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Advanced Methods in Mathematical Analysis

Winter Semester 2025/26 —— Sheet 6

Task 1

Let $(V, \|\cdot\|)$ be a normed space and $(V^*, \|\cdot\|_*)$ its dual. Let $\varphi\colon \mathbb{R} \to \mathbb{R} \cup \{+\infty\}$ be a convex l.s.c. function that is even ($\varphi(t) = \varphi(-t)$). Prove that the function

$$f\colon V \to \mathbb{R} \cup \{+\infty\}$$
$$f(v) = \varphi(\|v\|)$$

is a closed convex proper function, and

$$f^*(v^*) = \varphi^*(\|v^*\|_*)$$

Hint. Write $f^*(v^*) = \sup_{t \geq 0} \sup_{v \in V,\, \|v\|=t} [\langle v^*, v\rangle - \varphi(\|v\|)]$.

Task 2 (Monotone Convergence Theorem and Fatou's Lemma for measurable sets)

Let $(X, \mathcal{A}, \mu)$ be a measure space and $(E_n)_{n\in\mathbb{N}}$ be a monotone increasing sequence of sets in $\mathcal{A}$ (i.e. $E_n \subset E_{n+1}$ for all $n \in \mathbb{N}$).

(a) Show that

$$\lim_{n\to\infty} \mu(E_n) = \mu\left(\lim_{n\to\infty} E_n\right)$$

Hint. Express $\bigcup_{n\in\mathbb{N}} E_n$ as the countable union of $E_n \setminus \bigcup_{k=1}^{n-1} E_k$

(b) Prove that this is not necessarily true for decreasing sequences of sets, but it is true if one assumes additionally that $\mu(E_1) < \infty$.

Hint. Consider the counting measure and an infinite set.

(c) Let $(A_n)_{n\in\mathbb{N}}$ be a sequence of sets in $\mathcal{A}$. Prove with the help of (a) that

$$\mu\left(\liminf_{n\to\infty} A_n\right) \leq \liminf_{n\to\infty} \mu(A_n)$$

Task 3 (Properties of the Lebesgue integral)

Let $(\Omega, \mathcal{M}, \mu)$ be a measure space and $f, g\colon \Omega \to [0, +\infty]$ measurable functions. Prove the following statements:

(a) If $f \leq g$ pointwise μ-almost everywhere, then $\int_\Omega f \,\mathrm{d}\mu \leq \int_\Omega g \,\mathrm{d}\mu$.

(b) If $A \in \mathcal{M}$ is a measurable set, then $\int_A f \,\mathrm{d}\mu \leq \int_\Omega f \,\mathrm{d}\mu$.

(c) For any $\lambda \in [0, +\infty]$: $\int_\Omega f \,\mathrm{d}\mu = \lambda \int_\Omega f \,\mathrm{d}\mu$.

(d) If $f(x) = 0$ for μ-almost every x, then $\int_\Omega f \,\mathrm{d}\mu = 0$.

(e) For any $0 < \lambda < \infty$ one has

$$\mu(\{x \in \Omega \mid f(x) \geq \lambda\}) \leq \frac{1}{\lambda} \int_\Omega f \,\mathrm{d}\mu$$