Universal quadratic forms over number fields

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Outline

Universal forms over $\mathbb{Z}$

A quadratic form with integral coefficients is *universal* if it represents all positive integers.
There are many indefinite forms, eg. $x^2 - y^2 - dz^2$ with $4 \nmid d$.
Our topic are positive definite forms.

- No ternary universal positive forms
- Lagrange (1770): $x^2 + y^2 + z^2 + t^2$ is universal
- Ramanujan, Dickson (1916): classified quaternary universal positive forms, eg. $x^2 + 2y^2 + 4z^2 + dt^2$ with $d \leq 14$

Examples above are *diagonal* forms.

Theorem (Conway-Schneeberger, Bhargava-Hanke)

A diagonal positive form over $\mathbb{Z}$ is universal iff it represents $1, 2, 3, \dots, 15$.
An integral positive form over $\mathbb{Z}$ is universal iff it represents $1, 2, 3, \dots, 290$.

Outline

Real quadratic fields: Overview

Study totally positive classical universal forms over (real quadratic) number fields. Not much is known, eg.

- Siegel (1945): If K is totally real such that sum of M squares is universal, then $K = \mathbb{Q}$ ($M = 4$) or $K = \mathbb{Q}(\sqrt{5})$ ($M = 3$).
- Chan-Kim-Raghavan (1996): Determined all ternary universal forms over $\mathbb{Q}(\sqrt{2}), \mathbb{Q}(\sqrt{3})$, and $\mathbb{Q}(\sqrt{5})$
- Kim (1999): 8-ary universal form over $\mathbb{Q}(\sqrt{n^2-1})$, but only finitely many $\mathbb{Q}(\sqrt{D})$ admit *diagonal* 7-ary universal forms.

What if, say, each real quadratic field had a million-ary universal form?

Theorem (Blomer-K 2015, K 2016)

For each M, there are infinitely many real fields $\mathbb{Q}(\sqrt{D})$ that do not admit universal M-ary forms.

Want to explain the idea of the proof.

Preliminaries

Notation:

- $D > 1$ squarefree, $D \equiv 2, 3 \pmod 4$ (for simplicity)
- $\mathcal{O}_K = \mathbb{Z}[\sqrt{D}]$
- the conjugate of $\alpha = x + y\sqrt{D}$ is $\alpha' = x - y\sqrt{D}$
- $N(\alpha) = x^2 - y^2 D$
- $\alpha \succ \beta$ iff $\alpha > \beta$ and $\alpha' > \beta'$

A quadratic form

$$Q(x_1, \ldots, x_M) = \sum_{1 \le i \le j \le M} a_{ij} x_i x_j$$

with $a_{ij} \in \mathcal{O}_K$ is

- *(totally) positive* if all values $Q(x_1, \ldots, x_M) \succeq 0$
- *universal* if for each $\alpha \in \mathcal{O}_K, \alpha \succ 0$ there are $x_i \in \mathcal{O}_K$ with $Q(x_1, \ldots, x_M) = \alpha$

Indecomposable elements

Proposition

Assume that there are totally positive $a_1, \ldots, a_M \in \mathcal{O}_K$ *such that* $4a_ia_j \succeq c^2$ $(i \neq j,\ c \in \mathcal{O}_K)$ *implies* $c = 0$.
Then each universal positive form over $\mathcal{O}_K$ *requires at least* M *variables.*

Sketch of proof

- To a positive quadratic N-ary form Q can attach a lattice $L \subset \mathbb{R}^N$ with basis $\ell_1, \ldots, \ell_N$ such that $Q(x) = \langle \sum x_i\ell_i, \sum x_i\ell_i \rangle$.
- Q represents an integer z iff L contains a vector ℓ with $\langle \ell, \ell \rangle = z$.
- Assume L universal $\Rightarrow$ contains vectors $v_1, \ldots, v_M$ with $a_i = \langle v_i, v_i \rangle$.
- Assumption and Cauchy-Schwarz imply that v_i and v_j are orthogonal for $i \neq j$. Hence the rank N of L is at least M.

The assumption will be satisfied for certain indecomposable elements, i.e., $\alpha \succ 0$ such that $\alpha \neq \beta + \gamma$ with $\beta, \gamma \succ 0$. How to find them?

Outline

Continued fractions

- The continued fraction
$$\sqrt{D} = [k, \overline{u_1, u_2, \dots, u_{s-1}, 2k}] = k + \frac{1}{u_1 + \frac{1}{u_2 + \cdots}}$$
is periodic and the sequence $(u_1, u_2, \dots, u_{s-1})$ is symmetric.
- Friesen: conversely, given any symmetric sequence $(u_1, u_2, \dots, u_{s-1})$ (satisfying a mild parity condition), there are infinitely many k such that $\sqrt{D} = [k, \overline{u_1, u_2, \dots, u_{s-1}, 2k}]$.
- $p_i/q_i := [k, u_1, \dots, u_i] =$ ith approximation. Then $p_{i+1} = u_{i+1}p_i + p_{i-1}$ and $q_{i+1} = u_{i+1}q_i + q_{i-1}$.
- $\alpha_i := p_i + q_i\sqrt{D}$, $\alpha_{-1} := 1$
- The totally positive fundamental unit of $\mathbb{Q}(\sqrt{D})$, α_{s-1} or α_{2s-1}, is indecomposable! How about other α_i?

Estimates of norm

- $p_i/q_i = [k, u_1, \ldots, u_i] = i$th approximation = convergent
- $\alpha_i = p_i + q_i\sqrt{D}$
- $\alpha_{i+1} = u_{i+1}\alpha_i + \alpha_{i-1}$
- $\alpha_i \succ 0$ iff i is odd
- α_i is increasing
- α_i' is very small and is decreasing

Proposition

We have

$$\frac{2\sqrt{D}}{u_{i+1}+2} < |N(\alpha_i)| < \frac{2\sqrt{D}}{u_{i+1}}.$$

Conversely, let $\mu \in \mathbb{Z}[\sqrt{D}]$ be such that $0 < |N(\mu)| < \frac{\sqrt{D}}{2}$. Then $\mu = n\alpha_i$ or $\mu = n\alpha_i'$ for some $i \geq -1$ and $n \in \mathbb{Z}$.

This gives us lots of control over small elements.

Construction

Let $s \equiv 2 \pmod 3$ and $u_i = 3^{3^{i-1}}$ for $1 \leq i \leq s/2$.

Proposition

Let i, j be odd such that $1 \leq i < j \leq s/2$. If $4\alpha_i\alpha_j \succeq c^2$ for $c \in \mathcal{O}_K$, then $c = 0$.

Sketch of proof

- $|N(c)| = |cc'| \leq 4\sqrt{N(\alpha_i)N(\alpha_j)} < \frac{8\sqrt{D}}{\sqrt{u_{i+1}u_{j+1}}} < \frac{\sqrt{D}}{2}$, and so (wlog) $c = \alpha_h$
- $4\alpha_i\alpha_j \succeq \alpha_h^2$
- $h < j$ because α_t is (rapidly) increasing
- $h \geq j$ because $\frac{2\sqrt{D}}{u_{h+1}+2} < |N(c)| \leq 4\sqrt{N(\alpha_i)N(\alpha_j)} < \frac{8\sqrt{D}}{\sqrt{u_{i+1}u_{j+1}}}$
- Contradiction!

Comments

This proves our theorem:

Theorem (Blomer-K 2015, K 2016)

For each M, there are infinitely many real fields $\mathbb{Q}(\sqrt{D})$ that do not admit universal M-ary forms.

- K-Svoboda (2017): Generalize to multiquadratic fields
- Yatsyna (2017): Similar results over number fields possessing units of every signature

Outline

Octonary universal forms

- We have found some lower bounds on the arity of a universal form. How about more precise bounds?
- Upper bounds = constructing universal forms
- Kim:
$$x_1^2 + x_2^2 + x_3^2 + x_4^2 + \varepsilon(x_5^2 + x_6^2 + x_7^2 + x_8^2)$$
is universal over $\mathbb{Q}(\sqrt{n^2 - 1})$, where ε is the (totally positive) fundamental unit.
- In fact, each totally positive element is of the form $a\varepsilon^n + b\varepsilon^{n+1}$ with $a, b, n \in \mathbb{Z}$, $a, b \geq 0$.
- Can we generalize it to arbitrary $\mathbb{Q}(\sqrt{D})$?

Constructing universal forms

- *Semi-convergent* to $\sqrt{D} = [k, \overline{u_1, u_2, \ldots, u_{s-1}, 2k}]$ is
$$\frac{p_i + rp_{i+1}}{q_i + rq_{i+1}}$$
with $0 \leq r < u_{i+2}$ $(i \geq -1)$.
- Semi-convergents correspond precisely to indecomposable integers
- S = (finite) set of all semi-convergents σ satisfying $\varepsilon > \sigma > \sigma' > 0$.
- $M_D = \#S = \begin{cases} u_1 + u_3 + \cdots + u_{s-1} & \text{if } s \text{ is even} \\ 2k + u_1 + u_2 + \cdots + u_{s-1} & \text{if } s \text{ is odd} \end{cases}$

Theorem (Blomer-K 2017)

$$\sum_{\sigma \in S} \sigma \left(x_{1j}^2 + x_{2j}^2 + x_{3j}^2 + x_{4j}^2 + \varepsilon (x_{5j}^2 + x_{6j}^2 + x_{7j}^2 + x_{8j}^2) \right)$$

is universal and has $8M_D$ *variables*

Arity of diagonal forms & sums of coefficients

$\sqrt{D} = [k, \overline{u_1, u_2, \dots, u_{s-1}, 2k}]$

$$M_D = \#S = \begin{cases} u_1 + u_3 + \cdots + u_{s-1} & \text{if } s \text{ is even} \\ 2k + u_1 + u_2 + \cdots + u_{s-1} & \text{if } s \text{ is odd} \end{cases}$$

How large is M_D? Can we get a corresponding lower bound on arity?

Theorem (Blomer-K 2017)

- *Each diagonal universal quadratic form has at least $C_\varepsilon M^*_{D,\varepsilon}$ variables, where $M^*_{D,\varepsilon}$ is the same sum of u_i as M_D, but only over coefficients $u_i \geq D^{1/8+\varepsilon}$.*
- $M_D \leq c\sqrt{D}(\log D)^2$
- *If s is odd, then $M^*_{D,\varepsilon} \geq \sqrt{D}$ for every $\varepsilon \leq \frac{1}{8}$.*

Expect (very imprecisely) $M_D \approx M^*_{D,\varepsilon} \approx \frac{\sqrt{D}\log D}{h}(+\sqrt{D}$ if s is odd)

Thanks for your attention!

Slides at `sites.google.com/site/vitakala/`