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# Advanced Methods in Mathematical Analysis

Winter Semester 2025/26 —— Sheet 11

## Task 1 (The obstacle problem)

This task will focus on the obstacle problem posed on a bounded Lipschitz domain $\Omega \subset \mathbb{R}^d$, which consists in finding a function $u \in H_0^1(\Omega)$ that minimises the following energy

$$I\colon H_0^1(\Omega) \to \mathbb{R} \cup \{+\infty\}$$
$$I(v) := \frac{1}{2}\int_\Omega |\nabla v(x)|^2 \,\mathrm{d}x - \langle f, v\rangle_{H^{-1}(\Omega);H_0^1(\Omega)} + \chi_K(v),$$

where $f \in H^{-1}(\Omega) = (H_0^1(\Omega))^*$ is given, and $K$ is the admissible set associated to a given obstacle $\varphi \in H^1(\Omega)$ with $\varphi|_{\partial\Omega} \leq 0$:

$$K := \{v \in H_0^1(\Omega) \mid v \geq \varphi \text{ a.e. on } \Omega\}.$$

(a) Prove that $I$ is convex, proper and lower semicontinuous on $H_0^1(\Omega)$.

(b) Using the direct method of the calculus of variations, prove that there is a unique solution to the obstacle problem.

(c) Based on the optimality condition $0 \in \partial I(u)$, prove that the solution $u$ satisfies the following variational inequality:

$$\int_\Omega \nabla u \cdot \nabla(v-u) \geq \langle f, v-u\rangle_{H^{-1}(\Omega);H_0^1(\Omega)} \qquad \forall\, v \in K.$$

(d) Assuming additional regularity, e.g. $u \in C^2(\overline{\Omega})$, $\varphi, f \in C(\overline{\Omega})$ should suffice, prove that the obstacle problem can be re-written as:

$$\begin{cases} -\Delta u - f \geq 0 & \text{in } \Omega,\\ u - \varphi \geq 0 & \text{in } \Omega,\\ (u-\varphi)(\Delta u + f) = 0 & \text{in } \Omega. \end{cases}$$

How can you interpret these conditions?

(e) [**Bonus for those with knowledge of distribution theory**] Prove that any positive distribution $F \in \mathcal{D}'(\Omega)$ (meaning that $F(\psi) \geq 0$ for all $\psi \in \mathcal{D}(\Omega)$ with $\psi \geq 0$) can be extended to an element of $\mathcal{M}(\Omega)$.

(f) Prove that $\mu := -\Delta u - f \in H^{-1}(\Omega)$ is positive and therefore can be identified with a Radon measure.

(g) It can be shown that every $v \in H_0^1(\Omega)$ has a representative $\widetilde{v}$ that is $\mu$-measurable for which one can write

$$\langle \mu, v\rangle_{H^{-1}(\Omega);H_0^1(\Omega)} = \int_\Omega \widetilde{v}\,\mathrm{d}\mu.$$

Moreover, the obstacle constraint can be interpreted also as $\widetilde{u} \geq \widetilde{\varphi}$ $\mu$-a.e. in $\Omega$. In addition, there is a sequence $\{\varphi_k\}_{k\in\mathbb{N}} \subset K$ such that $\widetilde{\varphi}_k$ decreases pointwise $\mu$-a.e. to $\widetilde{\varphi}$.

Using these facts, prove that $u \in K$ is the solution to the obstacle problem if and only if $\mu := -\Delta u - f \in H^{-1}(\Omega)$ is a positive Radon measure and the following complementarity condition holds:

$$\int_\Omega (\widetilde{u} - \widetilde{\varphi}) \, \mathrm{d}\mu = 0$$

This is the generalisation of (d) for less regular solutions.

**Task 2 (The incompressible Navier–Stokes equations)**

The focus of this task is the incompressible Navier–Stokes system. Here we look for a velocity field $\boldsymbol{u}\colon \overline{\Omega} \to \mathbb{R}^d$ and a pressure $p\colon \Omega \to \mathbb{R}$ (again $\Omega \subset \mathbb{R}^d$ is bounded and Lipschitz and $d \in \{2, 3\}$), such that

$$\begin{cases} -\Delta \boldsymbol{u} + \operatorname{div}(\boldsymbol{u} \otimes \boldsymbol{u}) + \nabla p = \boldsymbol{f} & \text{in } \Omega, \\ \operatorname{div} \boldsymbol{u} = 0 & \text{in } \Omega, \\ \boldsymbol{u} = \boldsymbol{0} & \text{on } \partial\Omega. \end{cases}$$

Here $\boldsymbol{f}\colon \Omega \to \mathbb{R}^d$ is a given forcing term.

(a) Assume that $\boldsymbol{f} \in H^{-1}(\Omega)^d$. Derive a weak formulation of the Navier–Stokes equations posed on the space of divergence-free functions (i.e. $\boldsymbol{u} \in H^1_{0,\mathrm{div}}(\Omega)$)

$$H^1_{0,\mathrm{div}}(\Omega) := \{\boldsymbol{v} \in H^1_0(\Omega)^d \mid \operatorname{div} \boldsymbol{v} = 0 \text{ a.e. in } \Omega\}$$

(You should check that this is in fact a Hilbert space.)

(b) Prove that for any $\boldsymbol{v} \in H^1_{0,\mathrm{div}}(\Omega)$ one has:

$$\int_\Omega (\boldsymbol{v} \otimes \boldsymbol{v}) : \nabla \boldsymbol{v} = 0$$

(You should justify why this integral is well-defined in the first place.)

(c) Consider the following operator $T\colon H^1_{0,\mathrm{div}}(\Omega) \to H^1_{0,\mathrm{div}}(\Omega)$:

$$T(\boldsymbol{v}) = (-\Delta)^{-1}(\boldsymbol{f} - \operatorname{div}(\boldsymbol{v} \otimes \boldsymbol{v})),$$

where $(-\Delta)^{-1}\colon (H^1_{0,\mathrm{div}}(\Omega))^* \to H^1_{0,\mathrm{div}}(\Omega)$ is the solution operator to the Laplace problem with zero boundary conditions. Verify that $T$ is well-defined and is in fact a compact operator.

(d) Making use of $T$, reformulate the weak formulation from (a) as a fixed point problem and with the help of Schäffer's Fixed Point Theorem prove that a solution exists.

(e) [**Bonus**] Prove that if $\boldsymbol{f}$ is small enough, the solution $\boldsymbol{u}$ is unique.