UNIVERZITA KARLOVA
Matematicko-fyzikální
fakulta

Dr. Malte Kampschulte
Dr. Alexei Gazca

Advanced Methods in Mathematical Analysis

Winter Semester 2025/26 —— Sheet 9

Task 1 (Existence of Minimisers for convex functionals)

Let $\Omega \subset \mathbb{R}^n$ be a bounded Lipschitz domain and let $f\colon \Omega \times \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}$ be a Carathéodory function that is convex in its third argument. Suppose that the following growth condition is satisfied:

$$f(x, z, \xi) \geq \phi(|\xi|) \qquad \text{for a.e. } x \in \Omega,\ \forall\, z \in \mathbb{R},\ \xi \in \mathbb{R}^n.$$

where $\phi\colon \mathbb{R}_+ \to \mathbb{R}_+$ is a convex increasing function with superlinear growth: $\lim_{t\to\infty} \frac{\phi(t)}{t} = \infty$. Prove that there exists a function $u \in W_0^{1,1}(\Omega)$ that minimises the energy functional

$$u \in W^{1,1}(\Omega) \mapsto \int_\Omega f(x, u(x), \nabla u(x))\,\mathrm{d}x$$

Hint. Poincaré's inequality can be useful: $\|v\|_{L^1(\Omega)} \leq c\|\nabla u\|_{L^1(\Omega)}$ for some constant $c > 0$ and all $v \in W_0^{1,1}(\Omega)$.

Task 2 (Vitali's Convergence Theorem)

Let $(X, \mathcal{M}, \mu)$ be a measure space, and $u_k, u\colon X \to \mathbb{R}$ measurable for all $k \in \mathbb{N}$. Prove that $u_k \to u$ in $L^1(X)$ if

(i) $u_k \to u$ pointwise μ-a.e.

(ii) The sequence $(u_k)_{k\in\mathbb{N}}$ is equi-integrable. I.e. for all $\varepsilon > 0$ there is $\delta > 0$ such that if $\mu(E) < \delta$ for some $E \in \mathcal{M}$, one has

$$\int_E |u_k|\,\mathrm{d}\mu < \varepsilon \qquad \forall\, k \in \mathbb{N}.$$

(iii) For every $\varepsilon > 0$ there is $A \in \mathcal{M}$ with $\mu(A) < \infty$, such that

$$\int_{X\setminus A} |u_k|\,\mathrm{d}\mu < \varepsilon \qquad \forall\, k \in \mathbb{N}.$$

Conversely, show that if $u_k \to u$ in $L^1(X)$, then (ii) and (iii) hold, and (i) holds for a subsequence.

Task 3 (Cavalieri's Principle)

Let $A \subset \mathbb{R}^n$ be a Lebesgue measurable set and let $f\colon A \to [0, \infty]$ be a Lebesgue measurable function. Prove that

$$\int_A f\,\mathrm{d}x = \int_0^\infty \mu(\{x \in A \mid f(x) > t\})\,\mathrm{d}t.$$

Hint. Fubini.

Task 4

Consider the following sequence of measures on $\mathbb{R}$:

$$\mu_k = \frac{1}{k}(\delta_{\frac{1}{k}} + \delta_{\frac{2}{k}} + \cdots \delta_{\frac{k}{k}})$$

Does this sequence converge weakly-* in $\mathcal{M}$?

Task 5 (Biting vs. measure convergence)

Consider the following sequence of functions $u_k\colon (0,1) \to \mathbb{R}$, defined for $k \geq 2$:

$$u_k(x) = \begin{cases} \frac{k^2}{2} & \text{for } x \in \left(\frac{j}{k+1} - \frac{1}{k^3}, \frac{j}{k+1} + \frac{1}{k^3}\right), \quad j \in \{1, \dots, k\}, \\ 0 & \text{otherwise.} \end{cases}$$

(a) Prove that $\|u_k\|_{L^1} = 1$ and that $u_k \overset{*}{\rightharpoonup} 1$ in $\mathcal{M}(0,1)$ (in particular note that the limit actually lies in $L^1(0,1)$).

(b) Chacon's biting lemma guarantees the existence of a sequence of measurable sets E_n with $\lim_{n\to\infty} \mu(E_n) = 0$ such that (up to a subsequence) u_k converges weakly in $L^1((0,1) \setminus E_n)$ (for fixed n) to a function $u \in L^1(0,1)$. In this example one can take the following sequence:

$$E_n = \bigcup_{k \geq n} \{u_k \neq 0\}.$$

Verify that $\mu(E_n) \to 0$ as $n \to \infty$ and that $u_k \rightharpoonup 0$ weakly in $L^1((0,1) \setminus E_n)$ as $k \to \infty$ for fixed n (no subsequence needed).