Classifications of finitely generated semifields and lattice-ordered groups

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Supported by Neuron Impulse award

56th Summer School on General Algebra & Ordered Sets
September 6, 2018

Outline

1. Idempotent semifields
2. Finitely generated semirings
3. Non-idempotent semifields
4. What next?

Semifields

Generalize tropical semifield $\mathbb{R}(\max, +)$

Semifield $S(\oplus, \otimes)$

- $S(\oplus)$ is commutative semigroup
- $S(\otimes)$ is commutative group with inverse x^{-1}
- $\otimes$ distributes over $\oplus$, i.e., $a \otimes (b \oplus c) = a \otimes b \oplus a \otimes c$

Our semifields don't have 0 $(= -\infty)$ ... *not crucial*

Examples:

- $\mathbb{R}(\max, +)$, $\mathbb{Z}(\max, +)$, $\mathbb{Z}^2(\max_{lex}, +)$
- $\mathbb{R}^+(+, \cdot)$, $\mathbb{Q}^+(+, \cdot)$

Want:

- find (and understand) more interesting semifields
- Toy problem: Find all operations $\oplus$ such that $\mathbb{Z}^n(\oplus, +)$ is a semifield.

Idempotent semifields

Idempotent: $a \oplus a = a$
$\mathbb{R}(\max, +)$ Yes
$\mathbb{R}^+(+, \cdot)$ No

Semifields of functions

$\mathcal{M}([0,1]^n)$ = set of continuous functions $f : [0,1]^n \to \mathbb{R}$ that

- are piecewise linear
- each piece has $\mathbb{Z}$-coefficients
- $(f \vee g)(x) := \max(f(x), g(x))$ pointwise maximum
- $(f + g)(x) := f(x) + g(x)$ pointwise sum

$\mathcal{M}([0,1]^n)(\vee, +)$ is idempotent semifield

Ordering

$S(\oplus, \otimes)$ semifield
Define

- Canonical (pre-)ordering: $a \leq b$ if $a \oplus c = b$ for some c, or $a = b$
- Order-unit $u \in S$: for each $s \in S$ there is $n \in \mathbb{N}$ such that $s \leq u \otimes u \otimes \cdots \otimes u$ (n-times)

$\mathcal{M}([0,1]^n)(\vee, +)$: can take $u = \mathbf{1}$, where $\mathbf{1}(x) = 1$ for all x

$\mathcal{M}([0,1]^n)(\vee, +)$ is free n-generated idempotent semifield with order-unit

Generators $f_i(x_1, \ldots, x_n) = x_i$

Lattice-ordered groups $G(\vee, \wedge, +)$ are equivalent to idempotent semifields $G(\vee, +)$

Classification 1

$\mathcal{M}([0,1]^n)(\vee,+)$ is free n-generated idempotent semifield with order-unit

Classify all finitely generated idempotent semifield with order-unit?

For $[0,1]^n \supset D$ define

- $\mathcal{M}(D) =$ restrictions $f|D$ for $f \in \mathcal{M}([0,1]^n)$
- $I(D) =$ functions $f \in \mathcal{M}([0,1]^n)$ such that $f(D) = 0$

For $\mathcal{D} = \{[0,1]^n \supset D_1 \supset D_2 \supset \dots\}$ define

- $I(\mathcal{D}) =$ functions $f \in \mathcal{M}([0,1]^n)$ such that $f(D_i) = 0$ for some i
- $\mathcal{M}(\mathcal{D}) = \mathcal{M}([0,1]^n)/I(\mathcal{D})$
- $\mathcal{M}(\mathcal{D})$ is almost like functions on $\bigcap D_i$

Theorem (Busaniche-Cabrer-Mundici 2012)

Every finitely generated idempotent semifield with order-unit is isomorphic to $\mathcal{M}(\mathcal{D})$, where $\mathcal{D} = \{[0,1]^n \supset D_1 \supset D_2 \supset \dots\}$ is a suitable "stellar" sequence of simplicial complexes (& uniqueness).

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Finitely generated semifield is very strong: using $\oplus, \otimes, ^{-1}$, equivalently, using $\vee, \wedge, +$
What if we want to generate only using the polynomial operations $\oplus, \otimes$, i.e., as a semiring?

WAKE UP: RING MOTIVATION

Theorem (Folklore (Kaplansky))

Let $F(+, \cdot)$ be a field that is finitely generated as a ring, i.e., using only $+, \cdot$.
Then F is finite.

Example: $\mathbb{Q}$ is not finitely generated as a ring.
Are there infinite semifield examples?
YES! $\mathbb{Z}(\max, +)$

Classification 2

$\mathbb{Z}^n(\oplus, +)$ is finitely generated as a semigroup using $+$
Any operation $\oplus$ works to get finitely generated semiring

- T = "rooted forest" on n vertices
- $\vee_T$ = join on $\mathbb{Z}^n$ that is "lexicographic with respect to T"
  see flipchart

Are there more examples? No!

Theorem (K 2017)

Let $S(\oplus, \otimes)$ be an idempotent semifield, finitely generated as semiring. Then there is $n \in \mathbb{N}$ and a rooted forest T such that $S(\oplus, \otimes) \simeq \mathbb{Z}^n(\vee_T, +)$ (& uniqueness).

Idea of proof

- S has to be unital
- Can use classification of Busaniche-Cabrer-Mundici: $S \simeq \mathcal{M}(\mathcal{D})$
- If there is a line segment in $\bigcap D_i$, then $\mathcal{M}(\mathcal{D})$ is not f.g. semiring

Outline

Non-idempotent semifields

Theorem (Folklore (Kaplansky))

Let $F(+,\cdot)$ be a field that is finitely generated as a ring. Then F is finite.

No idempotency assumption, so why assume it in

Theorem (K 2017)

Every **idempotent** *semifield, finitely generated as semiring, is isomorphic to some $\mathbb{Z}^n(\vee_T, +)$.*

?

Because these are the only cases!

Theorem (K-Korbelar 2018)

Every semifield that is finitely generated as a semiring is idempotent.

Idea of proof

Theorem (K-Korbelar 2018)

Every semifield that is finitely generated as a semiring is idempotent.

Idea of proof

- Assume that $S(\oplus, \otimes)$ is not idempotent, generated by $x_1, \ldots, x_n$ as semiring
- $\mathbb{Q}^+ \subset S$
- $Q := \{s \in S \text{ such that } s \leq q \text{ for some } q \in \mathbb{Q}^+\}$
- $\mathcal{C} := \{(a_1, \ldots, a_n) \in \mathbb{N}_0^n \text{ such that } x_1^{a_1} \otimes \cdots \otimes x_n^{a_n} \in Q\}$
- $\mathcal{C} \subset \mathbb{N}_0^n$ is a semigroup with very nice properties – that are proved using the idempotent classification
- These properties are too nice $\Rightarrow$ contradiction!

Outline

What next?

Perfect Banach semifield $S(\oplus, \otimes)$ is

- idempotent semifield with order-unit
- with action of $\mathbb{Q}^+$, $s \mapsto s^q$, extending $s \mapsto s^n = s \otimes \cdots \otimes s$
- complete with respect to suitable distance

Theorem (Leichtnam 2017)

Every perfect Banach semifield is isomorphic to $C^0(X)(\max, +)$, where

- *X is a compact topological space (=set of characters of S),*
- *$C^0(X)$ = space of continuous functions $f : X \to \mathbb{R}$.*

Motivated by “geometry in characteristic one” of Connes with the goal of proving Riemann hypothesis

Continuous instead of discrete, but similar to $\mathbb{Z}^n(\vee_T, +)$ when all vertices of T are roots and $X = \{1, 2, \ldots, n\}$.

Perhaps can combine classifications and generalize Leichtnam's result to more semifields

Thanks

Thanks for your attention!

Slides at `sites.google.com/site/vitakala/`