Dr. Malte Kampschulte
Dr. Alexei Gazca

Advanced Methods in Mathematical Analysis

Winter Semester 2025/26 —— Sheet 7

Task 1 (Integral operators on $L^2(\Omega)$ vs. $C(\overline{\Omega})$)

Let $\Omega \subset \mathbb{R}^d$ be a smooth bounded domain.

(a) Let $a\colon \overline{\Omega} \times \overline{\Omega} \times \mathbb{R} \to \mathbb{R}$ be a continuous map and define

$$A(u)(x) := \int_\Omega a(x, y, u(y))\, \mathrm{d}y.$$

Prove that $A\colon C(\overline{\Omega}) \to C(\overline{\Omega})$ is well-defined and compact.

(b) Let $k \in L^2(\Omega \times \Omega)$ and define

$$K(u)(x) := \int_\Omega k(x, y)u(y)\, \mathrm{d}y.$$

Prove that $K\colon L^2(\Omega) \to L^2(\Omega)$ is well-defined and compact.

(c) Give an example of continuous a such that A is not well-defined as an operator $L^2(\Omega) \to L^2(\Omega)$.

(d) [**Bonus**] For $f \in C([-1, 1])$ consider the following boundary value problem:

$$-u'' = f \text{ in } (-1, 1), \qquad u(-1) = 0 = u(1).$$

Show that the solution to this problem is unique and that it is represented by the formula

$$u(x) = \int_{-1}^{x} \frac{(1+y)(1-x)}{2} f(y)\, \mathrm{d}y + \int_{x}^{1} \frac{(1-y)(1+x)}{2} f(y)\, \mathrm{d}y$$

Show that the solution operator $f \in L^2(-1, 1) \mapsto u \in L^2(-1, 1)$ is compact.

Task 2 (Nemytskiĭ operators)

Let Ω be a topological space with Borel measure μ and let $f\colon \Omega \times \mathbb{R}^m \to \mathbb{R}^n$ be a Carathéodory function such that

$$|f(x, z)| \le c|z|^{p/q} + g(x) \qquad \text{for } \mu\text{-a.e. } x \in \Omega, \text{and all } z \in \mathbb{R}^m.$$

where $c > 0$, $p, q \in [1, \infty)$, and $g \in L^q(\Omega)$. The associated Nemytskiĭ (or superposition) operator to f, denoted $F(u)$, maps a function $u\colon \Omega \to \mathbb{R}^m$ to the function

$$x \in \Omega \mapsto F(u)(x) := f(x, u(x)) \in \mathbb{R}^n.$$

(a) Prove that if $u\colon \Omega \to \mathbb{R}^m$ is measurable, then $F(u)$ is also measurable.

(b) Prove that $F(u) \in L^q(\Omega; \mathbb{R}^n)$ whenever $u \in L^p(\Omega; \mathbb{R}^m)$.

(c) Assume that f is uniformly continuous and let $(u_k)_{k\in\mathbb{N}} \subset L^p(\Omega; \mathbb{R}^m)$ be a sequence such that $u_k \to u$ in $L^p(\Omega; \mathbb{R}^m)$ and $|u_k(x)| \le R$ for some $R > 0$, μ-a.e. in Ω. Prove that $F(u_k) \to F(u)$ in $L^q(\Omega; \mathbb{R}^n)$.

(d) With the help of the Lusin-type theorem for Carathéodory functions, show that the previous statement holds also without the restrictions on f and u_k.

Hint: Show that $\int_A |f(x, u_k(x))|^q\, \mathrm{d}\mu$ is small if $\mu(A)$ is small, even if $|u_k(x)|$ is large.

Task 3 (Lower-semicontinuity of convex functionals)

Let Ω be a topological space with Borel measure μ and let $f\colon \Omega \times \mathbb{R}^m \times \mathbb{R}^n \to \mathbb{R}$ be a Carathéodory function that is convex in its third argument. Suppose that $u_k \to u$ in $L^p(\Omega; \mathbb{R}^m)$ and $v_k \rightharpoonup v$ weakly in $L^q(\Omega; \mathbb{R}^n)$. Prove that

$$\int_\Omega f(x, u(x), v(x))\,\mathrm{d}\mu \le \liminf_{k\to\infty} \int_\Omega f(x, u_k(x), v_k(x))\,\mathrm{d}\mu.$$