

Appendix to *The addition on totally positive integers uniquely determines the totally real number field*

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A.1 Introduction

We use the notation and terminology of [KR]. Thus V is an $\mathbb{R}$ -vector space of finite dimension $n \geq 1$ with a fixed norm $\|\cdot\|$; a *lattice* is a discrete subgroup $\Lambda \subset V$ with $\text{span}_{\mathbb{R}}(\Lambda) = V$; a *cone* is a subset $\mathcal{C} \subset V$ that is convex, closed under addition and under outward rays, and satisfies $0 \notin \mathcal{C}$, so that a cone need be neither closed nor closed under multiplication by scalars $t \in (0, 1)$; and a *semilattice* is a subsemigroup $\mathcal{S} \subset V$ with $0 \notin \mathcal{S}$ such that $\text{span}_{\mathbb{Z}}(\mathcal{S})$ is a lattice. We write $\mathcal{I} = \mathcal{I}(\mathcal{S})$ for the set of indecomposable elements of $\mathcal{S}$, $\langle X \rangle$ for the subsemigroup generated by a subset $X \subset \mathcal{S}$, and

$$D = D(\mathcal{S}) := \overline{\text{conv}(\mathcal{S})},$$

so that $\mathcal{S}$ is generated by its indecomposables precisely when $\langle \mathcal{I} \rangle = \mathcal{S}$.

By [KR, Proposition 2.6], $0 \notin D(\mathcal{S})$ implies $\langle \mathcal{I}(\mathcal{S}) \rangle = \mathcal{S}$. The converse is discussed in [KR, Remark 2.8], and the purpose of this appendix is to prove the three assertions made there. The first is that the converse fails for general semilattices; the relevant example is treated in Section A.2, where we also check that it is not of the form $\mathcal{C} \cap \Lambda$. For the semilattices of [KR, Example 2.4], obtained by intersecting a cone with a lattice, the converse does hold in low dimension.

Theorem A.1. *Let $n \leq 3$, let $\Lambda \subset V$ be a lattice and let $\mathcal{C} \subset V$ be a cone such that $\mathcal{S} := \mathcal{C} \cap \Lambda$ satisfies $\text{span}_{\mathbb{R}}(\mathcal{S}) = V$. If every element of $\mathcal{S}$ is a finite sum of indecomposables, then $0 \notin \overline{\text{conv}(\mathcal{S})}$.*

Theorem A.2. *For every $n \geq 4$ there exist a lattice $\Lambda \subset \mathbb{R}^n$ and a cone $\mathcal{C} \subset \mathbb{R}^n$ such that $\mathcal{S} = \mathcal{C} \cap \Lambda$ satisfies $\text{span}_{\mathbb{R}}(\mathcal{S}) = \mathbb{R}^n$, every element of $\mathcal{S}$ is a finite sum of indecomposables, and $0 \in \overline{\text{conv}(\mathcal{S})}$.*

The proofs run as follows. If $0 \in D$, then D is a closed convex cone in the usual sense, equal to $\overline{\mathcal{C}}$, and all indecomposables of $\mathcal{S}$ lie on its boundary (Section A.3). Writing L for the lineality space of D and $W = V/L$, so that D descends to a pointed closed convex cone in W , the relevant invariants turn out to be $\dim L$, $\dim W$ and $\text{rank}(\Lambda \cap L)$. Section A.4 treats three cases and deduces Theorem A.1; Section A.5 proves Theorem A.2. One case, possible only for $n \geq 4$, is settled neither way and is stated in Section A.6.

We use the standard terminology for cones containing the origin, which conflicts with the convention of [KR] recalled above: a *closed convex cone* in a finite-dimensional real vector space U is a nonempty closed convex $E \subset U$ with $tE \subset E$ for all $t \geq 0$, its *lineality space* is the largest linear subspace $E \cap (-E)$ contained in it, and E is *pointed* if $E \cap (-E) = \{0\}$. For convexity we refer to Rockafellar [Ro], using Theorem 6.3 (closure and interior of a convex set), Corollary 11.4.2 (strict separation of a point from a closed convex set), the description of the lineality space and of the quotient of a closed convex cone by it (§8), and the fact that the dual cone of a pointed closed convex cone has nonempty interior, its interior consisting of the functionals that are strictly positive on the cone minus the origin (§14).

A.2 The example of Remark 2.8

Example A.3. Let $V = \mathbb{R}^2$ and

$$\mathcal{S}_0 := \{(x, 0) \in \mathbb{Z}^2 \mid x \geq 1\} \cup \{(x, y) \in \mathbb{Z}^2 \mid y \geq 1, x \geq -y^2\}.$$

Then $\mathcal{S}_0$ is a semilattice, its indecomposables are $\mathcal{I}(\mathcal{S}_0) = \{(1, 0)\} \cup \{(-y^2, y) \mid y \in \mathbb{Z}, y \geq 1\}$, they generate $\mathcal{S}_0$, and $D(\mathcal{S}_0) = \{(x, y) \in \mathbb{R}^2 \mid y \geq 0\} \ni 0$.

Proof. Closure under addition is clear, the only point being $-y^2 - y'^2 \geq -(y + y')^2$; moreover $(1, 0), (-1, 1) \in \mathcal{S}_0$ generate $\mathbb{Z}^2$, so $\mathcal{S}_0$ is a semilattice. All its elements have nonnegative second coordinate, which is therefore additive in decompositions. Hence $(x, 0)$ is decomposable exactly for $x \geq 2$, while for $y \geq 1$ the element (x, y) admits a decomposition with a summand $(x_1, 0)$, $x_1 \geq 1$, exactly when $x + y^2 \geq 1$, and one with both summands of positive second coordinate exactly when $y \geq 2$ and $x \geq -1 - (y - 1)^2$, the maximum of $y_1^2 + y_2^2$ over $y_1 + y_2 = y$ with $y_i \geq 1$ being $1 + (y - 1)^2$. As $-1 - (y - 1)^2 > -y^2$ for $y \geq 2$, the second condition implies the first, so (x, y) is indecomposable exactly when $x = -y^2$. This gives $\mathcal{I}(\mathcal{S}_0)$, which generates because $(x, y) = (-y^2, y) + (x + y^2) \cdot (1, 0)$ for $y \geq 1$.

For the last claim only the inclusion $\supset$ needs an argument. Fix $y > 0$. For integers $Y > y$ and $x_0 \geq 1$ the convex combination $\frac{y}{Y}(-Y^2, Y) + (1 - \frac{y}{Y})(x_0, 0)$ has second coordinate y and first coordinate $-yY + (1 - \frac{y}{Y})x_0$, which is unbounded above as $x_0 \rightarrow \infty$ and unbounded below as $Y \rightarrow \infty$. So $\text{conv}(\mathcal{S}_0)$ meets the horizontal line at height y in an unbounded interval in both directions, that is, in the whole line; passing to the closure gives the remaining points $y = 0$. $\square$

Remark A.4. The semilattice $\mathcal{S}_0$ is not of the form $\mathcal{C} \cap \Lambda$ with Λ a lattice and $\mathcal{C}$ a cone: such a Λ contains $\text{span}_{\mathbb{Z}}(\mathcal{S}_0) = \mathbb{Z}^2$ and such a $\mathcal{C}$ contains $\text{conv}(\mathcal{S}_0)$, so that $\text{conv}(\mathcal{S}_0) \cap \mathbb{Z}^2 \subset \mathcal{S}_0$; but $(-1, 1), (-9, 3) \in \mathcal{S}_0$ while their midpoint $(-5, 2)$ is not.

A.3 The structure of $\mathcal{S} = \mathcal{C} \cap \Lambda$ when $0 \in D$

Lemma A.5. Let $\Lambda \subset V$ be a lattice and $\mathcal{C} \subset V$ a cone such that $\mathcal{S} := \mathcal{C} \cap \Lambda$ satisfies $\text{span}_{\mathbb{R}}(\mathcal{S}) = V$, and assume $0 \in D = D(\mathcal{S})$. Then D is a closed convex cone, $D = \overline{\mathcal{C}}$, $\text{int}(D) = \text{int}(\mathcal{C}) \neq \emptyset$, and $\text{int}(\mathcal{C}) \cap \Lambda \neq \emptyset$.

Proof. As $\mathcal{S} \subset \mathcal{C}$ and $\mathcal{C}$ is convex, $D \subset \overline{\mathcal{C}}$. The identity $\sum_i \lambda_i a_i + \sum_j \mu_j b_j = \sum_{i,j} \lambda_i \mu_j (a_i + b_j)$ for convex combinations shows that $\text{conv}(\mathcal{S})$, hence also D , is closed under addition; as $0 \in D$ and D is convex, it follows that $tD \subset D$ for every $t \geq 0$, so D is a closed convex cone.

Since $\mathcal{S}$ contains $\mathbb{R}$ -linearly independent $a_1, \dots, a_n$ and $\mathcal{C}$ is convex and closed under addition, $\mathcal{C}$ contains the n -simplex $\text{conv}\{a_1, \dots, a_n, 2a_1\}$, so $\text{int}(\mathcal{C}) \neq \emptyset$. All notions involved are invariant under linear isomorphisms, so we may assume $V = \mathbb{R}^n$ and $\Lambda = \mathbb{Z}^n$ and use the maximum norm. Let $x \in \text{int}(\mathcal{C})$ and choose $\rho > 0$ with $x + [-\rho, \rho]^n \subset \mathcal{C}$. If $t \geq 1$ and $\|y - tx\|_{\infty} \leq t\rho$, then $y/t \in \mathcal{C}$, so $y = t \cdot (y/t) \in \mathcal{C}$ because $\mathcal{C}$ is closed under outward rays; that is,

$$tx + [-t\rho, t\rho]^n \subset \mathcal{C} \quad \text{for all } t \geq 1. \quad (1)$$

Let $t \geq \max(1, 1/\rho)$. The point tx is a convex combination of the 2^n vertices of the unit cube containing it, and these are points of $\mathbb{Z}^n$ at $\|\cdot\|_{\infty}$ -distance at most $1 \leq t\rho$ from tx , hence lie in $\mathcal{C} \cap \mathbb{Z}^n = \mathcal{S}$ by (1). Therefore $tx \in \text{conv}(\mathcal{S}) \subset D$ and $x = t^{-1}(tx) \in D$. Moreover the open box $tx + (-t\rho, t\rho)^n$ lies in $\text{int}(\mathcal{C})$ by (1), and for $t > \max(1, 1/\rho)$ its side length exceeds 2, so it contains a point of $\mathbb{Z}^n$.

Since $\mathcal{C}$ is convex with nonempty interior, $\overline{\mathcal{C}} = \overline{\text{int}(\mathcal{C})}$ and $\text{int}(\overline{\mathcal{C}}) = \text{int}(\mathcal{C})$ [Ro, Theorem 6.3]. The first equality, together with $\text{int}(\mathcal{C}) \subset D$ and the closedness of D , gives $\overline{\mathcal{C}} \subset D$, so $D = \overline{\mathcal{C}}$; the second then gives $\text{int}(D) = \text{int}(\mathcal{C})$. $\square$

Standing notation. For the rest of Section A.3 and in Section A.4 we fix a lattice $\Lambda \subset V$ and a cone $\mathcal{C} \subset V$ such that $\mathcal{S} := \mathcal{C} \cap \Lambda$ satisfies $\text{span}_{\mathbb{R}}(\mathcal{S}) = V$, we write $D = \overline{\text{conv}(\mathcal{S})}$ and $\mathcal{I} = \mathcal{I}(\mathcal{S})$, and **we assume** $0 \in D$. Then $\mathcal{S}$ is a semilattice as in [KR, Example 2.4], and D is a closed convex cone by Lemma A.5. Let $L := D \cap (-D)$ be its lineality space, let

$$\pi : V \longrightarrow W := V/L$$

be the quotient map and let $\bar{\Lambda} := \pi(\Lambda)$. Since $L \subset D$ and D is closed under addition we have $D + L = D$, so $D = \pi^{-1}(P)$ for $P := \pi(D)$, and P is a pointed closed convex cone in W with $\text{span}_{\mathbb{R}}(P) = W$ [Ro, §8]. As π is continuous, open and surjective, the set $\pi^{-1}(\text{int}(P))$ is open and contained in D , while π maps every open subset of V contained in D into $\text{int}(P)$; combining this with Lemma A.5,

$$\text{int}(\mathcal{C}) = \text{int}(D) = \pi^{-1}(\text{int}(P)). \quad (2)$$

Lemma A.6. $\{0\} \neq L \neq V$.

Proof. If $L = V$ then $D = V$, so $0 \in \text{int}(D) = \text{int}(\mathcal{C})$ by Lemma A.5, contradicting $0 \notin \mathcal{C}$.

Suppose $L = \{0\}$, so that D is a pointed closed convex cone. Choose f in the interior of the dual cone of D , so that $f > 0$ on $D \setminus \{0\}$ [Ro, §14]; as $D \cap \{\|y\| = 1\}$ is compact, there is $c > 0$ with $f(y) \geq c\|y\|$ for all $y \in D$. Every $\alpha \in \mathcal{S}$ is a nonzero element of Λ , so $\|\alpha\| \geq \lambda_1 := \min\{\|\lambda\| \mid \lambda \in \Lambda \setminus \{0\}\} > 0$ and hence $f(\alpha) \geq c\lambda_1$. As f is linear and continuous, $f \geq c\lambda_1$ on $\text{conv}(\mathcal{S})$ and therefore on D , contradicting $0 \in D$ and $f(0) = 0$. $\square$

By Lemma A.6 we have $W \neq \{0\}$, so the argument just given, applied to the pointed closed convex cone P in W , provides a linear form $\bar{g}$ on W and a $c > 0$ with $\bar{g}(\bar{y}) \geq c\|\bar{y}\|$ for all $\bar{y} \in P$. We fix such $\bar{g}$ and c once and for all and set $g := \bar{g} \circ \pi$, so that $g \geq 0$ on D and

$$g(y) \geq c\|\pi(y)\| \quad \text{for all } y \in D. \quad (3)$$

Since g is linear and continuous, $g \geq 0$ on D and $0 \in D$, we also have

$$\inf_{\mathcal{S}} g = \inf_{\text{conv}(\mathcal{S})} g = \inf_D g = 0. \quad (4)$$

Lemma A.7. *Every element of $\text{int}(\mathcal{C}) \cap \Lambda$ is decomposable in $\mathcal{S}$; in particular $\mathcal{I} \subset \mathcal{C} \setminus \text{int}(\mathcal{C})$ and $\pi(\mathcal{I}) \subset P \setminus \text{int}(P)$.*

Proof. By (4) there are $\beta_k \in \mathcal{S}$ with $g(\beta_k) \rightarrow 0$, and $\pi(\beta_k) \rightarrow 0$ by (3). Let $\alpha \in \text{int}(\mathcal{C}) \cap \Lambda$. By (2) we have $\pi(\alpha) \in \text{int}(P)$, which is open, so $\pi(\alpha) - \pi(\beta_k) \in \text{int}(P)$ for all large k ; for such k , (2) gives $\alpha - \beta_k \in \text{int}(\mathcal{C}) \cap \Lambda \subset \mathcal{S}$, and $\alpha = \beta_k + (\alpha - \beta_k)$ is a decomposition. The remaining assertions follow from (2) and $\pi(\mathcal{C}) \subset \pi(D) = P$. $\square$

A.4 Three cases

Proposition A.8. *Assume $\dim W = 1$. Then $\langle \mathcal{I} \rangle \neq \mathcal{S}$.*

Proof. As P is a pointed closed convex cone with $\text{span}_{\mathbb{R}}(P) = W$ and $\dim W = 1$, we have $P = \mathbb{R}_{\geq 0}\bar{u}$ for some $\bar{u} \neq 0$ and $P \setminus \text{int}(P) = \{0\}$. So Lemma A.7 gives $\mathcal{I} \subset L$ and hence $\langle \mathcal{I} \rangle \subset L$. By Lemma A.6, $L \neq V = \text{span}_{\mathbb{R}}(\mathcal{S})$, so $\mathcal{S} \not\subset L$. $\square$

Proposition A.9. *Assume $\dim L = 1$ and $\text{rank}(\Lambda \cap L) = 1$. Then $\langle \mathcal{I} \rangle \neq \mathcal{S}$.*

Proof. Write $\Lambda_L := \Lambda \cap L$. As Λ_L has $\text{rank } 1 = \dim L$, it spans L over $\mathbb{R}$, so $\bar{\Lambda} = \pi(\Lambda)$ is a lattice in W . Put $\eta := c \cdot \min\{\|\bar{\lambda}\| \mid \bar{\lambda} \in \bar{\Lambda} \setminus \{0\}\} > 0$; by (3),

$$g(\alpha) \geq \eta \quad \text{for every } \alpha \in \mathcal{S} \text{ with } \pi(\alpha) \neq 0. \quad (5)$$

As $\inf_{\mathcal{S}} g = 0$ by (4), it follows that $\mathcal{S} \cap L \neq \emptyset$.

The set $\mathcal{C} \cap L$ is a nonempty convex subset of the line L that does not contain 0, so it lies in one of the two components of $L \setminus \{0\}$. Choose a linear map $\ell : V \rightarrow \mathbb{R}$ whose restriction to L is injective with $\ell(\Lambda_L) = \mathbb{Z}$ and $\ell > 0$ on $\mathcal{C} \cap L$. As $\mathcal{C} \cap L$ is convex and closed under multiplication by $t \geq 1$, the set $\ell(\mathcal{C} \cap L)$ is an interval unbounded above, so $\ell(\mathcal{S} \cap L) = \ell(\mathcal{C} \cap L) = \{k \in \mathbb{Z} \mid k \geq m\}$ for some $m \geq 1$; fix $\mu \in \mathcal{S} \cap L$ with $\ell(\mu) = m$.

For $\bar{v} \in \bar{\Lambda}$ put $S_{\bar{v}} := \mathcal{S} \cap \pi^{-1}(\bar{v})$. The set $\Lambda \cap \pi^{-1}(\bar{v})$ is a coset of Λ_L , on which ℓ is injective with image a coset of $\mathbb{Z}$ in $\mathbb{R}$, and $S_{\bar{v}}$ consists of the points of that coset lying in the convex subset $\mathcal{C} \cap \pi^{-1}(\bar{v})$ of the line $\pi^{-1}(\bar{v})$. Moreover $S_{\bar{v}} + \mu \subset S_{\bar{v}}$, so if $S_{\bar{v}} \neq \emptyset$ then $\ell(S_{\bar{v}})$ is unbounded above. Consequently $\ell(S_{\bar{v}})$ is empty, or all of the coset, or of the form $\{c_{\bar{v}} + k \mid k \in \mathbb{Z}_{\geq 0}\}$ for some $c_{\bar{v}} \in \mathbb{R}$. In the second case every $\alpha \in S_{\bar{v}}$ satisfies $\alpha - \mu \in S_{\bar{v}}$, so $\alpha = \mu + (\alpha - \mu)$ is decomposable and $\mathcal{I} \cap S_{\bar{v}} = \emptyset$. In the third case $\alpha - \mu \in S_{\bar{v}}$ whenever $\ell(\alpha) \geq c_{\bar{v}} + m$, so

$$\mathcal{I} \cap S_{\bar{v}} \subset \{\alpha \in S_{\bar{v}} \mid c_{\bar{v}} \leq \ell(\alpha) < c_{\bar{v}} + m\}, \quad (6)$$

a set of at most m elements. In all three cases $\mathcal{I} \cap S_{\bar{v}}$ is finite.

Suppose now that $\langle \mathcal{I} \rangle = \mathcal{S}$. By Lemma A.5 the set $\text{int}(\mathcal{C}) \cap \Lambda$ is nonempty, so by (2) we may fix $\bar{w} \in \bar{\Lambda} \cap \text{int}(P)$. Then $\pi^{-1}(\bar{w}) \subset \text{int}(\mathcal{C})$ by (2), so $S_{\bar{w}} = \Lambda \cap \pi^{-1}(\bar{w})$ is a full coset of Λ_L , and ℓ is unbounded below on $S_{\bar{w}}$.

Let $\mathcal{V} := \{\bar{v} \in \bar{\Lambda} \cap P \mid \bar{g}(\bar{v}) \leq \bar{g}(\bar{w})\}$, a set that is bounded because $\bar{g}(\bar{y}) \geq c\|\bar{y}\|$ on P , hence finite. Put

$$B := \min\left(\{0\} \cup \{\ell(\beta) \mid \beta \in \mathcal{I} \cap S_{\bar{v}}, \bar{v} \in \mathcal{V} \setminus \{0\}\}\right) \leq 0,$$

which is well defined because $\mathcal{V}$ and each $\mathcal{I} \cap S_{\bar{v}}$ are finite.

Let $\alpha \in S_{\bar{w}}$ and write $\alpha = \sum_{i=1}^N \beta_i$ with $\beta_i \in \mathcal{I}$. Then $\sum_i \pi(\beta_i) = \bar{w}$ with all $\pi(\beta_i) \in P$, so $\sum_i \bar{g}(\pi(\beta_i)) = \bar{g}(\bar{w})$ with all summands nonnegative. Let $J := \{i \mid \pi(\beta_i) \neq 0\}$. For $i \in J$ we have $\bar{g}(\pi(\beta_i)) = g(\beta_i) \geq \eta$ by (5), so $|J| \leq \bar{g}(\bar{w})/\eta$; moreover $\bar{g}(\pi(\beta_i)) \leq \bar{g}(\bar{w})$, so $\pi(\beta_i) \in \mathcal{V} \setminus \{0\}$ and hence $\ell(\beta_i) \geq B$. For $i \notin J$ we have $\beta_i \in \mathcal{S} \cap L$, so $\ell(\beta_i) \geq m > 0$. Therefore

$$\ell(\alpha) = \sum_{i=1}^N \ell(\beta_i) \geq |J| \cdot B \geq \frac{\bar{g}(\bar{w})}{\eta} \cdot B,$$

the last step because $B \leq 0$. This bound is independent of α , contradicting the fact that ℓ is unbounded below on $S_{\bar{w}}$. $\square$

Proposition A.10. *Assume $\dim L = 1$, $\Lambda \cap L = \{0\}$ and $\dim W = 2$. Then $\langle \mathcal{I} \rangle \neq \mathcal{S}$.*

Proof. Since $\Lambda \cap L = \{0\}$, the map π is injective on Λ , hence restricts to an isomorphism of semigroups from $\mathcal{S}$ onto $T := \pi(\mathcal{S})$; note $0 \notin T$. So $\pi(\mathcal{I})$ is the set of indecomposable elements of T , and $\langle \mathcal{I} \rangle = \mathcal{S}$ if and only if $\langle \pi(\mathcal{I}) \rangle = T$. Moreover $\bar{\Lambda} \cong \Lambda$ has rank $n = \dim L + \dim W = 3$, and by (2)

$$\bar{\Lambda} \cap \text{int}(P) \subset T, \quad (7)$$

since $\lambda \in \Lambda$ with $\pi(\lambda) \in \text{int}(P)$ lies in $\text{int}(\mathcal{C}) \cap \Lambda \subset \mathcal{S}$.

As P is a pointed closed convex cone in the plane W with $\text{span}_{\mathbb{R}}(P) = W$, there are $\mathbb{R}$ -linearly independent $u_1, u_2 \in W$ with $P = \{su_1 + tu_2 \mid s, t \geq 0\}$; then $\text{int}(P) = \{su_1 + tu_2 \mid s, t > 0\}$ and $P \setminus \text{int}(P) = R_1 \cup R_2$, where $R_i := \mathbb{R}_{\geq 0}u_i$. Put $T_i := T \cap R_i$ and $G_i := \bar{\Lambda} \cap \mathbb{R}u_i$. By Lemma A.7, $\pi(\mathcal{I}) \subset R_1 \cup R_2$. By Lemma A.5 and (2) we may fix $\bar{v}_0 = s_0u_1 + t_0u_2 \in \bar{\Lambda}$ with $s_0, t_0 > 0$.

Assume $\langle \mathcal{I} \rangle = \mathcal{S}$, so $\langle \pi(\mathcal{I}) \rangle = T$. We first prove

$$su_1 + tu_2 \in \bar{\Lambda} \text{ and } s, t > 0 \implies su_1 \in T_1 \text{ and } tu_2 \in T_2. \quad (8)$$

Indeed, let $\bar{v} = su_1 + tu_2 \in \bar{\Lambda}$ with $s, t > 0$. Then $\bar{v} \in T$ by (7), so $\bar{v} = \sum_k \beta_k$ with $\beta_k \in \pi(\mathcal{I}) \subset R_1 \cup R_2$. Each β_k is nonzero because $0 \notin T$, and $R_1 \cap R_2 = \{0\}$, so each β_k lies in exactly one of

R_1, R_2 . Let $\bar{a}$ be the sum of those lying in R_1 and $\bar{b}$ the sum of those lying in R_2 , an empty sum being 0; as T is a semigroup, $\bar{a} \in T_1$, respectively $\bar{b} \in T_2$, whenever the corresponding sum is nonempty. Comparing coordinates in the basis u_1, u_2 gives $\bar{a} = su_1$ and $\bar{b} = tu_2$, and both are nonzero because $s, t > 0$; this proves (8).

Next we show $\bar{\Lambda} = G_1 \oplus G_2$. Let $\bar{\lambda} = su_1 + tu_2 \in \bar{\Lambda}$ and choose $M \in \mathbb{Z}^+$ with $s + Ms_0 > 0$ and $t + Mt_0 > 0$. Applying (8) to $\bar{\lambda} + M\bar{v}_0$ and to $M\bar{v}_0$, both of which lie in $\bar{\Lambda}$ and have positive coordinates, yields $(s + Ms_0)u_1 \in T_1 \subset \bar{\Lambda}$ and $Ms_0u_1 \in T_1 \subset \bar{\Lambda}$; subtracting, $su_1 \in G_1$, and symmetrically $tu_2 \in G_2$. As $G_1 \cap G_2 \subset \mathbb{R}u_1 \cap \mathbb{R}u_2 = \{0\}$, the sum is direct, and therefore

$$\text{rank}G_1 + \text{rank}G_2 = \text{rank}\bar{\Lambda} = 3.$$

Both G_i are nonzero: if $G_2 = \{0\}$ then $T_2 \subset \bar{\Lambda} \cap R_2 \subset \{0\}$, so $T_2 = \emptyset$ because $0 \notin T$, contradicting (8) applied to $\bar{v}_0$; and symmetrically for G_1 . Consequently, given $s > 0$ with $su_1 \in G_1$, we may choose $t > 0$ with $tu_2 \in G_2$, so that $su_1 + tu_2 \in \bar{\Lambda}$ and hence $su_1 \in T_1$ by (8). Together with $T_1 \subset (\bar{\Lambda} \cap R_1) \setminus \{0\}$, and symmetrically for $i = 2$, this gives

$$T_i = \{su_i \mid su_i \in G_i, s > 0\} \quad (i = 1, 2). \quad (9)$$

By the rank identity, $\text{rank}G_i \geq 2$ for some i ; fix such an i . A subgroup of a line of rank at least 2 is not cyclic, hence is dense in that line, so $\{s \in \mathbb{R} \mid su_i \in G_i\}$ is dense in $\mathbb{R}$. Let $su_i \in T_i$, so $s > 0$ by (9), and choose s' with $0 < s' < s$ and $s'u_i \in G_i$. Then $(s - s')u_i \in G_i$ with $s - s' > 0$, so $s'u_i$ and $(s - s')u_i$ lie in T_i by (9), and $su_i = s'u_i + (s - s')u_i$ is a decomposition in T . Hence $\pi(\mathcal{I}) \cap R_i = \emptyset$.

Finally, $T_i \neq \emptyset$ by (8) applied to $\bar{v}_0$, so we may pick $\gamma \in T_i$ and write $\gamma = \sum_{k=1}^N \beta_k$ with $N \geq 1$ and $\beta_k \in \pi(\mathcal{I}) \subset R_1 \cup R_2$. Grouping the summands as in the proof of (8) and comparing coordinates shows that every β_k lies in R_i , the sum of those lying in the other ray being a sum of nonzero elements of that ray. As $N \geq 1$, this contradicts $\pi(\mathcal{I}) \cap R_i = \emptyset$. $\square$

Proof of Theorem A.1. Assume $0 \in D$. By Lemma A.6, $1 \leq \dim L \leq n - 1$; in particular $n \geq 2$. If $\dim W = 1$, then $\langle \mathcal{I} \rangle \neq \mathcal{S}$ by Proposition A.8; this covers $n = 2$, and for $n = 3$ it covers the case $\dim L = 2$. There remains the case $n = 3$, $\dim L = 1$, $\dim W = 2$, where $\Lambda \cap L$ is a discrete subgroup of a line and so has rank 0 or 1: apply Proposition A.9, respectively Proposition A.10. $\square$

A.5 Counterexamples in dimension at least four

Example A.11. Let $V = \mathbb{R}^4$ with coordinates (x, y, s, t) , let $\Lambda = \mathbb{Z}^4$ and let

$$\begin{aligned} \mathcal{C} := & \{s > 0, t > 0\} \cup \{s > 0, t = 0, y \geq 0\} \\ & \cup \{s = 0, t > 0, x \geq 0\} \cup \{s = t = 0, x > 0, y > 0\}. \end{aligned}$$

Then $\mathcal{C}$ is a cone, $\mathcal{S} := \mathcal{C} \cap \mathbb{Z}^4$ is a semilattice with $\text{span}_{\mathbb{R}}(\mathcal{S}) = \mathbb{R}^4$, every element of $\mathcal{S}$ is a finite sum of indecomposables, and $0 \in \overline{\text{conv}(\mathcal{S})}$.

Proof. Every point of $\mathcal{C}$ has $s \geq 0$ and $t \geq 0$. Let $q, q' \in \mathcal{C}$ and let r be either a proper convex combination of q and q' or their sum. If $s_r > 0$ and $t_r > 0$, then r lies in the first of the four sets. If $s_r > 0$ and $t_r = 0$, then $t_q = t_{q'} = 0$, so q and q' lie in the second or the fourth set and hence have nonnegative second coordinate; thus $y_r \geq 0$ and r lies in the second set, and the case $s_r = 0, t_r > 0$ is symmetric. If $s_r = t_r = 0$, then all four of $s_q, s_{q'}, t_q, t_{q'}$ vanish, so q, q' lie in the fourth set and so does r . Hence $\mathcal{C}$ is convex and closed under addition; each of the four sets is defined by conditions invariant under multiplication by positive scalars, and $0 \notin \mathcal{C}$.

The elements $(0, 0, 1, 0)$, $(0, 0, 0, 1)$, $(1, 1, 0, 0)$, $(1, 2, 0, 0)$ of $\mathcal{S}$ generate $\mathbb{Z}^4$, so $\text{span}_{\mathbb{Z}}(\mathcal{S}) = \mathbb{Z}^4$. For $N \in \mathbb{Z}^+$ the points $(-N, 0, 1, 0)$, $(0, -N, 0, 1)$, $(1, 1, 0, 0)$ lie in $\mathcal{S}$ and

$$\frac{1}{N+2}(-N, 0, 1, 0) + \frac{1}{N+2}(0, -N, 0, 1) + \frac{N}{N+2}(1, 1, 0, 0) = \left(0, 0, \frac{1}{N+2}, \frac{1}{N+2}\right) \in \text{conv}(\mathcal{S}),$$

so $0 \in \overline{\text{conv}(\mathcal{S})}$ on letting $N \rightarrow \infty$.

It remains to determine $\mathcal{I}$ and to check that it generates. Write $\varpi(x, y, s, t) := (s, t)$, so that $\varpi(\mathcal{S}) = \mathbb{Z}_{\geq 0}^2$ and ϖ is additive on decompositions. The fiber of $\mathcal{S}$ over (s, t) is $\mathbb{Z}^2$ if $s, t \geq 1$; it is $\{y \geq 0\}$ if $s \geq 1, t = 0$; it is $\{x \geq 0\}$ if $s = 0, t \geq 1$; and it is $\{x \geq 1, y \geq 1\}$ if $s = t = 0$.

If $s, t \geq 1$, then $(x, y, s, t) = (x, 0, s, 0) + (0, y, 0, t)$; if $s \geq 2$ and $t = 0$, then $(x, y, s, 0) = (0, 0, 1, 0) + (x, y, s-1, 0)$, and symmetrically for $t \geq 2, s = 0$. For $s = 1, t = 0$, a decomposition of $(x, y, 1, 0)$ must have one summand $(x_2, y_2, 0, 0)$ with $x_2, y_2 \geq 1$, the other being $(x-x_2, y-y_2, 1, 0)$, which lies in $\mathcal{S}$ if and only if $y \geq y_2$; such summands exist if and only if $y \geq 1$. Hence the indecomposables in this fiber are the $(x, 0, 1, 0)$, $x \in \mathbb{Z}$, and symmetrically those in the fiber over $(0, 1)$ are the $(0, y, 0, 1)$, $y \in \mathbb{Z}$. Finally $(x, y, 0, 0)$ with $x, y \geq 1$ decomposes only within the fiber over $(0, 0)$, hence precisely when $x \geq 2$ and $y \geq 2$. Altogether

$$\mathcal{I} = \{(x, 0, 1, 0) \mid x \in \mathbb{Z}\} \cup \{(0, y, 0, 1) \mid y \in \mathbb{Z}\} \cup \{(x, 1, 0, 0) \mid x \geq 1\} \cup \{(1, y, 0, 0) \mid y \geq 1\}.$$

These generate. If $s, t \geq 1$, then $(x, y, s, t) = \sum_{i=1}^s (x_i, 0, 1, 0) + \sum_{j=1}^t (0, y_j, 0, 1)$ for any $x_i, y_j \in \mathbb{Z}$ with $\sum_i x_i = x$ and $\sum_j y_j = y$; the same formula with $t = 0$ settles the case $s \geq 1, y = 0$, and for $s \geq 1, y \geq 1$ we have $(x, y, s, 0) = (1, y, 0, 0) + (x-1, 0, s, 0)$. The case $s = 0, t \geq 1$ is symmetric. Finally, $(x, y, 0, 0)$ with $x, y \geq 1$ lies in $\mathcal{I}$ if $x = 1$ or $y = 1$, and equals $(x-1, 1, 0, 0) + (1, y-1, 0, 0)$ otherwise. $\square$

Lemma A.12. *Let $\Lambda \subset V$ be a lattice and $\mathcal{C} \subset V$ a cone such that $\mathcal{S} := \mathcal{C} \cap \Lambda$ satisfies $\text{span}_{\mathbb{R}}(\mathcal{S}) = V$. Let $V' := V \oplus \mathbb{R}$, $\Lambda' := \Lambda \oplus \mathbb{Z}$, let $\mathcal{C}'$ be either $\mathcal{C} \times \mathbb{R}_{\geq 0}$ or $\mathcal{C} \times \mathbb{R}$, and let $\mathcal{S}' := \mathcal{C}' \cap \Lambda'$. Then $\mathcal{C}'$ is a cone in V' , $\text{span}_{\mathbb{R}}(\mathcal{S}') = V'$, and:*

- (i) $\mathcal{I}(\mathcal{S}') = \mathcal{I}(\mathcal{S}) \times \mathbb{Z}_{\geq 0}$, respectively $\mathcal{I}(\mathcal{S}) \times \mathbb{Z}$;
- (ii) if $\langle \mathcal{I}(\mathcal{S}) \rangle = \mathcal{S}$, then $\langle \mathcal{I}(\mathcal{S}') \rangle = \mathcal{S}'$;
- (iii) if $0 \in \text{conv}(\mathcal{S})$, then $0 \in \text{conv}(\mathcal{S}')$, and the lineality space of $\overline{\text{conv}(\mathcal{S}')}$ is $L \times \{0\}$, respectively $L \times \mathbb{R}$, where L is the lineality space of $\text{conv}(\mathcal{S})$.

Proof. That $\mathcal{C}'$ is a cone is immediate, and $\mathcal{S}' = \mathcal{S} \times \mathbb{Z}_{\geq 0}$, respectively $\mathcal{S} \times \mathbb{Z}$, which spans V' . An element (α, u) of $\mathcal{S}'$ is decomposable if and only if α is: a decomposition of (α, u) gives one of α , and conversely $\alpha = \beta + \gamma$ gives $(\alpha, u) = (\beta, u) + (\gamma, 0)$. This is (i), and (ii) follows by writing $\alpha = \sum_{i=1}^N \beta_i$ with $\beta_i \in \mathcal{I}(\mathcal{S})$ and $(\alpha, u) = (\beta_1, u) + \sum_{i \geq 2} (\beta_i, 0)$.

For (iii), the inclusion $\mathcal{S} \times \{0\} \subset \mathcal{S}'$ gives $0 \in \overline{\text{conv}(\mathcal{S}')}$. Lemma A.5, applied in V and in V' , then gives $\text{conv}(\mathcal{S}) = \overline{\mathcal{C}}$ and $\text{conv}(\mathcal{S}') = \overline{\mathcal{C}'} = \overline{\mathcal{C}} \times \mathbb{R}_{\geq 0}$, respectively $\overline{\mathcal{C}} \times \mathbb{R}$, whose lineality spaces are $L \times \{0\}$ and $L \times \mathbb{R}$. $\square$

Proof of Theorem A.2. Start from Example A.11 and apply Lemma A.12 $n-4$ times. $\square$

A.6 The remaining case

Keep the standing notation of Section A.3, so that $0 \in D$, and recall that $\dim L \geq 1$ and $\dim W \geq 1$ by Lemma A.6. The indecomposables of $\mathcal{S}$ cannot generate $\mathcal{S}$ when $\dim W = 1$ (Proposition A.8), nor when $\dim L = 1$ and $\text{rank}(\Lambda \cap L) = 1$ (Proposition A.9), nor when $\dim L = 1, \Lambda \cap L = \{0\}$ and $\dim W = 2$ (Proposition A.10). On the other hand, for every pair $(\dim L, \dim W) = (d, e)$ with $d, e \geq 2$ there are examples in which they do generate: in Example A.11 we have $\overline{\mathcal{C}} = \{s \geq 0, t \geq 0\}$ with lineality space $\{s = t = 0\}$, so that $\dim L = \dim W = 2$ there, and by Lemma A.12 the factor $\mathcal{C} \times \mathbb{R}_{\geq 0}$ raises $\dim W$ by one while $\mathcal{C} \times \mathbb{R}$ raises $\dim L$ by one. This leaves one case undecided, which requires $n \geq 4$.

Question A.13. Let $\Lambda \subset V$ be a lattice and $\mathcal{C} \subset V$ a cone with $\mathcal{S} = \mathcal{C} \cap \Lambda$, $\text{span}_{\mathbb{R}}(\mathcal{S}) = V$ and $0 \in \overline{\text{conv}(\mathcal{S})}$, and suppose that the lineality space L of $\overline{\text{conv}(\mathcal{S})}$ satisfies $\dim L = 1$, $\Lambda \cap L = \{0\}$ and $\dim(V/L) \geq 3$. Can every element of $\mathcal{S}$ be a finite sum of indecomposables?

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