-adjoint and set $a = \inf W(T)$ and $b = \sup W(T)$. Then $\sigma(T) \subset [a, b]$, $a, b \in \sigma(T)$, $\|T\| = \max\{|a|, |b|\}$ and $\sigma(T)$ contains one of the numbers $\|T\|, -\|T\|$.*
(c) *$W(T) \subset [0, \infty)$ if and only if T is self-adjoint and $\sigma(T) \subset [0, \infty)$.*

Remarks and definitions.

(1) Operators satisfying the two equivalent conditions from Proposition 7(c) are called **positive**.
(2) T^*T is a positive operator for any $T \in L(H)$.
(3) If $T \in L(H)$, we define $|T| = \sqrt{T^*T}$ (i.e., we apply the continuous function $t \mapsto \sqrt{t}$ to the positive operator T^*T).
(4) If T is normal, then the operator $|T|$ defined above coincides with the operator obtained by applying the continuous function $\lambda \mapsto |\lambda|$ to the operator T. If T is not normal, then $|T| \neq |T^*|$.
(5) An operator $U \in L(H)$ is said to be a **partial isometry** if there is a closed subspace $H_1 \subset H$ such that $U|_{H_1}$ is an isometry of H_1 into H and $U|_{H_1^\perp} = 0$.

Theorem 8 (polar decomposition). *Let H be a Hilbert space and $T \in L(H)$. Then there is a unique partial isometry $U \in L(H)$ such that $T = U|T|$ and $U = 0$ on $R(|T|)^\perp$.*

Moreover, U^* is also a partial isometry and $|T| = U^*T$ and $U^* = 0$ on $R(T)^\perp$.

Theorem 9 (Hilbert-Schmidt). *Let H be a Hilbert space and $T \in L(H)$ be a compact normal operator. Then there is an orthonormal basis of H consisting of eigenvectors of T. Moreover, if $T \neq 0$, then there exist an orthonormal system $(x_k)_{k\in N}$ and nonzero complex numbers $(\lambda_k)_{k\in N}$, where either $N = \mathbb{N}$ or $N = \{1, 2, \dots, m\}$ for some $m \in \mathbb{N}$, such that*

$$Tx = \textstyle\sum_{k\in N} \lambda_k \langle x, x_k\rangle x_k, \qquad x \in H.$$

Proposition 10. *Let H be an infinite-dimensional Hilbert space. Let $T \in L(H)$ be a compact normal operator represented as in Theorem 9. Then $\sigma(T) = \{0\} \cup \{\lambda_k; k \in N\}$. If $f \in \mathcal{C}(\sigma(T))$ is arbitrary, then*

$$\tilde{f}(T)x = f(0)x + \textstyle\sum_{k\in N}(f(\lambda_k) - f(0)) \langle x, x_k\rangle x_k, \qquad x \in H.$$

In particular, $\tilde{f}(T)$ is compact if and only if $f(0) = 0$.

Theorem 11 (Schmidt representation of compact operators). *Let H be a Hilbert space and $T \in L(H)$ be a nonzero compact operator. Then there are orthonormal systems $(e_k)_{k\in N}$, $(f_k)_{k\in N}$ and positive numbers $(\alpha_k)_{k\in N}$, where either $N = \mathbb{N}$ or $N = \{1, 2, \dots, m\}$ for some $m \in \mathbb{N}$, such that*

$$Tx = \textstyle\sum_{k\in N} \alpha_k \langle x, e_k\rangle f_k, \qquad x \in H.$$

Remarks: As specified above, all the statements hold for complex spaces. For real spaces some of the statements hold in the same way, some require a modification and some do not hold at all. More precisely:

- The adjoint operator may be defined in the real case in the same way. The polarization identity in the real case is simpler: $\langle x, y\rangle = \frac{1}{4}(\|x+y\|^2 - \|x-y\|^2)$. Proposition 1 and Proposition 6 hold in the same form for real spaces, a proof may be done in the same way. Proposition 5 requires a modification for real spaces.
- The spectrum is considered only in complex spaces, for real spaces (note that λ would be also real) it could be empty. The numerical range and radius may be of course defined in the real case as well. But Lemma 3 does not hold for real spaces (neither any analogue). This is related to the fact that assertions (a)-(c) from Proposition 4 and assertions (a),(c) from Proposition 7 fail in the real case. It may happen that a nonzero operator has zero numerical radius.
- Some statements remain to be true in the real case at least for self-adjoint operators (for example Proposition 7(b) and Theorem 9). We will analyze the situation later, at the end of Chapter VI.