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Advanced Methods in Mathematical Analysis

Winter Semester 2025/26 —— Sheet 8

Task 1

Let $(X, \mathcal{A}, \mu)$ be a finite measure space and $f \in L^p(X)$ for some $p \in [1, \infty]$. Show that $f \in L^q(\Omega)$ for all $q \in [1, p]$ with

$$\|f\|_{L^q} \leq \mu(X)^{\frac{1}{q}-\frac{1}{p}} \|f\|_{L^p}$$

When does equality occur?

Task 2

Let $(X, \mathcal{A}, \mu)$ be a measure space and $(f_k)_{k\in\mathbb{N}}$ be a sequence of integrable functions. Suppose that $f_k(x) \to f(x)$ for μ-a.e. $x \in X$, and $\lim\limits_{k\to\infty} \int_X |f_k| \mathrm{d}\mu \to \int_X |f| \mathrm{d}\mu$. Prove that

$$\lim_{k\to\infty} \int_X |f_k - f| \,\mathrm{d}\mu = 0.$$

Task 3

Consider the Lebesgue measure space $(\mathbb{R}, \mathcal{M}_L, \mu_L)$, and let ν be the counting measure on $\mathcal{M}_L$.

(a) Show that $\mu_L \ll \nu$ but $\frac{\mathrm{d}\mu_L}{\mathrm{d}\nu}$ does not exist.

(b) Show that ν does not have a Lebesgue decomposition with respect to μ_L.

Task 4

Let μ and ν be two positive measures on a measure space $(X, \mathcal{A})$. Suppose $\frac{\mathrm{d}\nu}{\mathrm{d}\mu}$ exists so that $\nu \ll \mu$.

(a) Show that if $\frac{\mathrm{d}\nu}{\mathrm{d}\mu} > 0$ μ-a.e. in X, then $\mu \ll \nu$ and thus $\mu \sim \nu$.

(b) Show that if $\frac{\mathrm{d}\nu}{\mathrm{d}\mu} > 0$ μ-a.e. in X and if μ and ν are σ-finite, then $\frac{\mathrm{d}\mu}{\mathrm{d}\nu}$ exists and

$$\frac{\mathrm{d}\mu}{\mathrm{d}\nu} = \left(\frac{\mathrm{d}\nu}{\mathrm{d}\mu}\right)^{-1} \qquad \mu\text{-a.e. and } \nu\text{-a.e. in } X.$$