EXAMPLE

Let $(A_n)_{n=1}^{\infty}$ be disjoint infinite sets covering $\mathbb{N}$

For example: $A_n = \{ (2k+1) \cdot 2^{n-1}, k \in \mathbb{N} \cup \{0\} \}$

Consider the topological space $T = \mathbb{N} \cup \{\infty\}$ with the following topology:

$U \subset T$ is open $\Leftrightarrow$ either $\infty \notin U$
or
$\{ n \in \mathbb{N} ; A_n \setminus U$ is infinite$\}$ is finite

Clearly: T is a Hausdorff topological space with a base of clopen sets, hence completely regular.

Moreover, • $\infty \in \overline{\mathbb{N}}$ (any open set containing ∞ intersects $\mathbb{N}$)

• No sequence from $\mathbb{N}$ converges to ∞

Let (n_k) be any sequence from $\mathbb{N}$. There are two possibilities:

• $\exists m$: $\{ k \in \mathbb{N} : n_k \in A_m \}$ is infinite
Then $T \setminus A_m$ is a neighborhood of ∞ s.t. infinitely many elements of the sequence are outside $T \setminus A_m$

• $\forall m$: $\{ k \in \mathbb{N} : n_k \in A_m \}$ is finite
Then $T \setminus \{ n_k ; k \in \mathbb{N} \}$ is a neighborhood of ∞ containing no element from the sequence.

- There is a subnet of $(n)_{n=1}^{\infty}$ converging to ∞

$\Lambda :=$ neighborhoods of ∞ ; $U \leq V \overset{df}{\equiv} U \supset V$

For each $U \in \Lambda$ let $x_U := \min(U \cap \mathbb{N})$

Then $(x_U)_{U\in\Lambda}$ is a subnet of $(n)_{n=1}^{\infty}$ converging to ∞

[• U nbhd of ∞ $\Rightarrow$ $\forall U \in \Lambda$, $\forall V \geq U$ we have $x_V \in V \subset U$

• $U \leq V \Rightarrow U \supset V \Rightarrow x_U \leq x_V$

• $n \in \mathbb{N} \Rightarrow U := \{k, k \geq n\} \cup \{\infty\} \in \Lambda$
and $\forall V \geq U$: $x_V \geq n$ ┘