Remarks on possible definitions of the resolvent set

$$\rho_a(T) = \{\lambda \in \mathbb{C};\ \lambda I - T \text{ is one-to-one, onto, } (\lambda I - T)^{-1} \in \mathcal{L}(X)\}$$
$$\rho_b(T) = \{\lambda \in \mathbb{C};\ \lambda I - T \text{ is one-to-one, onto}\}$$
$$\rho_c(T) = \{\lambda \in \mathbb{C};\ \lambda I - T \text{ is one-to-one, } R(\lambda I - T) \text{ is dense, } (\lambda I - T)^{-1} \text{ is continuous}\}$$

- T closed $\Rightarrow \rho_a(T) = \rho_b(T) = \rho_c(T)$ by Proposition 18

  [T closed, $\lambda \in \mathbb{C} \Rightarrow \lambda I - T$ closed by Prop. 16(a)]

- T not closed $\Rightarrow \rho_a(T) = \emptyset$

  [$(\lambda I - T)^{-1} \in \mathcal{L}(X) \Rightarrow (\lambda I - T)^{-1}$ closed $\underset{\text{Prop. 15 (c)}}{\Rightarrow} \lambda I - T$ closed

  $\overset{\text{Prop. 16(a)}}{\Rightarrow} T$ closed]

- T not closed but with a closed extension

  $\Rightarrow \rho_c(T) = \rho_c(\overline{T}) = \rho_a(\overline{T}) = \rho_b(\overline{T})$

  [by the last item in Remark at the end of Section V.2]

  $\rho_b(T) \cap \rho(\overline{T}) = \emptyset$

  [$\lambda I - \overline{T} = \overline{\lambda I - T}$. By the second item in Remark at the end of Section V.2: $\lambda \in \rho_b(T) \Rightarrow$

  $\lambda I - T$ is one-to-one, onto $\Rightarrow \overline{\lambda I - T}$ is not one-to-one

  Hence $\lambda I - \overline{T}$ is not one-to-one $\Rightarrow \lambda \notin \rho(\overline{T})$]